Chapter 4 Turbulence at Small Scales

Part 0 The Energy Cascade and Kolmogorov Hypotheses (Pope 6.1)

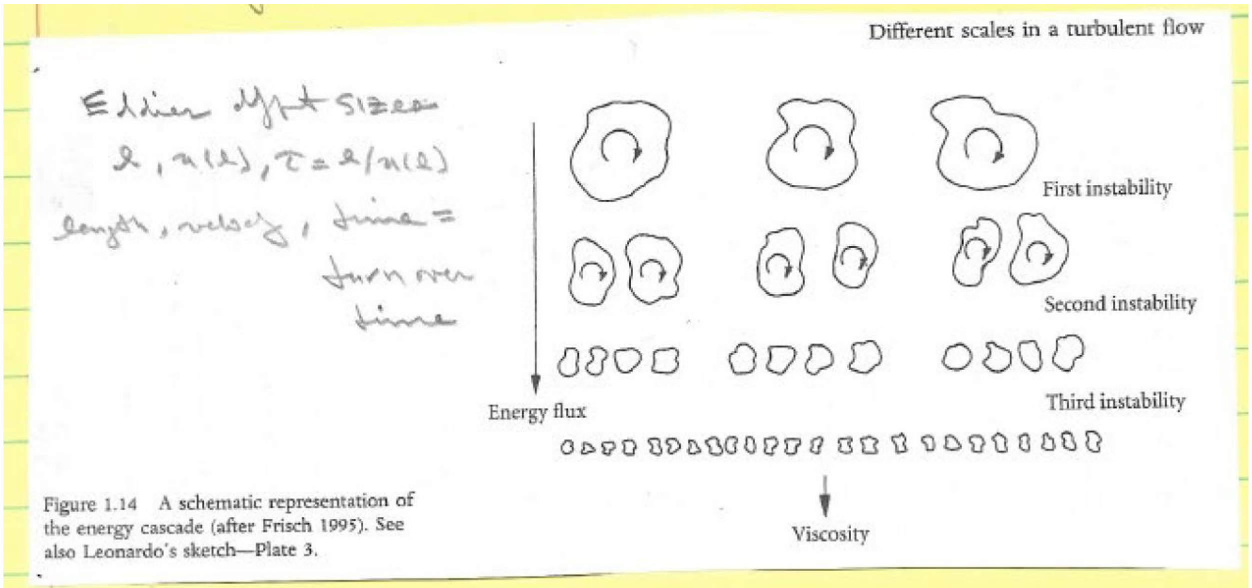


Figure 1.14 A schematic representation of the energy cascade (after Frisch 1995). See also Leonardo's sketch—Plate 3.

Assume large $Re = UL/\nu$ for large scale flow with geometries and flows of interest: wall (channel, pipe, or boundary layer) or free shear flows.

Largest size eddies:

$$l_0 \sim L, \quad u_0(l_0) \sim u' = u_{rms} \sim U, \quad \tau_0 = \frac{l_0}{u_0}$$

$$k = \frac{1}{2} < u^2 + v^2 + w^2 >$$

if isotropic = $k = \frac{3}{2} < u^2 >$

$$u_{rms} = \left[\frac{2}{3}k\right]^{1/2} \approx k^{1/2}$$

Smallest size eddies:

$$\eta, u_\eta, \tau_\eta$$ Kolmogorov scales

Energy Cascade:

Energy transferred from the largest to successively smaller scales until $Re_\eta = \frac{u_\eta \eta}{\nu} \sim 1$ such that eddy motion is stable and viscosity dissipates the TKE.

Leonardo's Da Vinci: sketch of water falling into a pool. Note the different scales of motion, suggestive of the energy cascade.

Rate of dissipation ε is determined by the largest scales with energy u_0^2 and time scale $\tau_0 = \frac{l_0}{u_0}$; therefore,

$$\varepsilon \approx \frac{u_0^2}{\tau_0} \approx \frac{u_0^3}{l_0} \neq f(\nu)! \quad (\mathrm{m^2/s^3})$$

Important assumption is that both $u(l)$ and $\tau(l)$ decrease as l decreases.

