Chapter 3: Overview of Turbulent Flow Physics and Equations

Part 3: Mean and Turbulent Kinetic Energy Equations

Kinetic Energy of the Mean Flow: transport and sources and sinks of mean KE

$$\frac{\partial \overline{U_i}}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -g\delta_{i3} + \frac{1}{\rho}\frac{\partial \overline{\sigma_{ij}}}{\partial x_i} \quad (1)$$

$$\overline{\sigma_{ij}} = -\overline{p}\delta_{ij} + 2\mu E_{ij} - \rho\overline{u_i u_j}$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial \overline{U_i}}{\partial x_j} + \frac{\partial \overline{U_j}}{\partial x_i}\right)$$

Notation:

$\underline{\Omega} = \overline{\underline{\Omega}} + \underline{\omega}$

$\underline{U} = \overline{\underline{U}} + \underline{u}$

$p = \overline{p} + p'$

$\overline{U_i} \times (1)$:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{U_i}^2\right) + \overline{U_j}\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{U_i}^2\right) = -g\overline{U_i}\delta_{i3} + \frac{1}{\rho}\overline{U_i}\frac{\partial \overline{\sigma_{ij}}}{\partial x_j}$$

$\frac{D}{Dt}\left(\frac{1}{2}\overline{U_i}^2\right) = -\frac{g}{\rho}\overline{U_i}\delta_{i3} + \frac{1}{\rho}\frac{\partial\left(\overline{U_i}\overline{\sigma_{ij}}\right)}{\partial x_j} - \frac{1}{\rho}\overline{\sigma_{ij}}\frac{\partial \overline{U_i}}{\partial x_j}$ As per instantaneous/deterministic ME equation.

$$\frac{D}{Dt}\left(\frac{1}{2}\overline{U_i}^2\right) = -\frac{g}{\rho}\overline{U_3} + \frac{\partial}{\partial x_j}\left(-\frac{\overline{U_i}\overline{p}}{\rho}\delta_{ij} + 2\nu\overline{U_i}E_{ij} - \overline{u_i u_j}\overline{U_i}\right) + \cancel{\frac{\overline{p}}{\rho}\delta_{ij}\frac{\partial \overline{U_i}}{\partial x_j}} - 2\nu E_{ij}\frac{\partial \overline{U_i}}{\partial x_j} + \overline{u_i u_j}\frac{\partial \overline{U_i}}{\partial x_j} \quad (2)$$

Note that:

$$E_{ij}\frac{\partial \overline{U_i}}{\partial x_j} = E_{ij}\left(E_{ij} + W_{ij}\right) = E_{ij}E_{ij}$$

$$\frac{\partial}{\partial x_j}\left(-\frac{\overline{U_i}\overline{p}}{\rho}\delta_{ij}\right) = -\frac{1}{\rho}\left(\frac{\partial \overline{U_i}}{\partial x_j}\overline{p}\delta_{ij} + \frac{\partial \overline{p}}{\partial x_j}\overline{U_i}\delta_{ij}\right) = -\frac{1}{\rho}\left(\frac{\partial \overline{U_j}}{\partial x_j}\overline{p} + \frac{\partial \overline{p}}{\partial x_j}\overline{U_j}\right) = \frac{\partial}{\partial x_j}\left(-\frac{\overline{U_j}\overline{p}}{\rho}\right)$$

Therefore, (2) can be rewritten as:

$$\frac{D}{Dt}\left(\frac{1}{2}\overline{U_i}^2\right) = \frac{\partial}{\partial x_j}\left(-\frac{\overline{p}\overline{U_j}}{\rho} + 2\nu\overline{U_i}E_{ij} - \overline{u_i u_j}\overline{U_i}\right) - 2\nu E_{ij}E_{ij} + \overline{u_i u_j}\frac{\partial \overline{U_i}}{\partial x_j} - g\overline{U_3}/\rho$$

$\boxed{\text{A}}$ $\boxed{\text{B}}$ $\boxed{\text{C}}$ $\boxed{\text{D}}$ $\boxed{\text{E}}$ $\boxed{\text{F}}$ $\boxed{\text{G}}$

A: rate of change of KE

B: due to the mean pressure

C: due to the mean viscous stresses

D: due to Reynolds stresses

E: viscous dissipation, $E_{ij} \times 2\upsilon E_{ij}$ =mean rate of strain × mean viscous stress = loss due to direct viscous dissipation

F: loss due to turbulence $\overline{u_i u_j}\frac{\partial \overline{U_i}}{\partial x_j} = \overline{u_i u_j} E_{ij}$ loss due to generation $\overline{u_i u_j}$ = gain in TKE

If $\overline{U_i} = \overline{U(y)}$, $\overline{u_i u_j}\frac{\partial \overline{U_i}}{\partial x_j} = \overline{uv}\frac{\partial \overline{U}}{\partial y}$ $\overline{uv} < 0$ $\frac{\partial \overline{U}}{\partial y} > 0$ $\overline{u_i u_j}\frac{\partial \overline{U_i}}{\partial x_j} < 0$ $\boxed{\text{Sign + in TKE equation}}$

G: loss due to potential energy, i.e. work done by gravity on mean vertical motion

B+C+D = transport or redistribution of energy region to region. Flux/divergence form, i.e., $\int \nabla \cdot \underline{b}\, d\forall = \int \underline{b} \cdot \underline{n} dA$ = 0 for U_i = 0 at large distances.

