Chapter 3: Overview of Turbulent Flow Physics and Equations

Part 4: Dissipation Rate, Reynolds Stress, Mean and Fluctuating Vorticity and Enstrophy Equations

Notation:

$\underline{\Omega} = \underline{\overline{\Omega}} + \underline{\omega}$

$\underline{U} = \underline{\overline{U}} + \underline{u}$

$\tilde{p} = \overline{p} + p$

Note in Part 4 notation for p differs and ' is dropped for fluctuating pressure

ε Equation

$$\varepsilon = 2\upsilon\overline{e_{ij}e_{ij}} = \upsilon\left(\overline{{u_{i,j}}^2 + u_{i,j}u_{j,i}}\right) = \tilde{\varepsilon} + \upsilon\frac{\partial^2\overline{u_i u_j}}{\partial x_i \partial x_j}$$

$$\tilde{\varepsilon} = \upsilon\overline{{u_{i,j}}^2} \approx \varepsilon$$

Since $\upsilon\overline{u_{i,j}u_{j,i}}$ represents a small percentage of ε and it is exactly 0 for homogenous isotropic turbulence.

To obtain an equation for $\boldsymbol{\varepsilon}$, the velocity fluctuation momentum equation,

$$\frac{\partial u_i}{\partial t} + \overline{U_j}u_{i,j} + u_j\overline{U_{i,j}} + u_j u_{i,j} - \left(\overline{u_i u_j}\right)_{,j} = -\frac{1}{\rho}\,p'_{,i} + \nu u_{i,jj}$$

is differentiated with respect to $x_j$and multiplied by $\upsilon\frac{\partial u_i}{\partial x_j}$ and time averaged, e.g.,

$$\upsilon\frac{\partial u_i}{\partial x_j}\frac{\partial u_{i,j}}{\partial t} = \frac{\partial}{\partial t}\left(\upsilon\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial\partial x_j}\right) = \frac{\partial\tilde{\varepsilon}}{\partial t}$$

The final equation is:

$$\frac{D\epsilon}{Dt} = P_\epsilon^1 + P_\epsilon^2 + P_\epsilon^3 + P_\epsilon^4 + \Pi_\epsilon + T_\epsilon + D_\epsilon - \Upsilon_\epsilon,$$

Where:

$$P_\epsilon^1 = -\epsilon_{ij}^c \frac{\partial \overline{U}_i}{\partial x_j}$$
$$P_\epsilon^2 = -\epsilon_{ij} \frac{\partial \overline{U}_i}{\partial x_j}$$
$$P_\epsilon^3 = -2\nu \overline{u_k \frac{\partial u_i}{\partial x_j}} \frac{\partial^2 \overline{U}_i}{\partial x_k \partial x_j}$$
$$P_\epsilon^4 = -2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_j}}$$
$$\Pi_\epsilon = -\frac{2\nu}{\rho} \frac{\partial}{\partial x_i} \left(\overline{\frac{\partial p}{\partial x_j} \frac{\partial u_i}{\partial x_j}} \right)$$
$$T_\epsilon = -\nu \frac{\partial}{\partial x_k} \left(\overline{u_k \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}} \right)$$
$$D_\epsilon = \nu \nabla^2 \epsilon$$
$$\Upsilon_\epsilon = 2\nu^2 \overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_k} \right)^2}.$$

$$\epsilon_{ij}^c = 2\nu \overline{\frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}}$$
$$\epsilon_{ij} = 2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}$$

ε_{ij}^c and ε_{ij} are referred to as the complimentary dissipation rate tensor and the dissipation rate tensor, respectively. For both quantities the trace satisfies $\varepsilon_{ii} = 2\varepsilon$.

Most terms in the equation don't have a known physical meaning.

$\Pi_\varepsilon, T_\varepsilon, D_\varepsilon$ are in gradient form ∴ mostly redistribution of ε in the domain, but their value is not zero at boundaries, so they may also function as sources or sinks of dissipation.

Y_ε is strictly negative ∴ contributes exclusively to viscous loss of the dissipation rate ε.

P_ε^i represent production terms only if they are positive, which is not always the case.

Much simplified for homogeneous isotopic turbulence and shear flows where many terms = 0; therefore, useful for development of model ε equation. In fact, for processes in the dissipation range the equation is exact vs. modeled equation used for large scale motions and energy cascade in the inertial subrange.

Reynolds stress equation

The $R_{ij} = \overline{u_i u_j}$ equation derived by averaging and adding together NS for U_i multiplied by u_j and for U_j multiplied by u_i.

$$\frac{\partial R_{ij}}{\partial t} + \overline{U_k}\frac{\partial R_{ij}}{\partial x_k} = -R_{ik}\frac{\partial \overline{U_j}}{\partial x_k} - R_{jk}\frac{\partial \overline{U_i}}{\partial x_k} - \varepsilon_{ij} - \frac{\partial \beta_{ijk}}{\partial x_k} + \Pi_{ij} + \upsilon\nabla^2 R_{ij}$$

Where:

$$\Pi_{ij} = \frac{1}{\rho}\overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

$$\beta_{ijk} = \frac{1}{\rho}\overline{pu_i}\delta_{jk} + \frac{1}{\rho}\overline{pu_j}\delta_{ik} + \overline{u_i u_j u_k}$$

The first two terms on the right-hand side may be interpreted as production terms. The third term ε_{ij} is the dissipation rate tensor defined for the ε equation:

$$\varepsilon_{ij} = 2\upsilon\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}$$

β_{ijk} is made up of two pressure work terms and a turbulent flux of Reynolds stress = flux or transport.

Π_{ij} is a pressure-strain term.

$\upsilon\nabla^2 R_{ij}$ is a viscous diffusion term.

If a contraction of indices is applied and the equation is multiplied by 1/2, we recover the TKE equation.

LHS of the equation and P terms do not require additional modeling than RANS equations; $\varepsilon_{ij}, \Pi_{ij}, \beta_{ijk}$ do require additional modeling.

The pressure-strain term Π_{ij}, under the incompressibility hypothesis, has the following property:

$$\Pi_{11} + \Pi_{22} + \Pi_{33} = 0$$

Consequently, the term Π_{ii} does not appear in the TKE equation.