Kolmogorov's hypothesis of local isotropy:

At high Reynolds number, the small-scale turbulent motions $(l \ll l_0)$ *are statistically isotropic* .

Define length scale l_{EI} as the demarcation between the anisotropic large eddies and the isotropic small eddies

$l > l_{EI} \sim \frac{1}{6} l_0$	anisotropic large eddies
$l < l_{EI}$	isotropic small eddies → information mean flow and BCs is lost → statistics of the small-scale motions are universal. i.e., similar all high-Reynolds number turbulent flows.

Two important parameters: energy transfer from large scales $\mathcal{T}_{EI} \approx \varepsilon$ and viscous diffusion ν (m^2/s).

Kolmogorov's first similarity hypothesis[1]:

at high Reynolds number, small-scale motions $(l < l_{EI})$ *have universal form uniquely* f(ε, ν) = universal equilibrium range.

The size range $l < l_{EI}$ is referred to as the *universal equilibrium range*.

[1] Paraphrased

Thus, Kolmogorov scales are only function of ε and ν, i.e., $\eta(\varepsilon, \nu)$, $u_\eta(\varepsilon, \nu)$, and $\tau_\eta(\varepsilon, \nu)$ can be determined by dimensional analysis:

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}}$$

$$u_\eta = (\varepsilon \nu)^{\frac{1}{4}}$$

$$\tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{\frac{1}{2}}$$

The ratios of the smallest to largest scales can be obtained using $\varepsilon = \frac{u_0^3}{l_0}$

$$\eta / l_0 \sim Re^{-3/4}$$

$$u_\eta / u_0 \sim Re^{-1/4}$$

$$\tau_\eta / \tau_0 \sim Re^{-1/2}$$

which shows how scales decrease with $Re = \frac{u_0 l_0}{\nu}$.

Thus,

$$Re_\eta = \frac{u_\eta \eta}{\nu} = 1$$

$$\varepsilon = \nu \left(u_\eta / \eta\right)^2 = \nu / {\tau_\eta}^2$$

$$\frac{u_\eta}{\eta} = 1/\tau_\eta$$

Velocity gradient of the dissipative eddies

= the inverse of the turnover time

Alternative reasoning:

$$\varepsilon = \frac{u_0^2}{\tau_0} = \frac{u_0^3}{l_0}$$ Largest scales

$$= \nu \varepsilon_{ij}\varepsilon_{ij} = \nu \left(u_\eta/\eta\right)^2$$ Smallest scales, as per TKE equation

$$\frac{u_0^3}{l_0} = \nu \left(u_\eta/\eta\right)^2 \text{ but } Re_\eta = \frac{u_\eta \eta}{\nu} = 1$$

$$\left.\begin{aligned} \eta &= l_0 Re^{-3/4} \\ u_\eta &= l_0 Re^{-1/4} \end{aligned}\right\} \rightarrow \begin{aligned} \eta &= (\nu^3/\varepsilon)^{1/4} \\ u_\eta &= (\varepsilon\nu)^{\frac{1}{4}} \end{aligned}$$

$$Re = \frac{u_0 l_0}{\nu}$$

How large is η?

Cases	*Re*	η / l_o	l_o	η
Educational experiments	10^3	5.6×10^{-3}	~ 1 cm	0.056 mm
Model-scale experiments	10^6	3.2×10^{-5}	~ 1 m	0.032 mm
Full-scale experiments	10^9	1.8×10^{-7}	~ 100 m	0.018 mm

The smallest fluid motion scales for ship and airplane:

	U(m/s)	L(m)	ν (m^2/s)	Re	η (mm)	u_η (m/s)	τ_η (s)
Ship (Container: ALIANCA MAUA)	11.8 (23.3 knots)	272	9.76E-7	3.3E09	0.02	0.05	4E-4
Airplane (Airbus A300)	216.8 (Ma=0.64)	56.2	3.7E-5 (z=10Km)	0.3E09	0.023	1.64	1.4E-5

Much of the energy in this flow is dissipated in eddies which are less than fraction of a millimeter in size!!