The two viscous terms $2\upsilon \frac{\partial}{\partial x_j}\left(\overline{U_i} E_{ij}\right)$ and $-2\upsilon E_{ij} E_{ij}$ are small for high Re turbulent flow, e.g.

$$\frac{2\upsilon E_{ij} E_{ij}}{\overline{u_i u_j}\frac{\partial \overline{U_i}}{\partial x_j}} \sim \frac{\nu \left(\frac{U}{L}\right)^2}{u_{rms}^2 \frac{U}{L}} \sim \frac{\nu}{UL} \ll 1$$

Where $u_{rms} \sim U$, i.e., same order of magnitude.

Therefore, direct influence viscous terms small in equation for mean kinetic energy, which is not true for TKE equation/budget.

The mean flow loss of energy to turbulence by shear production results in TKE which is dissipated by viscosity as per TKE equation.

$$\frac{D}{Dt}\left(\tfrac{1}{2}U_iU_i\right) = \frac{\partial}{\partial t}\left(\tfrac{1}{2}U_iU_i\right) + U_j\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}U_iU_i\right)$$

$$= \frac{\partial}{\partial t}\left[\tfrac{1}{2}\left(U^2+V^2+W^2\right)\right] + U\frac{\partial}{\partial x}(K) + V\frac{\partial}{\partial y}(K) + W\frac{\partial}{\partial z}(K)$$

$K = \frac{1}{2}U_iU_i$

$$= \frac{\partial K}{\partial t} + UK_x + VK_y + WK_z$$

$$= \frac{DK}{Dt} = \frac{\partial K}{\partial t} + \underline{U}\cdot\nabla K \qquad \text{Scalar}$$

$$\frac{\partial}{\partial x_i}\left(-\frac{\bar{p}U_i}{\rho}\right) = -\rho^{-1}\left[(\bar{p}U)_x + (\bar{p}V)_y + (\bar{p}W)_z\right] \qquad \nabla\cdot\underline{f} = \text{Scalar}$$

$$2\nu\frac{\partial}{\partial x_j}U_iE_{ij} = \nu\frac{\partial}{\partial x_j}\left[U_i\left(U_{i,j}+U_{j,i}\right)\right] \qquad \left[\frac{\partial}{\partial x_j}\right]\underbrace{\left[[U_i][E_{ij}]\right]}_{\text{vector}}$$

column, row, 3x3

$\nabla\cdot\underline{f} =$ Scalar

$$2(UU_x)_x + [U(U_y+V_x)]_y + [U(U_z+W_x)]_z$$
$$+ [V(V_x+U_y)]_x + 2(VV_y)_y + [V(V_z+W_y)]_z$$
$$+ [W(W_x+U_z)]_x + [W(W_y+V_z)]_y + 2(WW_z)_z$$

$$-\frac{\partial}{\partial x_j}\left(\overline{u_iu_j}\,U_i\right) = -\left[(\overline{u^2}U)_x + (\overline{uv}U)_y + (\overline{uw}U)_z\right.$$
$$+ (\overline{uv}V)_x + (\overline{v^2}V)_y + (\overline{vw}V)_z$$
$$\left.+ (\overline{uw}W)_x + (\overline{vw}W)_y + (\overline{w^2}W)_z\right]$$

Symmetric

$\therefore U_i\overline{u_iu_j}$

$$-2\nu E_{ij}E_{ij} = -2\nu\frac{1}{4}\left(U_{i,j}+U_{j,i}\right)^2 = -\frac{\nu}{2}\left(U_{i,j}+U_{j,i}\right)^2$$
$$= -\nu\left(U_{i,j}^2 + U_{i,j}U_{j,i}\right)$$

inner product two 2nd order tensor = 2nd order tensor & contraction over i,j

$$U_x^2 + U_y^2 + U_z^2 + U_x^2 + U_yV_x + U_zW_x$$
$$+ V_x^2 + V_y^2 + V_z^2 + V_xU_y + V_y^2 + V_zW_y$$
$$+ W_x^2 + W_y^2 + W_z^2 + W_xU_z + W_yV_z + W_z^2$$

$$\overline{u_iu_j}\frac{\partial U_i}{\partial x_j} = \overline{u^2}U_x + \overline{uv}U_y + \overline{uw}U_z$$
$$+ \overline{uv}V_x + \overline{v^2}V_y + \overline{vw}V_z$$
$$+ \overline{uw}W_x + \overline{vw}W_y + \overline{w^2}W_z$$

= Scalar

Turbulent Kinetic Energy Equation

Momentum equation for the mean flow:

$$\frac{D\overline{U_i}}{Dt} = \frac{\partial \overline{U_i}}{\partial t} + \overline{U_j}\frac{\partial \overline{U_i}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \upsilon \frac{\partial^2 \overline{U_i}}{\partial {x_j}^2} - \frac{\partial}{\partial x_j}\overline{u_i u_j} - g\delta_{i3}$$