Π_{ij} acts to redistribute energy between $\overline{u_i{}^2}$ components without a change in total energy. If the individual terms of Π_{ii} are non-zero, then at least one must be positive and one negative with the resultant action causing $\overline{u_i{}^2}$ to move closer towards an isotropic state, where $\Pi_{ij} = 0$. Π_{ij} is critical for modeling of anisotropic turbulence.

Mean vorticity equation

RANS equations include turbulent momentum flux via the Reynolds stresses:

$$\frac{\partial \overline{u_i u_j}}{\partial x_j} = \nabla \cdot \overline{u_i u_j}$$

An alternative form can be obtained using vector identities in which turbulent vorticity flux replaces the Reynolds stresses.

Mean flow momentum equation:

$$\frac{D\overline{U_i}}{Dt} = \frac{\partial \overline{U_i}}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{U_i}}{\partial {x_j}^2} - \frac{\partial}{\partial x_j}\overline{u_i u_j}$$

We can rewrite the Reynolds stress as:

$$\frac{\partial}{\partial x_j}\overline{u_i u_j} = \overline{u_j \frac{\partial u_i}{\partial x_j}} = \overline{\underline{u} \cdot \nabla \underline{u}} = \nabla k - \overline{\underline{u} \times \underline{\omega}}$$

$$k = \frac{1}{2}\overline{u_i u_i}$$

In vector form, the mean flow momentum equation becomes:

$$\frac{D\underline{\overline{U}}}{Dt} = -\nabla\left(\frac{\overline{p}}{\rho} + k\right) + \nu \nabla^2 \underline{\overline{U}} + \overline{\underline{u} \times \underline{\omega}}$$

Where:

$$\overline{(\underline{u} \times \underline{\omega})_i} = \varepsilon_{ijk}\overline{u_i \omega_j}$$

ε_{ijk} is the Levi-Civita symbol or permutation symbol.

$\overline{u_i \omega_j}$ represents the vorticity flux correlation.

$\frac{\overline{p}}{\rho} + k$ =modified mean pressure with isotropic Reynolds stress component.

Numerical methods for solving this alternative form of the mean momentum equation coupled with the continuity equation generally yield a solution for $\underline{\overline{U}}$ and the combined quantity $\frac{\overline{p}}{\rho} + k$,using closure for $\overline{u_i \omega_j}$.

This form of the equations was used by, e.g., Taylor to study atmospheric flows and vorticity of heat transport.

In this approach, the vorticity transport physics is embedded within a momentum balance. A more complete centering of the physics on vorticity can be had by developing an averaged form of the vorticity equation itself:

Focusing on vorticity physics only is to derive the mean vorticity transport equation.

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underbrace{\underline{v}\cdot\nabla\underline{v}}_{\nabla K - \underline{v}\times\underline{\Omega}}\right) = \rho\underline{g} + \nabla\cdot\sigma_{ij} = -\frac{\partial \hat{p}}{\partial x_i} + \mu\underbrace{\nabla^2\underline{v}}_{-(\nabla\times\underline{\Omega})}$$

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij} \qquad \varepsilon_{ij} = \tfrac{1}{2}\left(v_{i,j} + v_{j,i}\right)$$

$$(1)\quad \frac{\partial \underline{v}}{\partial t} + \nabla K - \underline{v}\times\underline{\Omega} = \frac{1}{\rho}\nabla\hat{p} - \nu\nabla\times\underline{\Omega} \qquad K = \tfrac{1}{2}\,\underline{v}\cdot\underline{v}$$

$$\nabla\times(1):\quad \frac{\partial\underline{\Omega}}{\partial t} + \underline{v}\cdot\nabla\underline{\Omega} = \frac{D\underline{\Omega}}{Dt} = \underline{\Omega}\cdot\nabla\underline{v} + \nu\nabla^2\underline{\Omega}$$

$$\text{or}\quad \underbrace{\frac{\partial\Omega_i}{\partial t} + v_j\frac{\partial\Omega_i}{\partial x_j}}_{\text{material derivative}} = \underbrace{\Omega_j\frac{\partial v_i}{\partial x_j}}_{\text{stretching / re-orientation}} + \overbrace{\nu\nabla^2\Omega_i}^{\text{viscous diffusion:}}$$

creation at solid boundaries due friction, i.e., no slip condition

$$\nabla(\underline{a}\cdot\underline{a}) = \underline{a}\cdot\nabla\underline{a} + \underline{a}\cdot\nabla\underline{a} + \underline{a}\times(\nabla\times\underline{a}) + \underline{a}\times(\nabla\times\underline{a})$$

$$\tfrac{1}{2}\nabla(\underline{a}\cdot\underline{a}) = \underline{a}\cdot\nabla\underline{a} + \underline{a}\times(\nabla\times\underline{a})$$

$$\therefore\quad \underline{v}\cdot\nabla\underline{v} = \tfrac{1}{2}\nabla(\underline{v}\cdot\underline{v}) - \underline{v}\times\underline{\omega}$$

$$\nabla\times(\nabla\times\underline{a}) = \nabla(\nabla\cdot\underline{a}) - \nabla^2\underline{a}$$

$$\nabla^2\underline{v} = -\nabla\times\underline{\omega}$$

$$\frac{\mathrm{D}\underline{\Omega}}{\mathrm{Dt}} = \underline{\Omega}\cdot\nabla\underline{U} + \upsilon\nabla^2\underline{\Omega}$$

Decomposition for the velocity and the vorticity:

$$\underline{\Omega} = \underline{\overline{\Omega}} + \underline{\omega}$$

$$\underline{U} = \underline{\overline{U}} + \underline{u}$$

$$\underline{\overline{\Omega}} = \nabla\times\underline{\overline{U}} = \left(\overline{W}_y - \overline{V}_z\right)\hat{\imath} + (\overline{U}_z - \overline{W}_x)\hat{\jmath} + \left(\overline{V}_x - \overline{U}_y\right)\hat{k}$$

$$\underline{\omega} = \nabla\times\underline{u} = \left(w_y - v_z\right)\hat{\imath} + (u_z - w_x)\hat{\jmath} + \left(v_x - u_y\right)\hat{k}$$

Substituting the decompositions and time-averaging:

$$\frac{\mathrm{D}\underline{\overline{\Omega}}}{\mathrm{Dt}} = \underline{\overline{\Omega}}\cdot\nabla\underline{\overline{U}} + \overline{\underline{\omega}\cdot\nabla\underline{u}} + \nabla\cdot\left(\nu\nabla\underline{\overline{\Omega}} - \overline{\omega_i u_j}\right)$$