For a ship with $L = 272m$, the BL thickness at L can be estimated using Prandtl's one-seventh power-law for the turbulent boundary layer thickness. Similarly, for an airplane, the BL thickness at the trailing edge of the wing can be estimated using the chord length $L = 5m$. The corresponding δ in each case is equal to:

$$\delta_{ship} = \frac{0.16L}{Re_L^{\frac{1}{7}}} \sim 1.9m$$

$$\delta_{wing} = \frac{0.16L}{Re_L^{\frac{1}{7}}} \sim 7cm$$

The corresponding boundary-layer Reynolds number are:

$$Re_{\delta_{ship}} = 2.3E + 7$$

$$Re_{\delta_{wing}} = 5E + 5$$

Using the relation $\eta/\delta \sim Re_\delta^{-3/4}$, the Kolmogorov scales can be estimated as:

$$\eta_{ship} = 0.0057mm = 5.7\ \mu m$$

$$\eta_{wing} = 0.004mm = 4\ \mu m$$

These represent the dissipation length scale based on the thickness of the boundary layer, instead of the length of the ship/airplane.

In terms of airplane turbulence, eddies that are roughly the same size as the aircraft itself are the ones that cause the most noticeable turbulence; meaning, smaller eddies create smaller bumps, while larger eddies might move the whole plane up and down, but the most impactful turbulence is felt when eddies are close in size to the airplane's wingspan and length, causing noticeable rolling and pitching motions.

Kolmogorov's second similarity hypothesis[2]:

at high Reynolds number, the statistics of the motions $l_0 \gg l \gg \eta$ are uniquely determined *by ε and not f(v).*

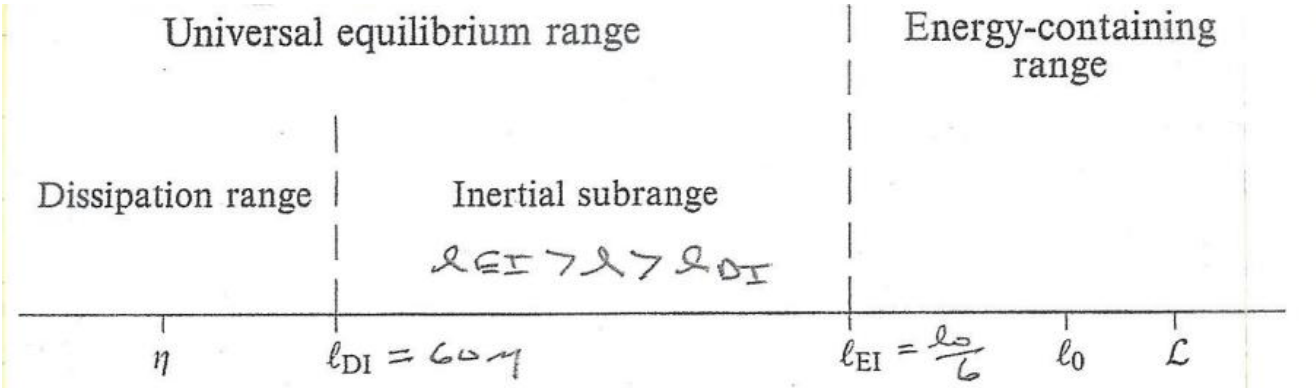


Fig. 6.1. Eddy sizes ℓ (on a logarithmic scale) at very high Reynolds number, showing the various lengthscales and ranges.

In the inertial subrange, viscous effects are negligible.