Which can also be written as:

$$\boxed{A_{,j} = \frac{\partial A}{\partial x_j}}$$

$$\frac{\partial \overline{U_i}}{\partial t} + \overline{U_j}\,\overline{U_{i,j}} = -\frac{1}{\rho}\overline{p}_{,i} + \upsilon \overline{U_{i,jj}} - \left(\overline{u_i u_j}\right)_{,j} - g\delta_{i3}$$

Mean

For the total flow, the moment equation is:

Instantaneous

$$\frac{\partial}{\partial t}\left(\overline{U_i} + u_i\right) + \left(\overline{U_j} + u_j\right)\left(\overline{U_i} + u_i\right)_{,j} = -\frac{1}{\rho}\left(\overline{p} + p'\right)_{,i} + \upsilon\left(\overline{U_i} + u_i\right)_{,jj}$$

Total = mean + fluctuation

To obtain the equation for the fluctuating part, subtract the mean momentum equation from the total equation:

$$\frac{\partial u_i}{\partial t} + \overline{U_j}u_{i,j} + u_j\overline{U_{i,j}} + u_j u_{i,j} - \left(\overline{u_i u_j}\right)_{,j} = -\frac{1}{\rho}\,p'_{,i} + \nu u_{i,jj} \qquad (3)$$

A B C D E F G

Multiply (3) by u_i, apply time average $\left(k = \frac{1}{2}\left(\overline{u_i u_i}\right)\right)$ and analyze A-G terms:

A: $\overline{u_i\frac{\partial u_i}{\partial t}} = \frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i^2}\right)$

B: $\overline{u_i\overline{U_j}u_{i,j}} = \overline{U_j}\left(\frac{1}{2}\overline{u_i^2}\right)_{,j}$

$$\boxed{\frac{\partial\left(\frac{1}{2}u_i u_i\right)}{\partial x_j} = \frac{1}{2}\left(u_{i,j}u_i + u_i u_{i,j}\right) = u_i u_{i,j}}$$

C: $\overline{u_i u_j \overline{U_{i,j}}} = \overline{u_i u_j}\,\overline{U_{i,j}}$

D: $\overline{u_i u_j u_{i,j}} = \left(\overline{\frac{1}{2}u_i^2 u_j}\right)_{,j} = \frac{1}{2}\overline{\left(2u_i\frac{\partial u_i}{\partial x_j}u_j + u_i^2\cancel{\frac{\partial u_j}{\partial x_j}}\right)}$ $\qquad \boxed{u_{j,j} = 0}$

E: $\overline{u_i\left(\overline{u_i u_j}\right)_{,j}} = \overline{u_i}\left(\overline{u_i u_j}\right)_{,j} = 0$ $\qquad \boxed{\overline{u_i} = 0}$

F: $-\frac{1}{\rho}\,\overline{u_i p'_{,i}} = -\frac{1}{\rho}\left(\overline{u_i p'}\right)_{,i}$ $\qquad \boxed{u_{i,i} = 0}$

G: $\nu\overline{u_i u_{i,jj}} = \nu\left[\overline{u_i u_{i,jj} + \underbrace{\frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\left(u_{i,j} - u_{j,i}\right)}_{\boxed{=0}}}\right]$

$\boxed{=0}$ ➔ Doubly contracted product of symmetric and anti-symmetric tensor

$$\overline{[u_i(u_{i,j}+u_{j,i})]}_{,j} = \frac{\partial}{\partial x_j}\left[\overline{u_i(u_{i,j}+u_{j,i})}\right] = \overline{\frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)} + \overline{u_i u_{i,jj}} + \overline{u_i \cancel{u_{jj,i}}}$$

Therefore:

$$\nu\overline{u_i u_{i,jj}} = \nu\left[\overline{[u_i(u_{i,j}+u_{j,i})]}_{,j} - \overline{u_{i,j}(u_{i,j}+u_{j,i}) + \tfrac{1}{2}(u_{i,j}+u_{j,i})(u_{i,j}-u_{j,i})}\right]$$

$$= \nu\left[\overline{[u_i(u_{i,j}+u_{j,i})]_{,j} - u_{i,j}^2 - u_{i,j}u_{j,i} + \tfrac{1}{2}(u_{i,j}^2 - u_{j,i}^2)}\right]$$

$$= \nu\left[\overline{[u_i(u_{i,j}+u_{j,i})]_{,j} - \tfrac{1}{2}u_{i,j}^2 - u_{i,j}u_{j,i} - \tfrac{1}{2}u_{j,i}^2}\right]$$

$$= \nu\left[\overline{[u_i(u_{i,j}+u_{j,i})]_{,j} - \frac{1}{2}(u_{i,j}+u_{j,i})^2}\right]$$

$$\nu\overline{u_i u_{i,jj}} = 2\nu\overline{(u_i e_{ij})}_{,j} - 2\nu\overline{e_{ij}e_{ij}}$$

Where the fluctuating rate of strain components e_{ij} are: $e_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$

Eq. (3) becomes:

TKE Equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i^2}\right) + \overline{U_j}\left(\frac{1}{2}\overline{u_i^2}\right)_{,j} = \frac{D}{Dt}\left(\frac{1}{2}\overline{u_i^2}\right) = -\frac{\partial}{\partial x_j}\underbrace{\left[\frac{1}{\rho}\left(\overline{p'u_j}\right) + \frac{1}{2}\overline{{u_i}^2 u_j} - 2\nu\overline{\left(u_i e_{ij}\right)}\right]}_{\text{Transport by pressure, turbulence and viscous effects}} - \underbrace{\overline{u_i u_j}\overline{U_{i,j}}}_{\text{Shear production}} - \underbrace{2\nu\overline{e_{ij}e_{ij}}}_{\varepsilon = \text{Viscous dissipation}} \quad (4)$$

First three terms on RHS are in flux divergence form and consequently represent spatial transport of TKE. First two are due to the turbulence itself, whereas the last is viscous transport.