$\overline{\underline{\omega}\cdot\nabla\underline{u}} = \nabla\cdot\overline{u_i\omega_j} =$ rate of deformation of vortex lines due to turbulence ($\nabla\cdot\underline{\omega}$=0)

$\nabla\cdot\left(\nu\nabla\underline{\overline{\Omega}} - -\overline{\omega_i u_j}\right) =$ viscous diffusion augmented by turbulent vorticity diffusion; similar as Reynolds stresses augment viscous momentum diffusion.

$$\frac{\mathrm{D}\underline{\overline{\Omega}}}{\mathrm{Dt}} = \underline{\overline{\Omega}}\cdot\nabla\underline{\overline{U}} - \frac{\partial}{\partial x_j}\left(\overline{\omega_i u_j} - \overline{u_i\omega_j}\right) + \nu\nabla^2\underline{\overline{\Omega}}$$

$$\frac{\mathrm{D}\underline{\overline{\Omega}}}{\mathrm{Dt}} = \underline{\overline{\Omega}}\cdot\nabla\underline{\overline{U}} - \overline{\underline{u}\cdot\nabla\underline{\omega}} + \overline{\underline{\omega}\cdot\nabla\underline{u}} + \nu\nabla^2\underline{\overline{\Omega}} \quad (\nabla\cdot\underline{u} = 0 \text{ and } \nabla\cdot\underline{\omega}\text{=0})$$

Or in index notation:

$$\frac{\partial\overline{\Omega_i}}{\partial t} + U_j\frac{\partial\overline{\Omega_i}}{\partial x_j} = \Omega_j\frac{\partial\overline{U_i}}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\overline{\omega_i u_j} - \overline{u_i\omega_j}\right) + \nu\frac{\partial^2\overline{\Omega_i}}{\partial {x_j}^2}$$

1 2 3

A Reynolds averaged vorticity transport equation can be derived by taking the curl of the Reynolds averaged Navier Stokes equations, which for steady flow is:

$$
\begin{aligned}
&\left(\bar{U}\frac{\partial\Omega_x}{\partial x}+\bar{V}\frac{\partial\Omega_x}{\partial y}+\bar{W}\frac{\partial\Omega_x}{\partial z}\right)=\left(\Omega_x\frac{\partial\bar{U}}{\partial x}+\Omega_y\frac{\partial\bar{U}}{\partial y}+\Omega_z\frac{\partial\bar{U}}{\partial z}\right)+\upsilon\left(\frac{\partial^2\Omega_x}{\partial x^2}+\frac{\partial^2\Omega_x}{\partial y^2}+\frac{\partial^2\Omega_x}{\partial z^2}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\overline{u'v'}}{\partial z}-\frac{\partial\overline{u'w'}}{\partial y}\right)+\frac{\partial^2}{\partial y\partial z}\left(\overline{v'v'}-\overline{w'w'}\right)+\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial y^2}\right)\overline{v'w'}\\
&\left(\bar{U}\frac{\partial\Omega_y}{\partial x}+\bar{V}\frac{\partial\Omega_y}{\partial y}+\bar{W}\frac{\partial\Omega_y}{\partial z}\right)=\left(\Omega_x\frac{\partial\bar{V}}{\partial x}+\Omega_y\frac{\partial\bar{V}}{\partial y}+\Omega_z\frac{\partial\bar{V}}{\partial z}\right)+\upsilon\left(\frac{\partial^2\Omega_y}{\partial x^2}+\frac{\partial^2\Omega_y}{\partial y^2}+\frac{\partial^2\Omega_y}{\partial z^2}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\overline{v'w'}}{\partial x}-\frac{\partial\overline{v'u'}}{\partial z}\right)+\frac{\partial^2}{\partial z\partial x}\left(\overline{w'w'}-\overline{u'u'}\right)+\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial z^2}\right)\overline{w'u'}\\
&\underbrace{\left(\bar{U}\frac{\partial\Omega_z}{\partial x}+\bar{V}\frac{\partial\Omega_z}{\partial y}+\bar{W}\frac{\partial\Omega_z}{\partial z}\right)}_{(A)}=\underbrace{\left(\Omega_x\frac{\partial\bar{W}}{\partial x}+\Omega_y\frac{\partial\bar{W}}{\partial y}+\Omega_z\frac{\partial\bar{W}}{\partial z}\right)}_{(B)}+\upsilon\underbrace{\left(\frac{\partial^2\Omega_z}{\partial x^2}+\frac{\partial^2\Omega_z}{\partial y^2}+\frac{\partial^2\Omega_z}{\partial z^2}\right)}_{(C)}+\underbrace{\frac{\partial}{\partial z}\left(\frac{\partial\overline{w'u'}}{\partial y}-\frac{\partial\overline{w'v'}}{\partial x}\right)}_{(D)}+\underbrace{\frac{\partial^2}{\partial x\partial y}\left(\overline{u'u'}-\overline{v'v'}\right)}_{(E)}+\underbrace{\left(\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right)\overline{u'v'}}_{(F)}
\end{aligned}
$$

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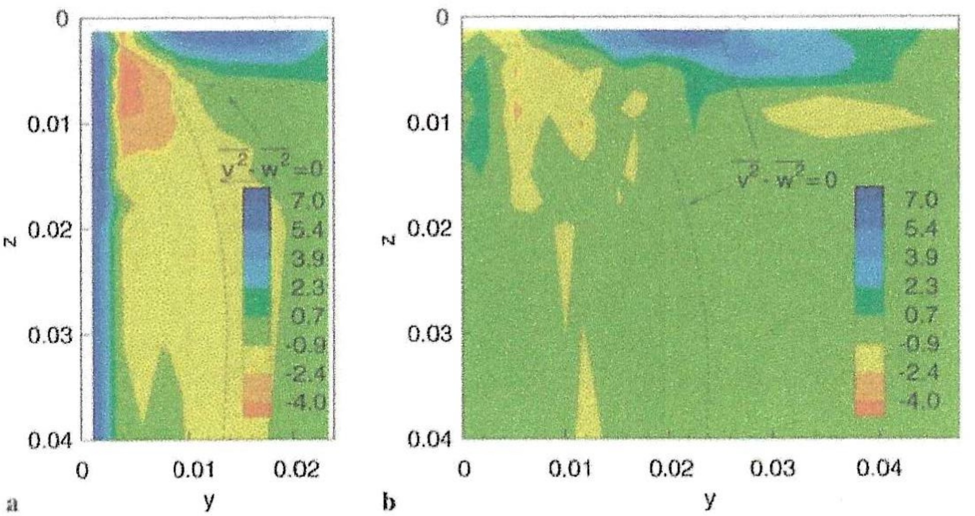
Term (A) represents the material derivative of the mean vorticity. The first term of term (B) is the vorticity amplification by the stretching and the other terms provide vortex-line bending effects. Term (C) presents the vorticity damping by the viscous diffusion, and the other terms (D), (E), and (F) are the vorticity production by inhomogeneity in the Reynolds stress.