$l_{EI} = l_0/6$ and $l_{DI} = 60\eta$ based on fact that 80% energy is for $l > l_0/6$ and most of the dissipation is for $l < 60\eta$ in the Energy and Dissipation spectrums, respectively, as per Chapter 4, Part 7.

E=energy range

I=inertial sub range

D=dissipation range

EI is boundary between E and I

DI is boundary between I and D

[2] Paraphrased

Length, velocity, and time scales l, u, τ cannot be formed using ε only, but using ε and l (in the inertial sub range)

$$u(l) = (\varepsilon l)^{1/3} = u_\eta (l/\eta)^{1/3} \sim u_0 (l/l_0)^{1/3}$$

$$\tau(l) = (l^2/\varepsilon)^{1/3} = \tau_\eta (l/\eta)^{2/3} \sim \tau_0 (l/l_0)^{2/3}$$

That is both $u(l)$ and $\tau(l)$ decrease as l decreases.

At the boundary of the energy containing and inertial sub ranges, i.e., EI where l = l_0/6:

$$u_{EI} = u\left(\frac{l_0}{6}\right) \sim u_0 (6)^{-\frac{1}{3}} = 0.5504 u_0$$

$$= u_0/1.8165$$

$$\sim u_0/2$$

$$\tau_{EI} = \tau\left(\frac{l_0}{6}\right) \sim \tau_0 (6)^{-\frac{2}{3}} = 0.3028 \tau_0$$

$$= \tau_0/3.3025$$

$$\sim \tau_0/3$$

That is at EI:

- l_0 is reduced by about 1/6
- u_0 is reduced by about 1/2
- τ_0 reduced by about 1/3

According to the Kolmogorov hypotheses.

$\mathcal{T}(l)$ is the rate at which energy is transferred from eddies larger than l to those smaller than l.

$\mathcal{T}(l) = \frac{u(l)^2}{\tau(l)} = \varepsilon \neq f(\nu)$ in the inertial subrange

$$\mathcal{T}_{EI}(l_{EI}) = \mathcal{T}(l) = \mathcal{T}_{DI}(l_{DI}) = \varepsilon \qquad \boxed{l_{EI} > l > l_{DI}}$$

Rate of energy transfer from the large scales determines the constant rate of energy transfer through the inertial subrange and that which enters the dissipation range.

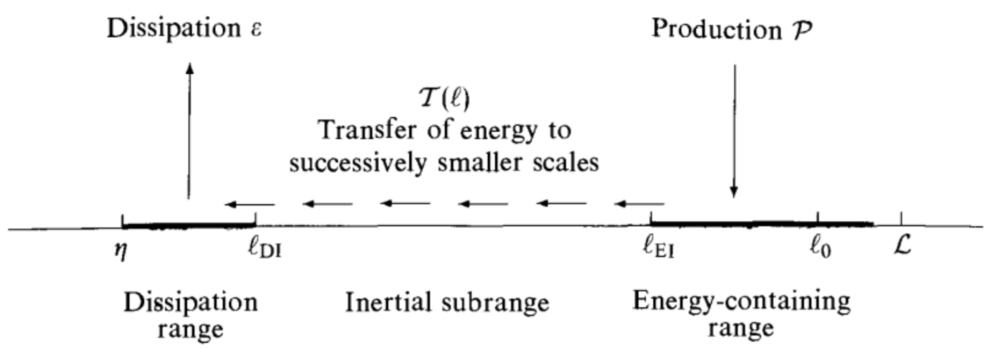
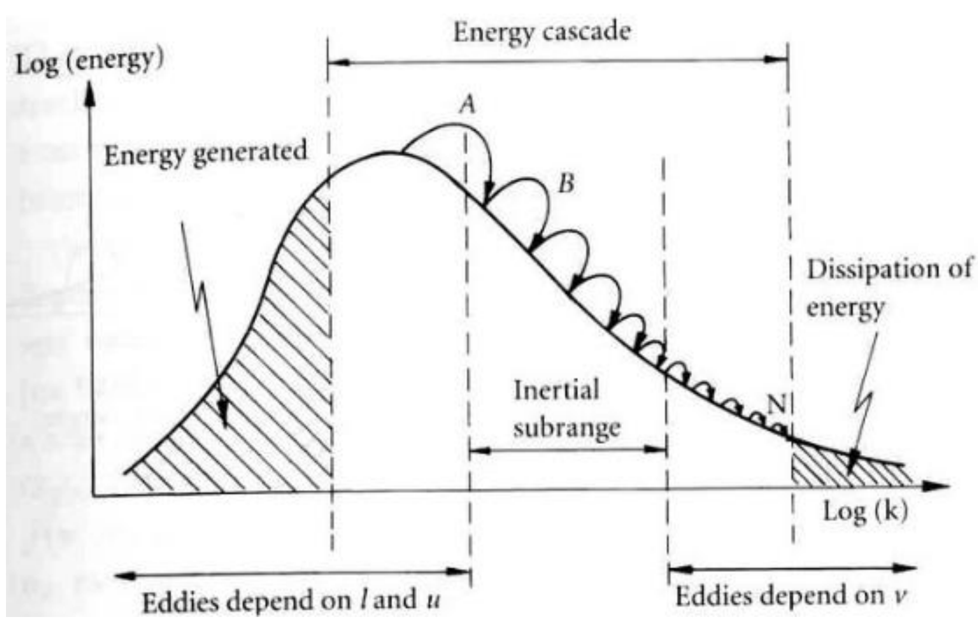