The shear production term appears in the mean KE equation with opposite sign. Usually > 0, therefore, represents loss of mean KE and gain of TKE. Viscous dissipation = ε = same order shear production.

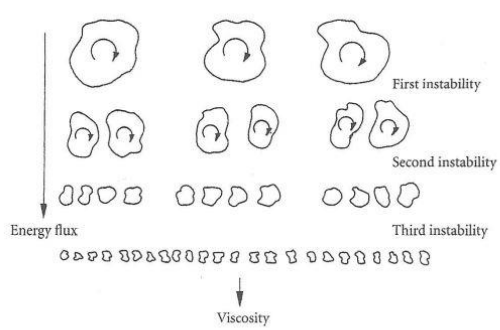


Figure 1.14 A schematic representation of the energy cascade (after Frisch 1995). See also Leonardo's sketch—Plate 3.

Shear production can be < 0, i.e., in some cases backscatter wherein small vortices combine to form larger ones, which can bring energy from the small to the large scales.

$$e_{ij}e_{ij} = e_{ij}\frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \underbrace{\frac{1}{2}e_{ij}\left(u_{i,j} - u_{j,i}\right)}_{\boxed{=0}}$$

$$= \frac{1}{2}e_{ij}\left(2u_{i,j}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\left(u_{i,j}\right)$$

True dissipation ε:

$$\varepsilon = 2\nu\overline{e_{ij}e_{ij}} = \nu\left(\overline{u_{i,j}^2 + u_{i,j}u_{j,i}}\right)$$

Pseudo-dissipation $\tilde{\varepsilon}$:

$$\tilde{\varepsilon} = \nu\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} = \upsilon\overline{u_{i,J}^2}$$

Relation between ε and $\tilde{\varepsilon}$ (Pope Prob. 5.25 for proof):

$$\varepsilon = \tilde{\varepsilon} + \nu\frac{\partial^2\overline{u_iu_j}}{\partial x_i\partial x_j} = \nu\left(\overline{u_{i,j}u_{i,j}} + \frac{\partial^2\overline{u_iu_j}}{\partial x_i\partial x_j}\right) = \nu\left(\overline{u_{i,j}^2 + u_{i,j}u_{j,i}}\right)$$

$$\frac{\partial^2\overline{u_iu_j}}{\partial x_i\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\overline{u_iu_j} = \overline{u_{i,j}u_{j,i}}$$

$$\frac{\partial}{\partial x_i}(u_iu_j) = \cancel{u_ju_{i,i}} + u_iu_{j,i}$$

$$\frac{\partial}{\partial x_j}(u_iu_{j,i}) = u_{i,j}u_{j,i} + u_i\cancel{\frac{\partial^2 u_j}{\partial x_i\partial x_j}}$$

The term $\upsilon\frac{\partial^2\overline{u_iu_j}}{\partial x_i\partial x_j}$ usually represents only a small percentage of ε. Additionally, for homogenous isotropic turbulence, this term is exactly 0 (Chapter 4, Part 1).

Reconsider term G to obtain an alternative form of the TKE equation as a function of $\tilde{\varepsilon}$:

G: $\nu\overline{u_iu_{i,jj}} = \nu\nabla^2k - \tilde{\varepsilon} = \nu\left(\overline{\left(\frac{1}{2}\overline{u_i^2}\right)_{,jj} - u_{i,j}^2}\right)$

$$\nu\nabla^2k = \nu\frac{1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j}\overline{u_i^2}\right) = \nu\frac{1}{2}\frac{\partial}{\partial x_j}\left(\overline{2u_i\frac{\partial u_i}{\partial x_j}}\right) = \nu\left(\overline{u_{i,j}^2 + u_iu_{i,jj}}\right) = \tilde{\varepsilon} + \nu\overline{u_iu_{i,jj}}$$

As shown before:

$$v\overline{u_i u_{i,jj}} = 2v\overline{\left(u_i e_{ij}\right)_{,j}} - 2v\overline{e_{ij}e_{ij}}$$

Or equivalently:

$$v\overline{u_i u_{i,jj}} = v\nabla^2 k - \tilde{\varepsilon} = v\overline{\left(\frac{1}{2}u_i^2\right)_{,jj}} - \tilde{\varepsilon} = \underbrace{v\frac{\partial}{\partial x_j}\overline{u_i u_{i,j}}}_{\text{Viscous transport}} - \underbrace{\tilde{\varepsilon}}_{\text{Pseudo dissipation}}$$

Eq. (4), then becomes:

TKE Equation Alternative form

$$\frac{D}{Dt}\overline{\left(\frac{1}{2}u_i^2\right)} + \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\overline{p' u_j}\right) + \frac{1}{2}\overline{u_i^2 u_j}\right] = v\nabla^2 k + P - \tilde{\varepsilon}$$

Where:

$$P = -\overline{u_i u_j}\,\overline{U_{i,j}}$$

$$\tilde{\varepsilon} = v\overline{u_{i,j}^2}$$

Finally, we can write Eq. (4) as:

$$\frac{D}{Dt}\overline{\left(\frac{1}{2}u_i^2\right)} = -\nabla\cdot T + P - \varepsilon = -\nabla\cdot T' + P - \tilde{\varepsilon}$$

Where:

$$T = \frac{1}{\rho}\left(\overline{p' u_j}\right) + \frac{1}{2}\overline{u_i^2 u_j} - 2v\overline{\left(u_i e_{ij}\right)}$$

$$T' = \frac{1}{\rho}\left(\overline{p' u_j}\right) + \frac{1}{2}\overline{u_i^2 u_j} - v\overline{u_i u_{i,j}}$$

$\tilde{\varepsilon}$ is also called isotropic dissipation rate. Note:

$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\overline{u_i u_j}$=0 for I = j and $\overline{u_i^2}$= constant and $\overline{u_i u_j}$=0 for i $\neq j$, i.e., for homogeneous isotropic turbulence

$$\frac{DR}{Dt} = \frac{\partial R}{\partial t} + \underline{v} \cdot \nabla R = \frac{\partial R}{\partial t} + \overline{u} R_x + \overline{v} R_y + \overline{w} R_z \quad \text{Scalar}$$

$$R = \frac{1}{2} u_i u_i = \frac{1}{2} \underline{u} \cdot \underline{u}$$

$$= -\frac{\partial}{\partial x_j}\left[\rho^{-1} \overline{p u_j} + \frac{1}{2} \overline{u_i^2 u_j} - 2\nu \overline{u_i \varepsilon_{ij}} \right] - \overline{u_i u_j} U_{i,j}$$
$$- 2\nu \overline{\varepsilon_{ij} \varepsilon_{ij}}$$

$$\rho^{-1} \frac{\partial}{\partial x_j}(\overline{p u_j}) = (\overline{pu})_x + (\overline{pv})_y + (\overline{pw})_z \qquad \nabla \cdot \underline{f} = \text{Scalar}$$

$$\frac{1}{2} \frac{\partial}{\partial x_j}(\overline{u_i^2 u_j}) = (\overline{u^3})_x + (\overline{u^2 v})_y + (\overline{u^2 w})_z$$

$u_i u_i$ = Scalar, $\underline{f}$

$\nabla \cdot \underline{f}$ = Scalar

$$+ (\overline{v^2 u})_x + (\overline{v^3})_y + (\overline{v^2 w})_z$$

$$+ (\overline{w^2 u})_x + (\overline{w^2 v})_y + (\overline{w^3})_z$$

$$-2\nu \frac{\partial}{\partial x_j}(\overline{u_i \varepsilon_{ij}}) = 2(u u_x)_x + [u(u_y + v_x)]_y + [u(u_z + w_x)]_z$$

$\nabla \cdot \underline{f}$ = Scalar

$$+ [v(v_x + u_y)]_x + 2(v v_y)_y + [v(v_z + w_y)]_z$$
$$+ [w(w_x + u_z)]_x + [w(w_y + v_z)]_y + 2(w w_z)_z$$

Contraction i,j $\quad - \overline{u_i u_j} U_{i,j} = P$ = Same term MKE equation

$n = 2$

tensors $\quad 2\nu \overline{\varepsilon_{ij} \varepsilon_{ij}} = \nu \left(u_{i,j}^2 + u_{ij} u_{ji} \right)$

= Scalar

$$u_x^2 + u_y^2 + u_z^2 \quad + u_x^2 + u_y v_x + u_z w_x$$
$$+ v_x^2 + v_y^2 + v_z^2 \quad + v_x u_y + v_y^2 + v_z w_y$$
$$+ w_x^2 + w_y^2 + w_z^2 \quad + w_x u_z + w_y v_z + w_z^2$$

Rate of Strain Principal Axes, Parallel Shear Flow, Turbulent Anisotropy, and Shear Production

$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ symmetric rate of strain tensor

Symmetric tensor obeys transformation laws that there are three invariants which are independent of the choice of the coordinate axes:

$$I_1 = e_{xx} + e_{yy} + e_{zz}$$

$$I_2 = e_{xx}e_{yy} + e_{yy}e_{zz} + e_{zz}e_{xx} - {e_{xy}}^2 - {e_{yz}}^2 - {e_{zx}}^2$$

$$I_3 = \begin{vmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{vmatrix}$$

Further property: one and only one set of axes exists for which $e_{ij} = 0 \quad i \neq j \rightarrow$ principal axes.

$$e_{ij} = \begin{Bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{Bmatrix}$$

Where e_1, e_2, e_3 represent the principal strain rates.

