Part 3 The Smallest Scales

Micro Scales

$\tilde{\varepsilon}/\nu = \langle u_{i,k} u_{i,k} \rangle = -\frac{\partial^2 R_{ii}(0)}{\partial r_k^2}$ Part 1

$R_{ij}(\underline{r}) = \langle u_i(\underline{x})\, u_j(\underline{x}+\underline{r}) \rangle$ two-point velocity correlation

$\overline{u^2} f(r) = R_{11}(r\hat{e}_1) = \langle u(x)\, u(x+r) \rangle$

$\overline{u^2} f''(0) = -\langle (\frac{\partial u}{\partial x})^2 \rangle$, $f''(0) = -2/\lambda_f^2$ Chapter 2

$\overline{u_x^2} = \frac{2\overline{u^2}}{\lambda_f^2}$

$\langle u_{i,j} u_{k,l} \rangle = \alpha \delta_{ij}\delta_{kl} + \beta \delta_{ik}\delta_{jl} + \gamma \delta_{il}\delta_{jk}$

$\alpha = \gamma$ $\beta = -4\gamma$ $\alpha = -\beta/4$

$\overline{u_{1,1}^2} = \beta/2$ $\overline{u_{1,2}^2} = \beta$ $\overline{u_{1,1}u_{2,2}} = -\beta/4$ $\overline{u_{1,2}u_{2,1}} = -\beta/4$

$\overline{u_{1,2}^2} = 2\overline{u_{1,1}^2}$ $\overline{u_{1,1}u_{2,2}} = \overline{u_{1,2}u_{2,1}} = -\frac{1}{2}\overline{u_{1,1}^2}$

$\tilde{\varepsilon} = 15\nu\overline{u_{1,1}^2} = 30\nu\overline{u^2}/\lambda_f^2$

Important result as can estimate $\tilde{\varepsilon}$ from $\overline{u^2}$ & λ_f which are easier obtain than $\overline{u_x^2}$ etc. ie directly obtaining $\varepsilon = \nu\langle u_{i,k} u_{i,k} \rangle$ or even more difficult $\varepsilon = \tilde{\varepsilon} + \langle u_{i,j} u_{j,i} \rangle$

Scaling relationship

$\eta/l_0 \sim Re^{-3/4}$ $u_\eta/u_0 \sim Re^{-1/4}$ $\tau_\eta/\tau_0 \sim Re^{-1/2}$ Part ②

Micro Scales

$u_{rms} \sim k^{1/2}$ $\varepsilon = \frac{k^{3/2}}{L}$ $Re_L = \frac{k^{1/2}L}{\nu} = \frac{k^2}{\varepsilon\nu}$

velocity $L =$ turbulent length scale turbulent Re

Relationship macro & micro scales

$\varepsilon = 15\nu\,\overline{u_1^2}/\lambda_g^2 = k^{3/2}L/\nu \Rightarrow \lambda_g/L = \sqrt{10}\,Re_L^{-1/2}$ $\quad \lambda_g = \lambda_f/\sqrt{2}$

$\eta = (\nu^3/\varepsilon)^{1/4} \Rightarrow \eta/L = Re_L^{-3/4}$ $\quad \eta/\lambda_g = \frac{1}{\sqrt{10}} Re_L^{-1/4}$

Defining $R_\lambda = u_{rms}\lambda/\nu \Rightarrow R_\lambda = \left(\frac{20}{3} Re_L\right)^{1/2}$

Important result: ε determined from macro scales ie $k^{3/2}$ & L. $k^{3/2}$ straight forward but must estimate L eg for SOUP = D_{vortex}

ε can also be determined from micro scales & equating provides estimates of scaling depending on turbulent $Re = Re_L$, which shows that $\eta < \lambda < L$.

Also shows that $\lambda_g = L\sqrt{10}\,Re_L^{-1/2}$ can be estimated from the macro scales alone as can η.

Part 4 Inertial Subrange

Absolutely amazing that $-5/3$ power law can be visually observed in nature by artists

$K(t) = \int_0^\infty E(k,t)\,dk$ $\quad E(k,t)$ = energy spectrum

TKE $\quad k$ = wave number

Kolomogorov Hypotheses

Universal equilibrium range: $E(k) = \varepsilon^{2/3} k^{-5/3} \psi(k\eta)$

Inertial subrange: $E(k) = f(\varepsilon)$ $\quad \psi(k\eta)$ = compensated Kolomogorov spectrum function

ie independent $k\eta$

$\therefore$ $E(k) = C\,\varepsilon^{2/3} k^{-5/3}$ $\quad C = 1.5$

Most famous prominent feature high Re turbulence = universal power law form energy spectrum in inertial subrange