Chapter 4: Turbulence at Small Scales

Part 1: Spectral Representation of $\tilde{\varepsilon}$

$$\tilde{\varepsilon} = \upsilon \overline{u_{i,j} u_{i,j}}$$

For homogeneous/isotropic turbulence $\tilde{\varepsilon} = \varepsilon$, since

$$\varepsilon = \tilde{\varepsilon} + \upsilon \frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j} = \upsilon \left(\overline{u_{i,j} u_{i,j}} + \frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j} \right)$$

$$\frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j} = \overline{u_{i,j} u_{j,i}} = 0$$

Consider the following terms:

$$\frac{\partial}{\partial x_i}\left(\overline{u_i u_j}\right) = \overline{u_i u_{j,i}} + \overline{u_{i,i} u_j} \quad (1)$$

$$\frac{\partial}{\partial x_j}\left(\overline{u_i \frac{\partial u_j}{\partial x_i}}\right) = \overline{u_{i,j} u_{j,i}} + \overline{u_i \frac{\partial^2 u_j}{\partial x_i \partial x_j}} \quad (2)$$

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\left(\overline{u_i u_j}\right) = \overline{u_{i,j} u_{j,i}} \quad (3)$$

Substituting (1) into LHS term in parentheses in (2) shows that (2) = (3).

In homogeneous turbulence, the fluctuating velocity $\underline{u}(\underline{x}, t)$ is statistically homogeneous, and the time-averaged properties of the flow are uniform and independent of position, i.e., $\frac{\partial}{\partial x_j} \overline{fluctuating\ terms} = 0$. Therefore,

$$(1) = (2) = (3) = 0$$

For homogeneous turbulence, the two-point velocity correlation tensor $\mathcal{R}_{ij}(\underline{x}, \underline{y}, t)$ depends only on the relative position of $\underline{x}, \underline{y}$ and not their absolute positions:

$$\mathcal{R}_{ij}\left(\underline{x}, \underline{y}, t\right) = \overline{u_i(\underline{x}, t) u_j(\underline{y}, t)} = \mathcal{R}_{ij}\left(\underline{r}, t\right) \quad (4)$$

where $\underline{r} = \underline{y} - \underline{x} \rightarrow r_l = y_l - x_l$.

$$\mathcal{R}_{ij}\left(\underline{r}\right) = \overline{u_i(\underline{x}) u_j(\underline{y})}$$

where the dependence on time is implied for ensemble averaging. Differentiating $\mathcal{R}_{ij}\left(\underline{r}\right)$ with respect to x_k (gradient of 2nd order tensor is third order tensor).

Chain rule

Third-order tensor

$$\frac{\partial}{\partial x_k} \overline{u_i(\underline{x}) u_j(\underline{y})} = \frac{\partial \mathcal{R}_{ij}}{\partial x_k} = \frac{\partial \mathcal{R}_{ij}}{\partial r_l} \frac{\partial r_l}{\partial x_k} \quad (5)$$

$$\frac{\partial}{\partial x_k} \overline{u_i(\underline{x}) u_j(\underline{y})} = \frac{\partial \mathcal{R}_{ij}}{\partial r_l} \frac{\partial (y_l - x_l)}{\partial x_k} = -\frac{\partial \mathcal{R}_{ij}}{\partial r_l} \underbrace{\frac{\partial x_l}{\partial x_k}}_{\delta_{lk}}$$

$$\frac{\partial}{\partial x_k} \overline{u_i(\underline{x}) u_j(\underline{y})} = -\frac{\partial \mathcal{R}_{ij}}{\partial r_k} \quad (6)$$

Note:

$$\frac{\partial (y_l - x_l)}{\partial x_k} = -\frac{\partial x_l}{\partial x_k} = -\delta_{lk}$$

$$\frac{\partial \mathcal{R}_{ij}}{\partial r_l} \delta_{lk} = \frac{\partial \mathcal{R}_{ij}}{\partial r_k}$$

Third order tensor

Kronecker delta function

Third order tensor with change of index

Applying $\partial/\partial y_l$ to Eq. (6) (gradient of 3rd order tensor is fourth order tensor):

$$\text{Fourth-order tensor} \longleftarrow \overline{\frac{\partial u_i(\underline{x})}{\partial x_k}\frac{\partial u_j(\underline{y})}{\partial y_l}} = -\frac{\partial^2 \mathcal{R}_{ij}}{\partial r_k \partial r_l}\underbrace{\frac{\partial r_l}{\partial y_l}}_{1} = -\frac{\partial^2 \mathcal{R}_{ij}}{\partial r_k \partial r_l} \quad (7)$$

Applying the limit for $\underline{y} \to \underline{x}$, $\underline{r} = 0$,

$$\overline{u_{i,k}(\underline{x})u_{j,l}(\underline{x})} = -\frac{\partial^2 \mathcal{R}_{ij}(0)}{\partial r_k \partial r_l} = \text{Constant} \neq f(\underline{x}) \quad (8)$$

RHS of (7) is only a function of relative position $f(\underline{r})$ and not $\underline{x}$ or $\underline{y}$, whereas for $\underline{r} = 0$, RHS of (8) is not a function of any position; therefore, must be constant. Thus, the LHS of (8) must also be constant both mathematically and invoking the fact that for homogenous turbulence the statistics for fluctuating terms are constant.