The invariants represented in this set of axes are:

$$I_1 = e_1 + e_2 + e_3$$

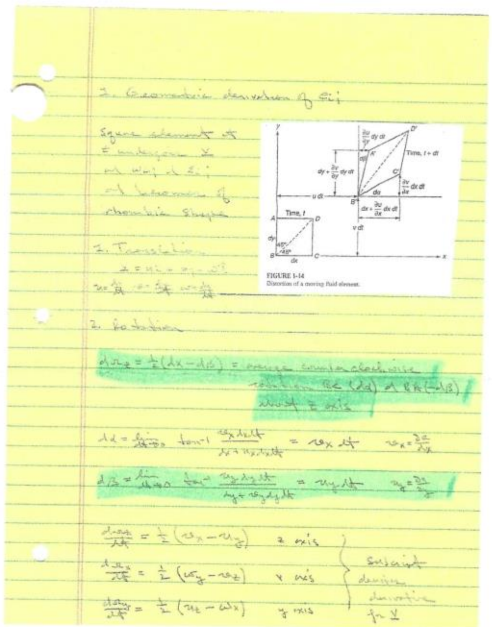
$$I_2 = e_1e_2 + e_2e_3 + e_3e_1$$

$$I_3 = e_1e_2e_3$$

If I_1, I_2, I_3 known these equations can be solved for e_1, e_2, e_3.

http://user.engineering.uiowa.edu/~me_260/Viscous_flow_main.htm

Chapters 1 & 2 (6.3)



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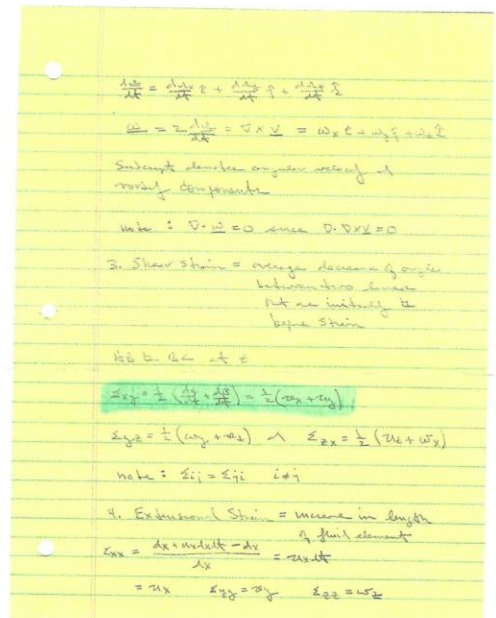

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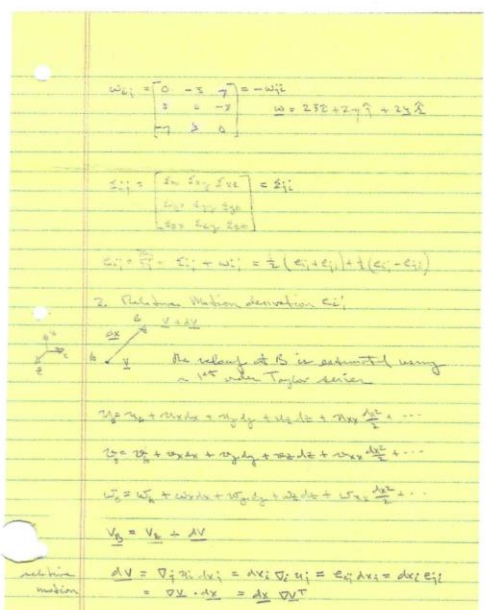

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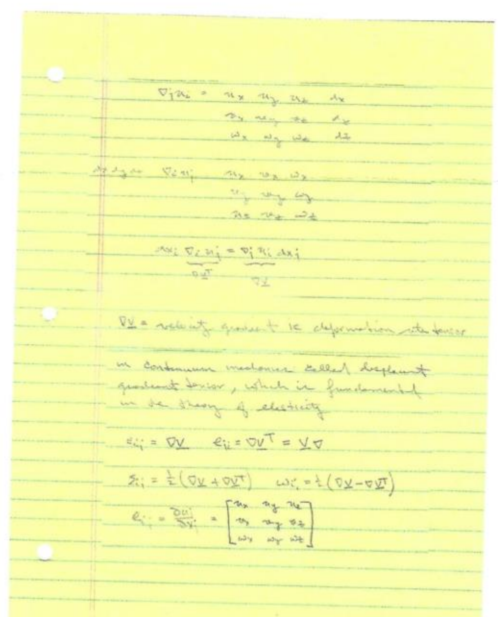

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Parallel Shear Flow Principal Axes

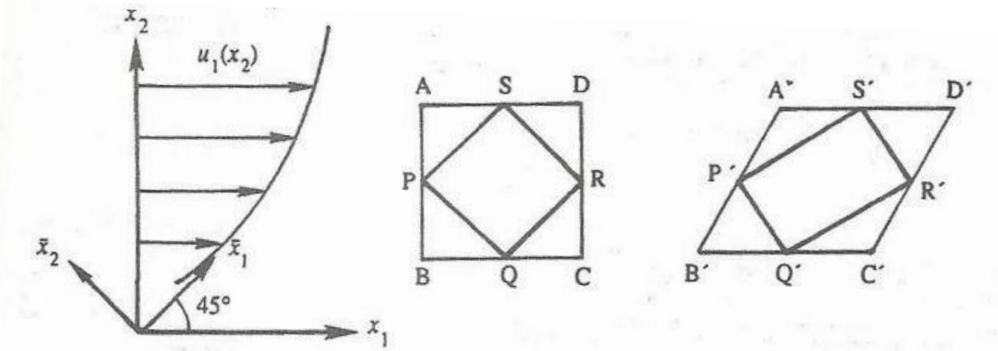

Figure 3.14 Deformation of elements in a parallel shear flow. The element is stretched along the principal axis $\bar{x}_1$ and compressed along the principal axis $\bar{x}_2$.