Fig. 6.2. A schematic diagram of the energy cascade at very high Reynolds number.



The energy spectrum: TKE distribution as $f(eddy\ size)$, as per Chapter 2

$$k_{(\kappa_a,\kappa_b)} = \int_{\kappa_a}^{\kappa_b} E(\kappa)d\kappa$$

Where $\kappa = 2\pi/l$= wave number and k =TKE; and similarly, the dissipation spectrum, as per Chapter 4, Part 1

$$\varepsilon_{(\kappa_a,\kappa_b)} = \int_{\kappa_a}^{\kappa_b} 2\nu\kappa^2 E(\kappa)d\kappa$$

In the universal equilibrium range ($\kappa > \kappa_{EI} \equiv 2\pi/l_{EI}$), E($\varepsilon$,$\nu$); whereas in the inertial subrange ($\kappa_{EI} < \kappa < \kappa_{DI} \equiv 2\pi/l_{DI}$) E($\varepsilon$) only such that using dimensional analysis, as per Chapter 4, Part 4

$$E(\kappa) = C\varepsilon^{2/3}\kappa^{-5/3} \quad (\mathrm{m^3/s^2})$$

where C is a universal constant. Which is in the form of a power-law spectrum

$$E(\kappa) = A\kappa^{-p}$$

with A and p constants. The energy contained in wave numbers greater than κ is

$$k_{(\kappa,\infty)} = \int_{\kappa}^{\infty} E(\kappa')d\kappa' = \frac{A}{p-1}\kappa^{-(p-1)}$$

for $p > 1$, whereas the integral diverges for $p \leq 1$.

Similarly, the dissipation in wavenumbers less than κ is

$$\varepsilon_{(0,\kappa)} = \int_0^\kappa 2\nu\kappa'^2 E(\kappa')d\kappa' = \frac{2\nu A}{3-p}\kappa^{3-p}$$

for $p < 3$, whereas the integral diverges for $p \geq 3$.

Thus, $p = 5/3$ = the Kolmogorov spectrum, is around the middle of the range (1,3) for which the integrals for $k_{(\kappa,\infty)}$ and $\varepsilon_{(0,\kappa)}$ converge.

