# Chapter 3: Overview of Turbulent Flow Physics and Equations

## Part 2: Reynolds-Averaged Navier-Stokes Equations

For convenience of notation use lower case with over squiggle, uppercase for mean, and lowercase for fluctuation in Reynolds decomposition.

$$\tilde{u}_i = U_i + u_i$$

$$\tilde{p} = P + p$$

$$\frac{\partial \tilde{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_i} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} - g\delta_{i3}$$

→ NS equation

Mean Continuity Equation

$$\overline{\frac{\partial}{\partial x_i}(U_i + u_i)} = \frac{\partial U_i}{\partial x_i} + \frac{\partial \overline{u}_i}{\partial x_i} = \frac{\partial U_i}{\partial x_i} = 0$$

$$\frac{\partial \tilde{u}_i}{\partial x_i} = \frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 0 \quad \rightarrow \quad \frac{\partial u_i}{\partial x_i} = 0$$

Both mean and fluctuation satisfy divergence = 0 condition.

## Mean Momentum Equation

$$\frac{\partial}{\partial t}(U_i + u_i) + (U_j + u_j)\frac{\partial}{\partial x_j}(U_i + u_i)$$
$$= -\frac{1}{\rho}\frac{\partial}{\partial x_i}(P + p) + \nu\frac{\partial^2}{\partial x_j x_j}(U_i + u_i) - g\delta_{i3}$$

$$\overline{\frac{\partial}{\partial t}(U_i + u_i)} = \frac{\partial U_i}{\partial t} + \frac{\partial \overline{u}_i}{\partial t} = \frac{\partial U_i}{\partial t}$$

$$\overline{(U_j + u_j)\frac{\partial}{\partial x_j}(U_i + u_i)}$$
$$= U_j\frac{\partial U_i}{\partial x_j} + \cancel{U_j\frac{\partial \overline{u}_i}{\partial x_j}} + \cancel{\overline{u_j}\frac{\partial U_i}{\partial x_j}} + \overline{u_j\frac{\partial u_i}{\partial x_j}}$$
$$= U_j\frac{\partial U_i}{\partial x_j} + \frac{\partial}{\partial x_j}\overline{u_i u_j}$$

Since $\frac{\partial}{\partial x_j}\overline{u_i u_j} = \cancel{\overline{u_i\frac{\partial u_j}{\partial x_j}}} + \overline{u_j\frac{\partial u_i}{\partial x_j}} = \overline{u_j\frac{\partial u_i}{\partial x_j}}$

$$\overline{\frac{\partial}{\partial x_i}(P + p)} = \frac{\partial P}{\partial x_i} + \frac{\partial \overline{p}}{\partial x_i} = \frac{\partial P}{\partial x_i}$$

$$-\overline{g\delta_{i3}} = -g\delta_{i3}$$

$$\overline{\nu \frac{\partial^2}{\partial {x_j}^2}(U_i + u_i)} = \nu \frac{\partial^2 U_i}{\partial x_j^2} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j^2} = \nu \frac{\partial^2 U_i}{\partial x_j^2}$$

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} + \frac{\partial(\overline{u_i u_j})}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} - g\delta_{i3}$$

Or $\frac{DU_i}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - g\delta_{i3} + \frac{\partial}{\partial x_j}\left[\nu \frac{\partial U_i}{\partial x_j} - \overline{u_i u_j}\right]$

$$\frac{DU_i}{Dt} = -g\delta_{i3} + \frac{1}{\rho}\frac{\partial}{\partial x_i}\overline{\sigma}_{ij}$$

$$\overline{\sigma} = -P\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \rho\overline{u_i u_j}$$

with $\frac{\partial U_i}{\partial x_i} = 0$

→ RANS Equations

The difference between the NS and RANS equations is the Reynolds stresses $-\rho\overline{u_i u_j}$, which acts like additional stress.