Longo, J., Huang, H.P., and Stern, F., "Solid-Fluid Juncture Boundary Layer and Wake," Experiments in Fluids, Vol. 25, No. 4, September 1998, pp. 283 –297



Anisotropy

Frederick Stern, "Integrated High-Fidelity Validation Experiments and LES for a Surface-Piercing Truncated Cylinder for Sub-and Critical Reynolds and Froude Numbers," AVT-246: Progress and Challenges in Validation Testing for CFD, Avila, Spain, 26-28 September 2016.

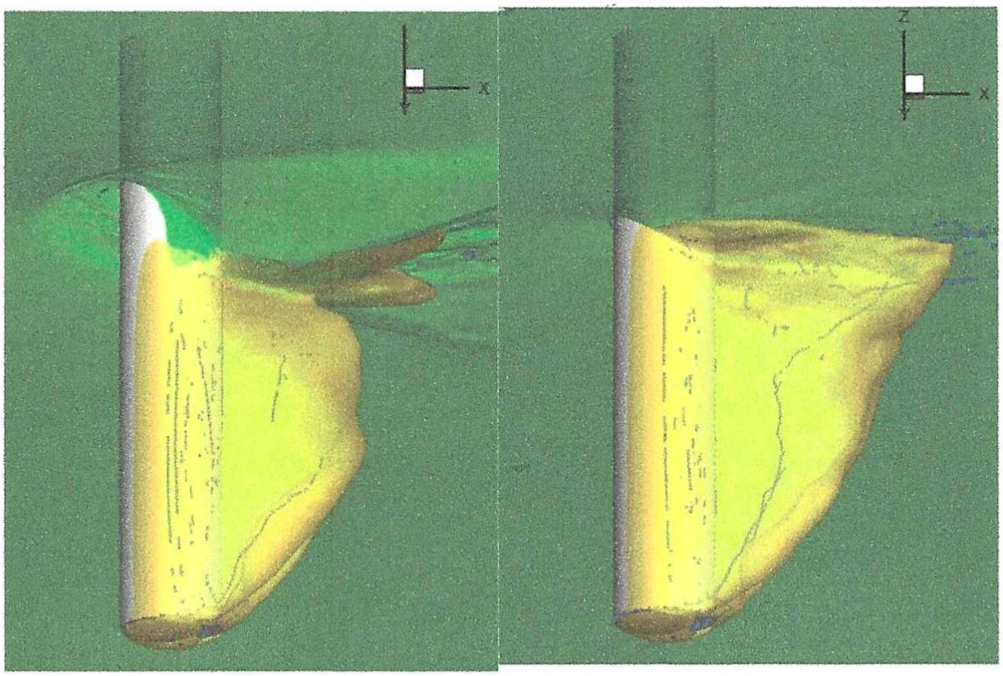


Figure 11-19: Iso-surface of instantaneous Q-criterion (Q=2): (a) sub-critical Re, (b) critical Re; Mean flow separation pattern with vortex core line: visualized approximately using the iso-surfaces of the stagnation C_p=-0.3, (c) sub-critical Re, (d) critical Re.

$$\overset{(1)}{\frac{D\underline{\Omega}}{\Delta t}} = \overset{(2)}{\underline{\Omega}\cdot\nabla\underline{U}} - \overset{(4)}{\frac{\partial}{\partial x_i}\left(\overline{\omega_i u_j} - u_i\overline{\omega}_j\right)} + \overset{(3)}{\nu\nabla^2\underline{\Omega}}$$

$$\frac{D\underline{\Omega}}{Dt} = \overline{\underline{\Omega}}\cdot\nabla\overline{\underline{U}} - \overline{\underline{u}\cdot\nabla\underline{\omega}} + \overline{\underline{\omega}\cdot\nabla\underline{u}} + \nu\nabla^2\overline{\underline{\Omega}}$$

Closed vs. Open Channel Turbulence Anisotropy

Non-circular ducts exhibit corner flows (secondary flow/streamwise vortices towards the corner) due to turbulence anisotropy. Open channel flows have similar corner flows not only at the bottom juncture but also at the free surface juncture due to turbulence anisotropy, which all leads to velocity-dip phenomena, i.e., maximum velocity occurs not at but just below free surface.

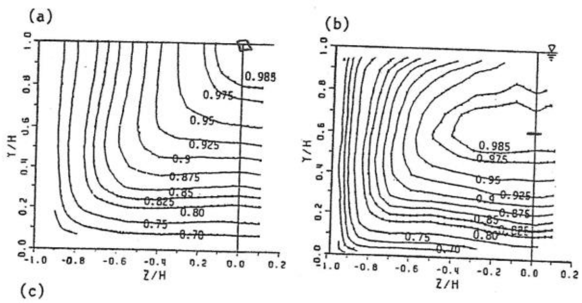
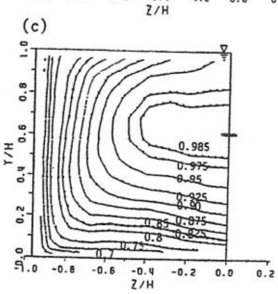



Figure 5.6 Isovels of mean primary velocity $U(y, z)$ for closed and open channels. a) Closed-channel. b) Subcritical open-channel. c) Supercritical open-channel.