For $p = \frac{5}{3} = 1.667$, $k_{(\kappa,\infty)} = \frac{A}{2/3}\kappa^{-2/3}$, i.e., amount of energy for high wave numbers decreases as $k_{(\kappa,\infty)} \sim \kappa^{-2/3}$ as κ increases, whereas the dissipation for low wave numbers decreases as $\varepsilon_{(0,\kappa)} = \frac{2\nu A}{4/3}\kappa^{4/3}$ as κ decreases towards zero.

Thus, the bulk of energy is at large scales $l > l_{EI}$ or $\kappa < 2\pi/l_{EI}$

and the bulk of dissipation is in small scales $l < l_{DI}$ or $\kappa > 2\pi/l_{DI}$

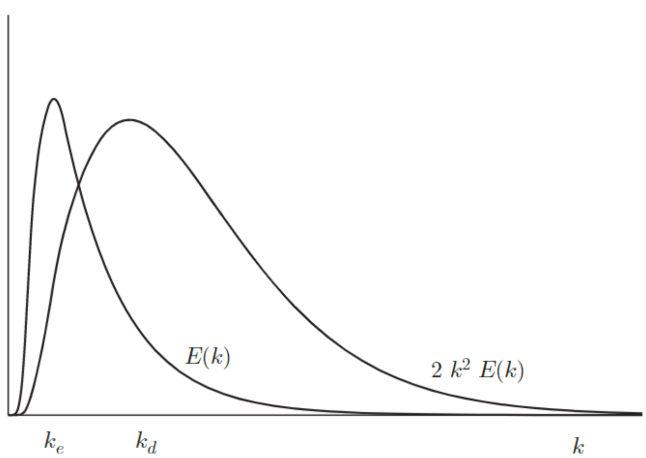


Figure 4.1 Spectral ranges of $E(k)$ and $2k^2E(k)$, with k_e and k_d marking their respective peaks.

6.1.4 Restatement of the Kolmogorov hypotheses

In order to deduce precise consequences from them, it is worthwhile to provide here more precise statements of the Kolmogorov (1941) hypotheses. Kolmogorov presented these in terms of an N-point distribution in the four-dimensional $\mathbf{x}$–t space. Here, however, we consider the N-point distribution in physical space ($\mathbf{x}$) at a fixed time t – which is sufficiently general for most purposes.

Consider a simple domain $\mathcal{G}$ within the turbulent flow, and let $\mathbf{x}^{(0)}$, $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}$ be a specified set of points within $\mathcal{G}$. New coordinates and velocity differences are defined by

$$\mathbf{y} \equiv \mathbf{x} - \mathbf{x}^{(0)}, \tag{6.20}$$

$$\mathbf{v}(\mathbf{y}) \equiv \mathbf{U}(\mathbf{x}, t) - \mathbf{U}(\mathbf{x}^{(0)}, t), \tag{6.21}$$

and the joint PDF of $\mathbf{v}$ at the N points $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(N)}$ is denoted by f_N.

The definition of local homogeneity. The turbulence is locally homogeneous in the domain $\mathcal{G}$, if for every fixed N and $\mathbf{y}^{(n)}(n = 1, 2, \ldots, N)$, the N-point PDF f_N is independent of $\mathbf{x}^{(0)}$ and $\mathbf{U}(\mathbf{x}^{(0)}, t)$.

The definition of local isotropy. The turbulence is locally isotropic in the domain $\mathcal{G}$ if it is locally homogeneous and if in addition the PDF f_N is invariant with respect to rotations and reflections of the coordinate axes.

The hypothesis of local isotropy. In any turbulent flow with a sufficiently large Reynolds number ($\mathrm{Re} = \mathcal{UL}/\nu$), the turbulence is, to a good approximation, locally isotropic if the domain $\mathcal{G}$ is sufficiently small (i.e., $|\mathbf{y}^{(n)}| \ll \mathcal{L}$, for all n) and is not near the boundary of the flow or its other singularities.

The first similarity hypothesis. For locally isotropic turbulence, the N-point PDF f_N is uniquely determined by the viscosity ν and the dissipation rate ε.