$-\rho\overline{u_i u_j} = -\rho\overline{u_j u_i}$ (i.e. Reynolds stresses are symmetric)

$$= \begin{bmatrix} -\rho\overline{u^2} & -\rho\overline{uv} & -\rho\overline{uw} \\ -\rho\overline{uv} & -\rho\overline{v^2} & -\rho\overline{vw} \\ -\rho\overline{uw} & -\rho\overline{vw} & -\rho\overline{w^2} \end{bmatrix}$$

$\overline{u_i^2}$ are normal stresses
$\overline{u_i u_j}$ $\quad i \neq j$ are shear stresses
6 new unknowns

For homogeneous/isotropic turbulence $\overline{u_i u_j}$ $\quad i \neq j = 0$ and $\overline{u^2} = \overline{v^2} = \overline{w^2} =$ constant; however, turbulence is generally non-isotropic.

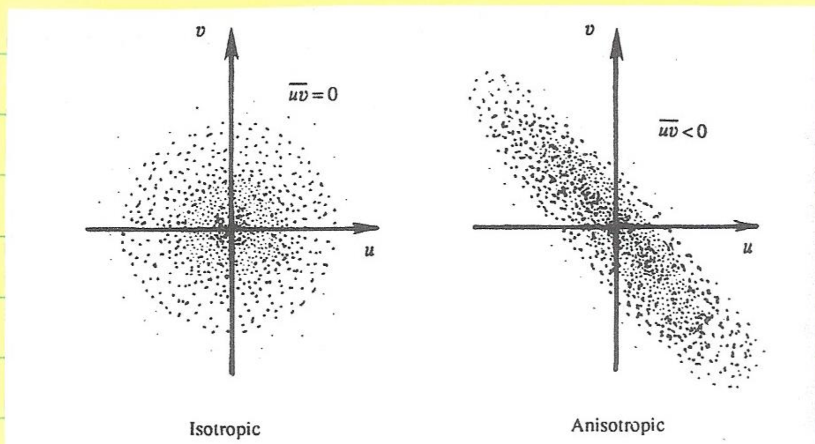
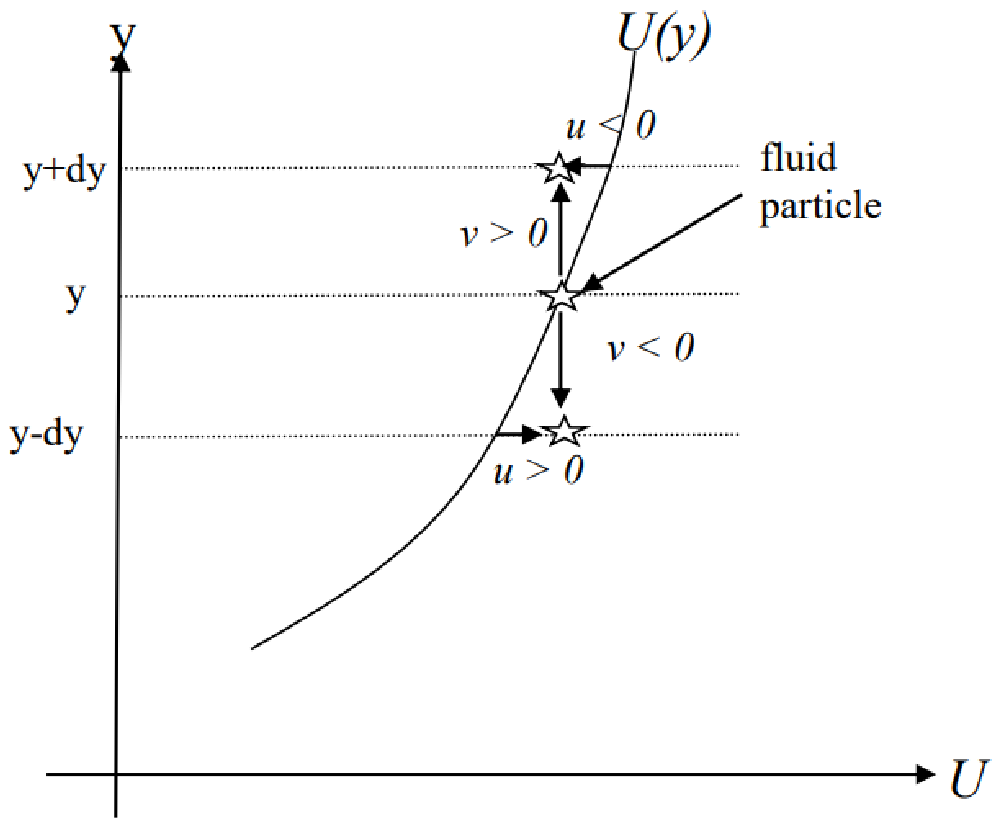