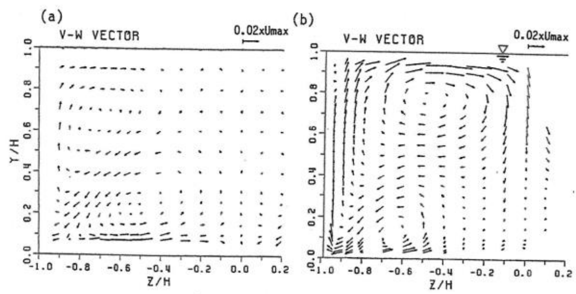
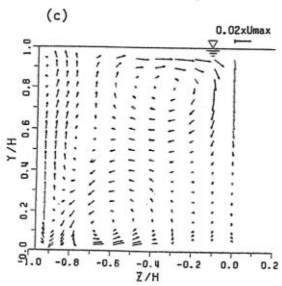



Figure 5.9 Vector descriptions (V, W) of secondary currents for closed and open channels; $B/h = 2$. a) Closed-channel. b) Subcritical open-channel. c) Supercritical open-channel.

Closed and open channel u’ and v’ similar near bottom juncture but different near symmetry plane vs. free surface where open channel shows u’ minimum near velocity dip and v’ monotonically damped near free surface.

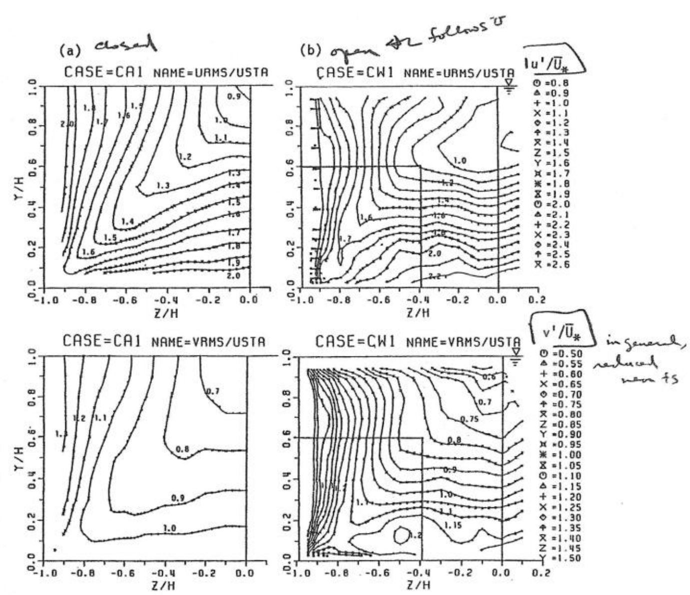


Figure 5.13 Contour lines of turbulence intensities u' and v' for closed and open channels. a) Closed-channel. b) Open-channel.

Mean enstrophy equation

To obtain the mean enstrophy equation, multiply the mean vorticity equation by $\overline{\Omega_i}$ (scalar equation):

$$\overline{\Omega_i}\cdot\left[\frac{\partial\overline{\Omega_i}}{\partial t}+\overline{U_j}\frac{\partial\overline{\Omega_i}}{\partial x_j}=\overline{\Omega_j}\frac{\partial\overline{U_i}}{\partial x_j}-\frac{\partial}{\partial x_j}\left(\overset{\boxed{\text{a}}}{\overline{\omega_i u_j}}-\overset{\boxed{\text{b}}}{\overline{u_i\omega_j}}\right)+\upsilon\frac{\partial^2\overline{\Omega_i}}{\partial x_j\partial x_j}\right]$$

$$\underbrace{\frac{\partial\left(\overline{\Omega_i^2}/2\right)}{\partial t}+\overline{U_j}\frac{\partial\left(\overline{\Omega_i^2}/2\right)}{\partial x_j}}=\overline{\Omega_i}\,\overline{\Omega_j}\frac{\partial\overline{U_i}}{\partial x_j}-\overline{\Omega_i}\frac{\partial}{\partial x_j}\left(\overline{\omega_i u_j}-\overline{u_i\omega_j}\right)+\upsilon\left(\frac{\partial^2\left(\overline{\Omega_i^2}/2\right)}{\partial x_j\partial x_j}-\frac{\partial\overline{\Omega_i}}{\partial x_j}\frac{\partial\overline{\Omega_i}}{\partial x_j}\right)$$

$$\boxed{\frac{D\left(\overline{\Omega_i^2}/2\right)}{Dt}=material\ derivative\ \overline{\Omega_i^2}/2}$$

$$\underset{\boxed{\text{I}}}{\frac{D\left(\overline{\Omega_i^2}/2\right)}{Dt}}=\underset{\boxed{\text{II}}}{\overline{\Omega_i}\,\overline{\Omega_j}\frac{\partial\overline{U_i}}{\partial x_j}}-\underset{\boxed{\text{III}}}{\frac{\partial}{\partial x_j}\left(\overline{\Omega_i}\overline{\omega_i u_j}\right)}+\underset{\boxed{\text{IV}}}{\overline{\omega_i u_j}\frac{\partial\overline{\Omega_i}}{\partial x_j}}+\underset{\boxed{\text{V}}}{\overline{\Omega_i}\frac{\partial}{\partial x_j}\left(\overline{u_i\omega_j}\right)}+\upsilon\left(\underset{\boxed{\text{VI}}}{\frac{\partial^2\left(\overline{\Omega_i^2}/2\right)}{\partial x_j\partial x_j}}-\underset{\boxed{\text{VII}}}{\frac{\partial\overline{\Omega_i}}{\partial x_j}\frac{\partial\overline{\Omega_i}}{\partial x_j}}\right)$$

I: mean enstrophy.

II: mean flow stretching.

III: transport by velocity/vorticity interaction.

IV: gradient production fluctuating vorticity.

III + IV: turbulence advection.

V: stretching turbulent vorticity by turbulence deformation rate = turbulence stretching.

VI: viscous transport (diffusion).

VII: dissipation of mean vorticity.

$$\underset{\boxed{\text{III}}}{-\frac{\partial}{\partial x_j}\left(\overline{\Omega_i}\overline{\omega_i u_j}\right)}=\underset{\boxed{\text{IV}}}{-\overline{\omega_i u_j}\frac{\partial\overline{\Omega_i}}{\partial x_j}}-\underset{\boxed{\text{a}}}{\overline{\Omega_i}\frac{\partial}{\partial x_j}\left(\overline{u_i\omega_j}\right)}$$

III + IV = a

V = b

Fluctuating vorticity equation

To obtain an equation for the fluctuating vorticity, subtract the mean vorticity equation from the instantaneous vorticity equation.