The second similarity hypothesis. If the moduli of the vectors $\mathbf{y}^{(m)}$ and of their differences $\mathbf{y}^{(m)} - \mathbf{y}^{(n)}$ ($m \neq n$) are large compared with the Kolmogorov scale η, then the N-point PDF f_N is uniquely determined by ε and does not depend on ν.

It is important to observe that the hypotheses apply specifically to velocity differences. The use of the N-point PDF f_N allows the hypotheses to be applied to any turbulent flow, whereas statements in terms of wavenumber spectra apply only to flows that are statistically homogeneous (in at least one direction).

For inhomogeneous flows, local isotropy is possible only 'to a good approximation' (as stated in the hypothesis). For example, taking $\mathbf{y}^{(1)} = \mathbf{e}\ell$ and $\mathbf{y}^{(2)} = -\mathbf{e}\ell$ (where ℓ is a specified length and $\mathbf{e}$ a specified unit vector), we have

$$\begin{aligned} \langle \mathbf{v}(\mathbf{y}^{(1)}) - \mathbf{v}(\mathbf{y}^{(2)}) \rangle &= \langle \mathbf{U}(\mathbf{y}^{(1)}) \rangle - \langle \mathbf{U}(\mathbf{y}^{(2)}) \rangle \\ &\approx 2\frac{\ell}{\mathcal{L}} \mathbf{e} \cdot \mathcal{L} \nabla \langle \mathbf{U} \rangle. \end{aligned} \tag{6.22}$$

Evidently this simple statistic is not exactly isotropic, but instead has a small anisotropic component – of order $\ell/\mathcal{L}$ – arising from large-scale inhomogeneities.

6.2 Structure functions

To illustrate the correct application of the Kolmogorov hypotheses, we consider – as did Kolmogorov (1941b) – the second-order velocity structure functions. The predictions of the hypotheses are deduced, and then compared with experimental data.

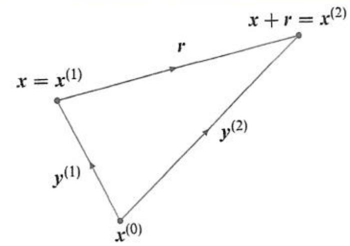


Fig. 6.3. A sketch showing the points x and $x + r$ in terms of $x^{(n)}$ and $y^{(n)}$. All points are within the domain $\mathcal{G}$.

By definition, the second-order velocity structure function is the covariance of the difference in velocity between two points $x + r$ and x:

$$D_{ij}(r, x, t) \equiv \langle [U_i(x + r, t) - U_i(x, t)][U_j(x + r, t) - U_j(x, t)] \rangle. \tag{6.23}$$

6.3 Two-point correlation

The Kolmogorov hypotheses, and deductions drawn from them, have no direct connection to the Navier–Stokes equations (although, as in the previous section, the continuity equation is usually invoked). Although, in the description of the energy cascade, the transfer of energy to successively smaller scales has been identified as a phenomenon of prime importance, the precise mechanism by which this transfer takes place has not been identified or quantified. It is natural, therefore, to try to extract from the Navier–Stokes equations useful information about the energy cascade. The earliest attempts (outlined in this section) are those of Taylor (1935a) and of von Kármán and Howarth (1938), which are based on the two-point correlation. The next two sections give the view from wavenumber space in terms of the energy spectrum – the Fourier transform of the two-point correlation.

Autocorrelation functions

Consider homogeneous isotropic turbulence, with zero mean velocity, r.m.s. velocity $u'(t)$, and dissipation rate $\varepsilon(t)$. Because of homogeneity, the two-point correlation

$$R_{ij}(r, t) \equiv \langle u_i(x + r, t) u_j(x, t) \rangle, \tag{6.41}$$

is independent of x. At the origin it is

$$R_{ij}(0, t) = \langle u_i u_j \rangle = u'^2 \delta_{ij}. \tag{6.42}$$