Setting $j = i$ and $l = k$ and using the definition of ε for isotropic turbulence $\varepsilon = \upsilon\overline{u_{i,k}u_{i,k}}$, we obtain:

$$\frac{\varepsilon}{\upsilon} = \overline{u_{i,k}u_{i,k}} = -\frac{\partial^2 \mathcal{R}_{ii}(0)}{\partial {r_k}^2} \quad (9)$$

Scalar

Recall

$$\mathcal{E}_{ij}(\underline{\kappa},t) = \frac{1}{(2\pi)^3}\int_{\forall} e^{i\underline{\kappa}\cdot\underline{r}}\,\mathcal{R}_{ij}(\underline{r},t)\,d\underline{r}$$

$$\mathcal{R}_{ij}(\underline{r},t) = \int_{\forall} e^{-i\underline{\kappa}\cdot\underline{r}}\,\mathcal{E}_{ij}(\underline{\kappa},t)\,d\underline{\kappa}$$

Fourier transform pair

Where:

$\mathcal{E}_{ij}$ = velocity spectrum tensor (Φ_{ij} Pope, energy spectrum tensor Bernard).

$\mathcal{R}_{ij}$ = 2-point, 2-velocity correlation tensor.

Using

$$\mathcal{R}_{ii}(\underline{r},t) = \int_{\forall} \mathcal{E}_{ii}(\underline{\kappa},t)e^{-i\underline{\kappa}\cdot\underline{r}}d\underline{\kappa}$$

where $d\underline{\kappa} = d\kappa_1 d\kappa_2 d\kappa_3$ and ∀ = infinite volume $(-\infty,\infty)$ in wave number space. Taking the first derivative with respect to r_k

$$\frac{\partial\mathcal{R}_{ii}}{\partial r_k} = \frac{\partial}{\partial r_k}\left[\int_{\forall} \mathcal{E}_{ii}(\underline{\kappa},t)e^{-i\underline{\kappa}\cdot\underline{r}}d\underline{\kappa}\right] = \int_{\forall} \mathcal{E}_{ii}(\underline{k},t)e^{-i\underline{\kappa}\cdot\underline{r}}\frac{\partial(-i\underline{\kappa}\cdot\underline{r})}{\partial r_k}d\underline{\kappa}$$

Which is a vector (since $\mathcal{R}_{ii}$ is a scalar and its gradient $\frac{\partial}{\partial r_k}$ is a vector) and evaluate for $k = 1,2,3$

$$\frac{\partial\mathcal{R}_{ii}}{\partial r_1} = \int_{\forall} \mathcal{E}_{ii}(\underline{\kappa},t)e^{-i\underline{\kappa}\cdot\underline{r}}\frac{\partial[-i(r_1\kappa_1 + r_2\kappa_2 + r_3\kappa_3)]}{\partial r_1}d\underline{\kappa}$$

$$= \int_{\forall} -i\kappa_1\mathcal{E}_{ii}(\underline{\kappa},t)e^{-i\underline{\kappa}\cdot\underline{r}}d\underline{\kappa}$$

$$\frac{\partial \mathcal{R}_{ii}}{\partial r_2} = \int_{\forall} -i\kappa_2 \mathcal{E}_{ii}(\underline{\kappa}, t) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{\kappa}$$

$$\frac{\partial \mathcal{R}_{ii}}{\partial r_3} = \int_{\forall} -i\kappa_3 \mathcal{E}_{ii}(\underline{\kappa}, t) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{\kappa}$$

Therefore, for any k, we obtain

Vector ←

$$\frac{\partial \mathcal{R}_{ii}}{\partial r_k} = \int_{\forall} -i\kappa_k \mathcal{E}_{ii}(\underline{\kappa}, t) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{\kappa} \qquad (10)$$

And evaluating at $\underline{r} = 0$

$$\frac{\partial \mathcal{R}_{ii}(0)}{\partial r_k} = \int_{\forall} -i\kappa_k \mathcal{E}_{ii}(\underline{\kappa}, t) d\underline{\kappa}$$

Taking another derivative of Eq. (10) with respect to r_k yields

$$\frac{\partial^2 \mathcal{R}_{ii}}{\partial {r_k}^2} = \frac{\partial}{\partial r_k}\left[\int_{\forall} -i\kappa_k \mathcal{E}_{ii}(\underline{\kappa}, t) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{\kappa}\right] \qquad (11)$$