$\underline{u} = (u_1(x_2), 0,0)$

$\gamma(x_2) = \dfrac{du_1}{dx_2}$

$\omega_3 = -\gamma$ represents the only non-zero component of the vorticity.

Angular velocity for AB= $-\gamma$, BC=0, ∴ average =$-\frac{\gamma}{2}$

The average angular velocity represents the rate of rotation which is independent of the coordinate system. Whereas e_{ij}depends on the coordinate system.

For ABCD with axes parallel the x_1x_2 plane e_{ij} only has shear elements:

$$e_{ij} = \begin{Bmatrix} 0 & \gamma/2 & 0 \\ \gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$

However, representation in principal axes (45° rotation PQRS) results in e_{ij} only having normal elements:

$$\widetilde{e_{ij}} = \begin{Bmatrix} \gamma/2 & 0 & 0 \\ 0 & -\gamma/2 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$

$\widetilde{e_{11}}$ =linear rate of extension = $\gamma/2$

$\widetilde{e_{22}}$ =linear rate of compression = -$\gamma/2$

PQRS oriented at 45° deforms to P'Q'R'S' -> PS elongates and PQ contracts, but the angle between the sides remain 90°: spherical element becomes ellipsoidal.

In parallel shear flow, the element ABCD undergoes shear only, whereas element PQRS undergoes only normal strain.

Recall also for parallel shear flow that the main loss term in mechanical energy equation < 0 and gain/production in TKE equation is $\overline{uv}\frac{\partial \overline{U}}{\partial y} < 0$.

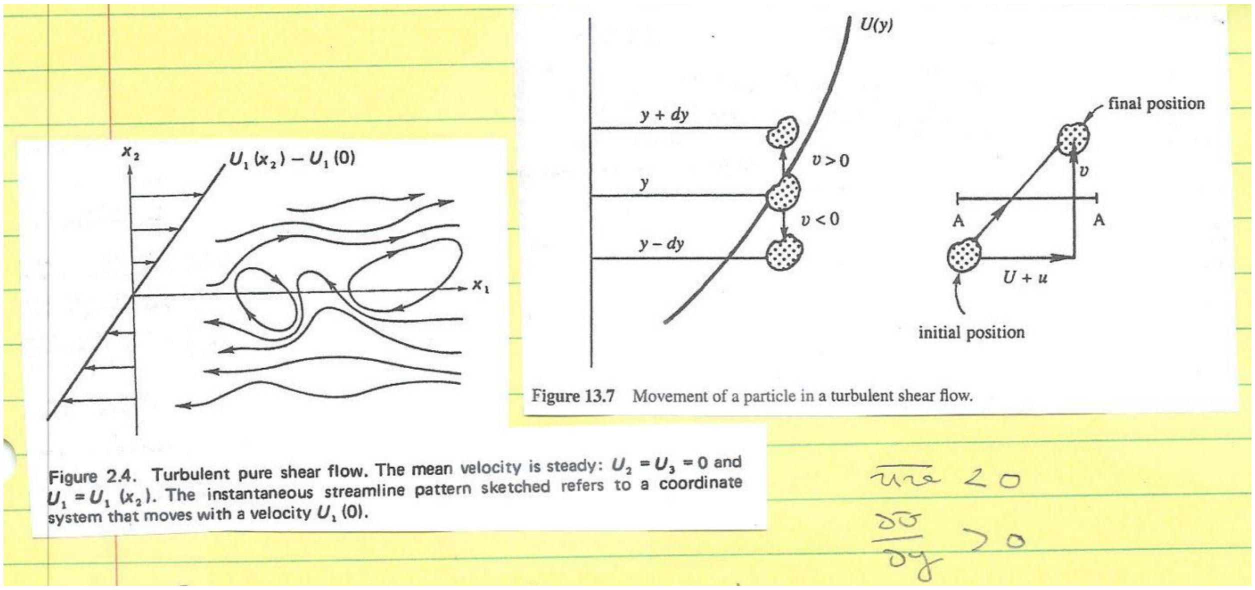

Figure 2.4. Turbulent pure shear flow. The mean velocity is steady: $U_2 = U_3 = 0$ and $U_1 = U_1(x_2)$. The instantaneous streamline pattern sketched refers to a coordinate system that moves with a velocity $U_1(0)$.

Figure 13.7 Movement of a particle in a turbulent shear flow.

Thus, eddies that are most effective in maintaining the $\overline{UV}$<0 correlation and extraction energy from the mean flow are those aligned with the principal axes, which is essentially a 3D process.