Instantaneous vorticity equation:

$$\frac{\partial(\overline{\Omega_i}+\omega_i)}{\partial t}+(\overline{U_j}+u_j)\frac{\partial(\overline{\Omega_i}+\omega_i)}{\partial x_j}=(\overline{\Omega_j}+\omega_j)\frac{\partial(\overline{U_i}+u_i)}{\partial x_j}+\upsilon\frac{\partial^2(\overline{\Omega_i}+\omega_i)}{\partial x_j\partial x_j}$$

Mean vorticity equation:

$$\frac{\partial\overline{\Omega_i}}{\partial t}+\overline{U_j}\frac{\partial\overline{\Omega_i}}{\partial x_j}=\overline{\Omega_j}\frac{\partial\overline{U_i}}{\partial x_j}-\frac{\partial}{\partial x_j}(\overline{\omega_i u_j}-\overline{u_i\omega_j})+\upsilon\frac{\partial^2\overline{\Omega_i}}{\partial x_j\partial x_j}$$

Resulting fluctuating equation:

$$\frac{\partial\omega_i}{\partial t}+\overline{U_j}\frac{\partial\omega_i}{\partial x_j}=-u_j\frac{\partial\overline{\Omega_i}}{\partial x_j}+\overline{\Omega_j}\frac{\partial u_i}{\partial x_j}+\omega_j\frac{\partial\overline{U_i}}{\partial x_j}-u_j\frac{\partial\omega_i}{\partial x_j}+\omega_j\frac{\partial u_i}{\partial x_j}+\frac{\partial}{\partial x_j}(\overline{\omega_i u_j}-\overline{u_i\omega_j})+\ \upsilon\frac{\partial^2\omega_i}{\partial x_j\partial x_j}$$

$$\frac{D\omega_i}{Dt}=-u_j\frac{\partial\overline{\Omega_i}}{\partial x_j}+\overline{\Omega_j}\frac{\partial u_i}{\partial x_j}+\omega_j\frac{\partial\overline{U_i}}{\partial x_j}-\frac{\partial}{\partial x_j}(\omega_i u_j-u_i\omega_j-\overline{\omega_i u_j}+\overline{u_i\omega_j})+\ \upsilon\frac{\partial^2\omega_i}{\partial x_j\partial x_j}$$

Multiply the resulting equation by u_i and apply time average to obtain turbulent enstrophy equation.

Note that for total enstrophy:

$$\frac{\overline{\Omega_i\Omega_i}}{2}=\frac{\overline{\Omega_i}\,\overline{\Omega_i}}{2}+\frac{\overline{\omega_i^2}}{2}$$

Fluctuating vorticity equation

Subtract mean vorticity equation from instantaneous equation

$$\frac{\partial(\Omega_i+\omega_i)}{\partial t} + (U_j+u_j)\frac{\partial}{\partial x_j}(\Omega_i+\omega_i) = (\Omega_j+\omega_j)\frac{\partial}{\partial x_j}(U_i+u_i) + \nu\frac{\partial^2(\Omega_i+\omega_i)}{\partial x_j \partial x_j}$$

$$\cancel{\frac{\partial \Omega_i}{\partial t}} + \frac{\partial \omega_i}{\partial t} + \cancel{U_j\frac{\partial \Omega_i}{\partial x_j}} + U_j\frac{\partial \omega_i}{\partial x_j} + u_j\frac{\partial \Omega_i}{\partial x_j} + u_j\frac{\partial \omega_i}{\partial x_j} =$$

$$\cancel{\Omega_j\frac{\partial U_i}{\partial x_j}} + \Omega_j\frac{\partial u_i}{\partial x_j} + \omega_j\frac{\partial U_i}{\partial x_j} + \omega_j\frac{\partial u_i}{\partial x_j} + \cancel{\nu\frac{\partial^2 \Omega_i}{\partial x_j \partial x_j}} + \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

instantaneous

$$\cancel{\frac{\partial \Omega_i}{\partial t}} + \cancel{U_j\frac{\partial \Omega_i}{\partial x_j}} = \cancel{\Omega_j\frac{\partial U_i}{\partial x_j}} - \frac{\partial}{\partial x_j}\left(\overline{\omega_i u_j} - \overline{u_i \omega_j}\right) + \cancel{\nu\frac{\partial^2 \Omega_i}{\partial x_j \partial x_j}}$$

Mean

$$\frac{\partial \omega_i}{\partial t} + U_j\frac{\partial \omega_i}{\partial x_j} = -u_j\frac{\partial \Omega_i}{\partial x_j} - u_j\frac{\partial \omega_i}{\partial x_j} + \Omega_j\frac{\partial u_i}{\partial x_j} + \omega_j\frac{\partial U_i}{\partial x_j} + \omega_j\frac{\partial u_i}{\partial x_j} + \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

$$+ \frac{\partial}{\partial x_j}\left(\overline{\omega_i u_j} - \overline{u_i \omega_j}\right)$$

$$= \Omega_j\frac{\partial u_i}{\partial x_j} + \omega_j\frac{\partial U_i}{\partial x_j} - u_j\frac{\partial \Omega_i}{\partial x_j} - u_j\frac{\partial \omega_i}{\partial x_j} + \omega_j\frac{\partial u_i}{\partial x_j}$$

$$+ \frac{\partial}{\partial x_j}\left(\overline{\omega_i u_j} - \overline{u_i \omega_j}\right) + \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

$$\underbrace{\frac{\partial \omega_i}{\partial t} + U_j\frac{\partial \omega_i}{\partial x_j}}_{\frac{D\omega_i}{Dt}} = \Omega_j\frac{\partial u_i}{\partial x_j} + \omega_j\frac{\partial U_i}{\partial x_j} - u_j\frac{\partial \Omega_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\omega_i u_j - \omega_j u_i - \overline{\omega_i u_j} + \overline{u_i \omega_j}\right)$$

$$+ \nu\frac{\partial^2 \omega_i}{\partial x_j \partial x_j}$$

multiply ω_i & average to obtain turbulent enstrophy equation

note total enstrophy $\overline{\Omega_i \Omega_i}/2 = \Omega_i \Omega_i/2 + \underbrace{\overline{\omega_i \omega_i}}_{\overline{\omega_i^2}}/2$