Which is a scalar since $\frac{\partial^2}{\partial {r_k}^2}$ is a scalar which when operating on a scalar is also a scalar. Also note that

$$\underline{\kappa} \cdot \underline{r} = \kappa_k r_k$$

Such that

$$\frac{\partial^2 \mathcal{R}_{ii}}{\partial {r_k}^2} = \frac{\partial}{\partial r_k}\left[\int_{\forall} (-i\kappa_k) \mathcal{E}_{ii}(\underline{\kappa}, t) e^{-i\kappa_k r_k} d\underline{\kappa}\right]$$

$$= \int_{\forall} (-i\kappa_k)\mathcal{E}_{ii}\left(\underline{\kappa}, t\right) e^{-i\kappa_k r_k} \frac{\partial(-i\kappa_k r_k)}{\partial r_k} d\underline{\kappa}$$

$$= \int_{\forall} (-i\kappa_k)(-i\kappa_k)\mathcal{E}_{ii}\left(\underline{\kappa}, t\right) e^{-i\kappa_k r_k} d\underline{\kappa}$$

$$= \int_{\forall} i^2 {\kappa_k}^2 \mathcal{E}_{ii}\left(\underline{\kappa}, t\right) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{\kappa}$$

At $r = 0, i^2 = -1$

$$-\frac{\partial^2 \mathcal{R}_{ii}(0)}{\partial {r_k}^2} = \frac{\varepsilon}{\upsilon} = \int_{\forall} \kappa_k \kappa_k \mathcal{E}_{ii}\left(\underline{\kappa}, t\right) e^{-i\underline{\kappa}\cdot\underline{r}} d\underline{\kappa}$$
$$= \int_{\forall} (\kappa_1\kappa_1 + \kappa_2\kappa_2 + \kappa_3\kappa_3)\mathcal{E}_{ii}\left(\underline{\kappa}, t\right) d\underline{\kappa}$$

$$-\frac{\partial^2 \mathcal{R}_{ii}(0)}{\partial {r_k}^2} = \frac{\varepsilon}{\upsilon} = \int_{\forall} \kappa^2 \mathrm{tr}\mathcal{E}\left(\underline{\kappa}, t\right) d\underline{\kappa} = \int_{\forall} \kappa^2 \underbrace{\mathcal{E}_{ii}\left(\underline{\kappa}, t\right)} d\underline{\kappa}$$

Density energy k space

$$= \frac{\varepsilon}{\upsilon} = 2\int_{\forall} \kappa^2 \underbrace{\left[\frac{1}{2}\int_{|\underline{\kappa}|=\kappa} \mathcal{E}_{ii}\left(\underline{\kappa}, t\right) d\Omega\right]} d\kappa$$

Energy spectrum = $E(\kappa, t)$

$d\underline{k} = d\Omega d\kappa$

$d\Omega$ =elemental solid angle

$E(\kappa, t)$ shows how KE is distributed across different scales, i.e., collects energy onto shells of radius $|\underline{\kappa}| = \kappa$.

$$\varepsilon = 2\,\upsilon \int_0^\infty \kappa^2 E(\kappa, t)\, d\kappa$$

Note

$$K(t) = KE = \frac{1}{2}\int_\forall \mathcal{E}_{ii}(\underline{\kappa}, t)\, d\underline{\kappa}$$

$$= \int_0^\infty \left[\frac{1}{2}\mathcal{E}_{ii}(\underline{\kappa}, t)\, d\Omega\right] d\kappa$$

$$K(t) \quad = \int_0^\infty E(\kappa, t)\, d\kappa$$

Motions making the biggest contributions to K and ε come from different wave number ranges: κ_e and κ_d.

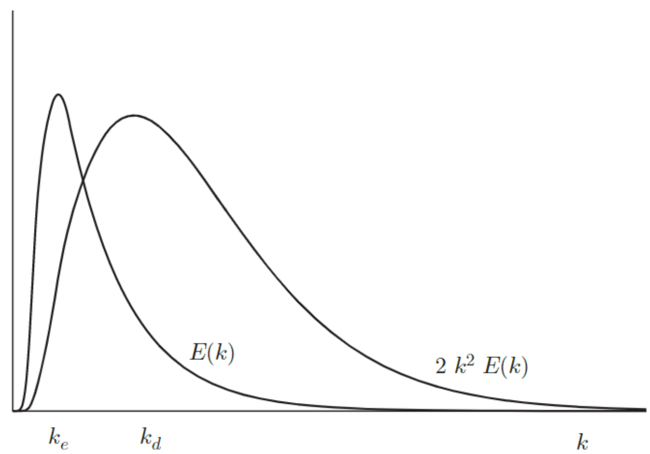

Figure 4.1 Spectral ranges of $E(k)$ and $2k^2E(k)$, with k_e and k_d marking their respective peaks.

Energy containing eddies size: $l_e = 1/\kappa_e$

Dissipation eddies size: $l_d = 1/\kappa_d$

Physics of transfer process i.e., energy cascade fundamental to theory of turbulence. Separation between scales increases with Re and for sufficiently high Re the transfer occurs without dissipation at intermediate scales.