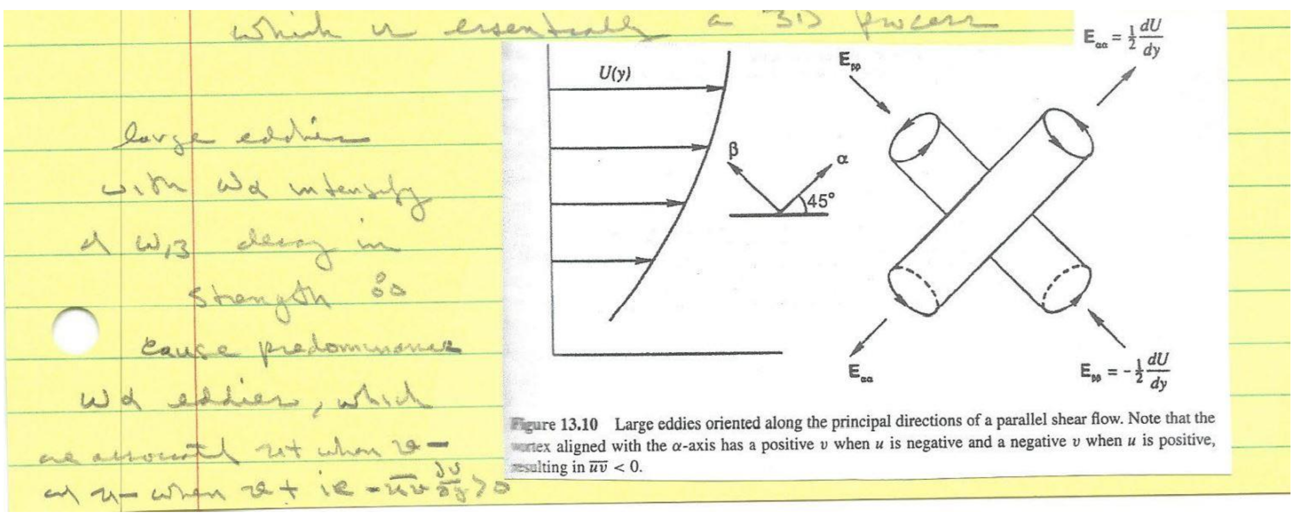

Figure 13.10 Large eddies oriented along the principal directions of a parallel shear flow. Note that the vortex aligned with the α-axis has a positive v when u is negative and a negative v when u is positive, resulting in $\overline{uv} < 0$.

Correlation functions can be used to calculate the form and distribution of the eddies

Objective procedure Lumley (1965) using wake measurements (1966). Dominant eddies have same pattern as first eigenfunction equation $\int R_{ij}(\underline{x}, \underline{r})\, \varphi_j^{(n)}(\underline{x}+\underline{r})\, d\underline{r} = \lambda^{(n)} \varphi_i^{(n)}(\underline{x})$

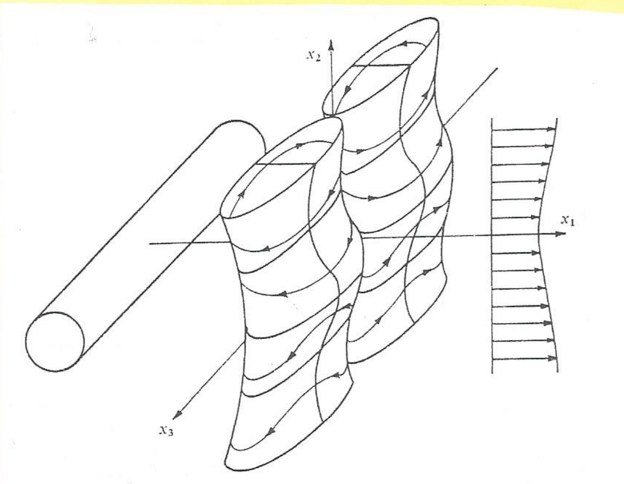

Fig. 4.10. Sketch of the velocity pattern of the eddy giving best fit to the correlation measurements of Grant, obtained by Lumley & Payne.

(1) $u_1(x_2)$
(2) convergence/divergence in x_2 direction
(3) $u_2(x_1)$
(5) $u_3(x_1)$

Simpler approach is to construct simplest eddy pattern based observations:

(1) $R_{11}(0,r,0) < 0$, $R_{11}(0,0,r) > 0$
(2) $R_{12}(0,r,0) < 0$ large r
(3) $R_{22}(r,0,0) < 0$
(4) $R_{13}(0,r,0)$ initially < 0 but changes sign > 0
(5) $R_{33}(r,0,0) < 0$, but $R_{33}(0,r,0) > 0$

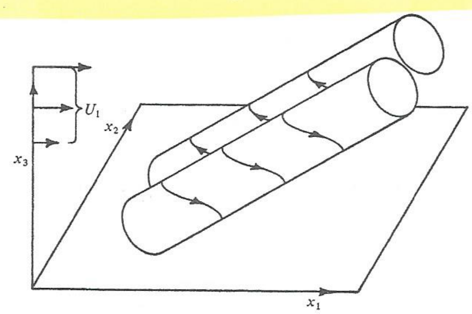

Fig. 4.11. Sketch of an inclined double-roller eddy. (Arrows on the lines around the cylinders indicate the eddy streamlines. From Townsend 1970.)