Enstrophy equation

$\zeta = \overline{\underline{\omega}\cdot\underline{\omega}} = \overline{\omega_i^2} = \frac{\varepsilon}{\nu} - \overline{u_{i,j}u_{j,i}} = \boxed{\frac{\tilde{\varepsilon}}{\nu}}$

$$\frac{D\zeta}{Dt} = P_\zeta^1 + P_\zeta^2 + P_\zeta^3 + P_\zeta^4 + T_\zeta + D_\zeta - \Upsilon_\zeta, \tag{3.82}$$

where

$$P_\zeta^1 = 2\overline{\omega_i\omega_k}\frac{\partial \overline{U}_i}{\partial x_k} \tag{3.83}$$

$$P_\zeta^2 = 2\overline{\omega_i\frac{\partial u_i}{\partial x_k}}\overline{\Omega}_k \tag{3.84}$$

$$P_\zeta^3 = -2\overline{u_k\omega_i}\frac{\partial \overline{\Omega}_i}{\partial x_k} \tag{3.85}$$

$$P_\zeta^4 = 2\overline{\omega_i\omega_k\frac{\partial u_i}{\partial x_k}} \tag{3.86}$$

$$T_\zeta = -\frac{\partial}{\partial x_k}\overline{(u_k\omega_i\omega_i)} \tag{3.87}$$

$$D_\zeta = \nu\nabla^2\zeta \tag{3.88}$$

$$\Upsilon_\zeta = 2\nu\overline{\frac{\partial \omega_i}{\partial x_k}\frac{\partial \omega_i}{\partial x_k}}. \tag{3.89}$$

Similar TKE, but in this case dissipation of vorticity

ζ equation derived from $\underline{\Omega}$ equation by writing Ω_i form $\times$ ω_i and average

Note absence p, but many difficulties in modeling as per ε

P_ζ terms = production ; P_ζ^2 & P_ζ^3 include transport & stretching

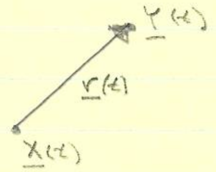
T_ζ term = transport

D_ζ term = diffusion

Υ_ζ term = dissipation

Vortex Stretching and Turning

Consider two neighboring fluid particles



$$\underline{r} = \underline{Y} - \underline{X}$$

$$\hat{\underline{r}} = \underline{r} / R \qquad \text{where } R = |\underline{r}|$$

$$\frac{d\underline{X}}{dt} = \underline{u}(\underline{X}, t) \qquad \underline{X}(t) = \underline{X}(0) + \int_0^t \underline{u}\, dt$$

$$\underline{X}(\Delta t) = \underline{X}(0) + \underline{u}(\underline{X}(0), 0)\,\Delta t \qquad \Delta t \text{ small}$$

$$\underline{Y}(\Delta t) = \underline{Y}(0) + \underline{u}(\underline{Y}(0), 0)\,\Delta t$$

$$\underline{r}(\Delta t) = \underline{Y}(\Delta t) - \underline{X}(\Delta t) = \underbrace{\underline{Y}(0) - \underline{X}(0)}_{r(0)} + \underbrace{\left[\underline{u}(\underline{Y}(0), 0) - \underline{u}(\underline{X}(0), 0)\right]}_{\triangle}\Delta t$$

As per derivation $d\underline{V} = \nabla \underline{V} \cdot d\underline{v}$,

$$\underline{u}(\underline{Y}(0), 0) = \underline{u}(\underline{X}(0), 0) + \nabla \underline{u} \cdot \underline{r}(0) \Leftarrow \begin{cases} Y(0) = X(0) + r(0) \\ r(0) \text{ small use} \\ \text{TS} \end{cases}$$

$$\underline{r}(\Delta t) = \underline{r}(0) + \nabla \underline{u} \cdot \underline{r}(0)\,\Delta t \qquad \begin{aligned} \underline{r}(\Delta t) &= \hat{\underline{r}}(\Delta t)\, R(\Delta t) \\ \underline{r}(0) &= \hat{\underline{r}}(0)\, R(0) \end{aligned}$$

$$\hat{\underline{r}}(\Delta t)\, R(\Delta t) = \hat{\underline{r}}(0)\, R(0) + \nabla \underline{u} \cdot \hat{\underline{r}}(0)\, R(0)\,\Delta t$$

$\div\ R(\Delta t)\,\Delta t$

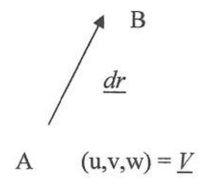
and $-\hat{r}(0)/\Delta t$

$$\frac{\hat{r}(\Delta t) - \hat{r}(0)}{\Delta t} = -\frac{\hat{r}(0)}{R(\Delta t)} \frac{[R(\Delta t) - R(0)]}{\Delta t} + \frac{R(0)}{R(\Delta t)} \nabla \underline{u} \cdot \hat{r}(0)$$

$$\lim \Delta t \to 0 \qquad \frac{d\hat{r}}{dt} + \alpha \hat{r} = \nabla \underline{u} \cdot \hat{r}$$

$$\alpha = \frac{1}{R}\frac{dR}{dt} = \text{fractional rate of change of line element } \underline{r}$$

Relative motion between two neighboring fluid particles.



@ B: $\underline{V} + \underline{dV} = \underline{V} + \nabla \underline{V} \cdot \underline{dr}$ 1^{st} order Taylor Series

$u_B = u_A + u_x dx + u_y dy + u_z dz + u_{xx} \frac{dx^2}{2} + \cdots$

$v_B = v_A + v_x dx + v_y dy + v_z dz + v_{xx} \frac{dx^2}{2} + \cdots$

$w_B = w_A + w_x dx + w_y dy + w_z dz + w_{xx} \frac{dx^2}{2} + \cdots$

$$\underline{dV} = \nabla \underline{V} \cdot \underline{dr} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = e_{ij}\, dx_j$$

relative motion

deformation rate tensor = e_{ij}

let $\hat{v} = \underline{\omega}/|\underline{\omega}|$ unit vector in direction vorticity

$$\underbrace{\nabla \underline{u} \cdot \underline{\omega}} = \underset{(1)}{\alpha \underline{\omega}} + \underset{(2)}{|\underline{\omega}| \frac{d\, \underline{\omega}/|\underline{\omega}|}{dt}}$$

vortex stretching
& turning term
in vorticity transport equation

(1) = stretching or contraction in direction $\underline{\omega}$ due $1/R \frac{dR}{dt}$, which strengthens or weakens $\underline{\omega}$