**Figure 13.6** Isotropic and anisotropic turbulent fields. Each dot represents a $uv$-pair at a certain time

scatter plot = instantaneous values $uv(t)$
isotropic = no directional preference
= equally likely all four quadrants
ie $\overline{uv} = 0$ uncorrelated

anisotropic = $u > 0$ correlated $v < 0$
$u < 0$ correlated $v > 0$
ie $\overline{uv} < 0$

Consider shear flow with $\frac{dU}{dy} > 0$ as below,




The fluid velocity is: $\underline{V} = (U + u, v, w)$

If fluid particle retains its total velocity $\underline{V}$ from $y$ to $y \pm dy$ gives,
$U + u = constant$ → If $U$ increases, $u$ decreases and vice versa.

$$\left.\begin{array}{lll} v > 0 & \rightarrow & u < 0 \\ v < 0 & \rightarrow & u > 0 \end{array}\right] \rightarrow \overline{uv} < 0$$

x-momentum tends towards decreasing y as turbulence diffuses gradients and decreases $\frac{dU}{dy}$

x-momentum transport in y direction, i.e., across y = constant AA per unit area

$M_{xy} = \int \rho \tilde{u} \underline{V} \cdot \underline{n} \, dA$, where $\tilde{u} = (U + u)$

$$\frac{d\overline{M_{xy}}}{dA} = \rho\overline{(U+u)v} = \rho U\overline{v} + \rho\overline{uv} = \rho\overline{uv}$$

i.e. $\rho\overline{u_i u_j}$ = average flux of j-momentum in i-direction = average flux of i-momentum in j-direction

Reynolds averaged pressure Poisson equation

$$-\frac{1}{\rho}\nabla^2 \overline{P} = \overline{U_{i,j} U_{j,i}} \qquad P = \overline{P} + p$$

$$= \overline{U_{i,j}}\,\overline{U_{j,i}} + \frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j}$$

can be obtained by taking mean of $\nabla^2 P$ equation or $\nabla \cdot$ RANS equations

## Anisotropy

Symmetry of the Reynolds stress tensor:

$$\overline{u_i u_j} = \overline{u_j u_i}$$

The diagonal components of the Reynolds stress tensor represent the normal stresses, which in general are not equal:

$$\overline{u_1^2}, \overline{u_2^2}, \overline{u_3^2}$$

The off-diagonal components represent the shear stresses:

$$\overline{u_i u_j} \quad i \neq j$$

Turbulent kinetic energy per unit mass:

$$k(\underline{x}, t) = \frac{1}{2}\overline{\underline{u} \cdot \underline{u}} = \frac{1}{2}\overline{u_i u_i} = \frac{1}{2}\left(\overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}\right)$$

Anisotropy tensor:

$$a_{ij} = \overline{u_i u_j} - \underbrace{\frac{2}{3} k \delta_{ij}}$$

Isotropic stress, i.e., assuming

$$\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}$$

Normalized anisotropy tensor:

$$b_{ij} = \frac{a_{ij}}{2k} = \frac{\overline{u_i u_j}}{\overline{u_i u_i}} - \frac{1}{3}\delta_{ij}$$

Role of the anisotropy tensor in the Mean Momentum Equation:

$$\rho \frac{D\overline{U_i}}{Dt} = -\rho g \delta_{i3} + \frac{\partial \overline{\sigma_{ij}}}{\partial x_j}$$

$$\overline{\sigma_{ij}} = -\overline{p}\delta_{ij} + \mu\left(\frac{\partial \overline{U_i}}{\partial x_j} + \frac{\partial \overline{U_j}}{\partial x_i}\right) - \rho\overline{u_i u_j}$$

$$\overline{u_i u_j} = \frac{2}{3}k\delta_{ij} + a_{ij} = 2k\left(\frac{1}{3}\delta_{ij} + b_{ij}\right)$$

$$\rho \frac{\partial \overline{u_i u_j}}{\partial x_j} + \frac{\partial \overline{p}}{\partial x_i} = \rho \frac{\partial a_{ij}}{\partial x_j} + \frac{\partial}{\partial x_i}\left(\overline{p} + \frac{2}{3}\rho k\right)$$

Therefore, only the anisotropic component $a_{ij}$ is effective in transporting momentum since $\frac{2}{3}\rho k$ can be absorbed in a modified mean pressure.