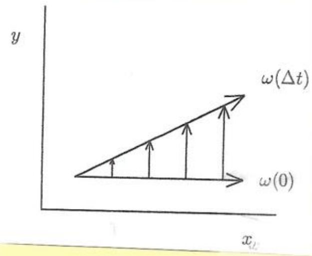
(2) using $\frac{\underline{\omega}}{|\underline{\omega}|} \cdot \frac{\underline{\omega}}{|\underline{\omega}|} = 1$

$$\frac{d}{dt}\left[\frac{\underline{\omega}}{|\underline{\omega}|} \cdot \frac{\underline{\omega}}{|\underline{\omega}|}\right] = 0 = 2\frac{\underline{\omega}}{|\underline{\omega}|} \cdot \frac{d(\underline{\omega}/|\underline{\omega}|)}{dt}$$

shows (2) is vector perpendicular $\underline{\omega}$ so represents turning of $\underline{\omega}$ into two orthogonal directions.

This analysis shows how $\nabla \underline{u} \cdot \underline{\omega}$ term can stretch & turn vorticity as it evolves in time & can transfer energy between scales via the Richardson energy cascade

Figure 18.1. Shearing of ω_1 in a velocity field $v(x)$ to create ω_2 vorticity.



$$\underline{\omega} = \omega_1 \hat{i} + 0\,\hat{j} + 0\,\hat{k} \qquad \text{ie } \omega_2 = \omega_3 = 0$$

neglecting $\nu \nabla^2 \underline{\omega}$

$$\frac{D\omega_1}{Dt} = \omega_1 u_x \qquad \frac{D\omega_2}{Dt} = \omega_1 v_x \qquad \frac{D\omega_3}{Dt} = \omega_1 w_x$$

using previous analysis:

$$\omega_1 u_x = \alpha \omega_1 \qquad \alpha = \frac{\partial u}{\partial x}$$

$$\omega_1 v_x = \omega_1 \frac{d(\underline{\omega}/|\underline{\omega}|)}{dt}\,\hat{j}$$

$$\omega_1 w_x = \omega_1 \frac{d(\underline{\omega}/|\underline{\omega}|)}{dt}\,\hat{k}$$

creation ω_2 & ω_3 due v_x & w_x
Fig illustrates production ω_2 from ω_1, where v_x is component of $d(\underline{\omega}/|\underline{\omega}|)/dt$ projected onto y direction

In the general case, arbitrarily oriented vorticity filaments are simultaneously stretched or compressed and reoriented by the shearing motions. The propensity for vorticity to stretch and reorient is the main driving force behind the appearance and maintenance of turbulence in flowing fluids. In essence, this physical process is the means by which energy is transferred to small scales, where the action of viscous forces in smoothing the flow and dissipating energy become important. A discussion of these and other aspects of turbulent flow may be found in a number of books (e.g., Bernard & Wallace 2002; Pope 2000).

Vortex stretching and reorientation: fundamental importance energy cascade

Cartesian coordinates:

$\underline{\Omega} \cdot \nabla \underline{u} =$

$\underbrace{(\Omega_x u_x}_{\text{stretching}} + \underbrace{\Omega_y u_y + \Omega_z u_z)}_{\text{turning}} \hat{i}$

$(\Omega_x v_x + \Omega_y v_y + \Omega_z v_z) \hat{j}$

$(\Omega_x w_x + \Omega_y w_y + \Omega_z w_z) \hat{k}$

Ω_i = component, whereas

subscript $\underline{v}$ = derivative

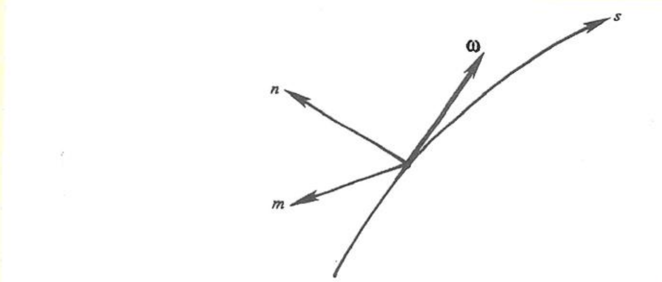


FIGURE 5.11 Natural coordinate system aligned with the vorticity vector.

Curvilinear coordinates tangent $\underline{\Omega}$: $\hat{e}_s = \underline{\Omega}/|\underline{\Omega}|$

$$\underline{\Omega} \cdot \nabla \underline{u} = \underline{\Omega} \cdot \left(\hat{e}_s \frac{\partial}{\partial s} + \hat{e}_n \frac{\partial}{\partial n} + \hat{e}_m \frac{\partial}{\partial m}\right) \underline{u} \quad \text{where } \underline{\Omega} \cdot \hat{e}_n = \underline{\Omega} \cdot \hat{e}_m = 0$$

$$= \Omega \frac{\partial \underline{u}}{\partial s} = \Omega \times \text{derivative} \quad \text{and } \underline{\Omega} \cdot \hat{e}_s = \Omega = |\underline{\Omega}|$$

$\underline{u}$ direction $\underline{\Omega}$

$$= \Big(\underbrace{\Omega \frac{\partial u_s}{\partial s}}_{\text{stretching}}, \underbrace{\Omega \frac{\partial u_n}{\partial s}, \Omega \frac{\partial u_m}{\partial s}}_{\text{turning about n and m axes}}\Big)$$

$\underline{\omega} \cdot \underline{\omega} = \omega_1^2 + \omega_2^2 + \omega_3^2 = \omega$

$\frac{D\underline{\omega}}{Dt} = \underline{\Omega} \cdot \nabla \underline{u}$ neglect ν

$|\underline{\omega}| = \sqrt{\omega}$

$\hat{e}_s = \underline{\omega}/|\underline{\omega}|$

$\frac{D\omega_s}{Dt} = \Omega \frac{\partial u_s}{\partial s}$ important turbulent flow $\underline{\omega} \cdot \hat{e}_s = \omega/\sqrt{\omega} = \sqrt{\omega}$

$\frac{D\omega_n}{Dt} = \Omega \frac{\partial u_n}{\partial s}$ $\quad \frac{D\omega_m}{Dt} = \Omega \frac{\partial u_m}{\partial s}$