## Irrotational motion

Consider an irrotational random velocity field:

$$\underline{\widetilde{\omega}} = \underline{\Omega} + \underline{\omega} = 0$$

Both mean and fluctuating vorticity are equal to zero. For example, water waves and outer portion of turbulent jet (free shear flow) or boundary layer.

Notation: $u_{i,j} = \frac{\partial u_i}{\partial x_j}$, $u_i$ = velocity fluctuation with $\underline{\omega} = 0$, i.e., $\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) = 0$ where $\omega_{ij}$ is the fluctuation rotation rate tensor.

$$u_{i,j} - u_{j,i} = 0$$

$$u_i\left(u_{i,j} - u_{j,i}\right) = 0 = u_i u_{i,j} - u_i u_{j,i}$$

$$\frac{\partial}{\partial x_i}\left(u_i u_j\right) = u_{i,i} u_j + u_i u_{j,i}$$

$$\frac{\partial}{\partial x_j}\left(\frac{1}{2} u_i u_i\right) = \frac{1}{2}\left(u_{i,j} u_i + u_i u_{i,j}\right) = u_i u_{i,j}$$

$\frac{\partial}{\partial x_i}\left(\overline{u_i u_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i u_i}\right) = \frac{\partial k}{\partial x_j}$ = gradient of a scalar, i.e., same form as fluctuating pressure: Corrsin-Kistler equation. Therefore, for potential flow, fluctuating velocity is equivalent to pressure fluctuations → do not affect the mean velocity.

Closure Problem:

1. RANS equations differ from the NS equations due to the Reynolds stress terms
2. RANS equations are for the mean flow $(U_i, P)$; thus, represent 4 equations with 10 unknowns due to the additional 6 unknown Reynolds stresses $\overline{u_i u_j}$
3. Equations can be derived for $\overline{u_i u_j}$ by summing products of velocity and momentum components and time averaging, but these include additionally 10 triple products $\overline{u_i u_j u_l}$ unknowns. Triple products represent Reynold stress transport.
4. Again, equations for triple products can be derived that involve higher order correlations leading to fact that RANS equations are inherently non-deterministic, which requires turbulence modeling.
5. Turbulence closure models render deterministic RANS solutions.
6. The NS and RANS equations have paradox that NS equations are deterministic but have nondeterministic solutions for turbulent flow due to inherent stochastic nature of turbulence, whereas the RANS equations are nondeterministic, but have deterministic solutions due to turbulence closure models.

## Closure Problem

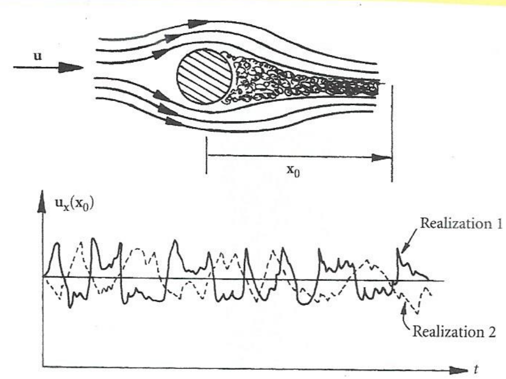



Figure 1.8 A cylinder is towed through a quiescent fluid and $\mathbf{u}_x$ is measured at location $x_0$. The records of $\mathbf{u}_x(t)$ in two nominally identical realizations of the experiment are quite different.

$\underline{u}(\underline{x}, t)$ random

$\bar{u}(\underline{x})$ and $\overline{u^2}(x)$ reproducible ie statistical properties should be predictable

∴ How can equations be derived for statistical properties

$$\rho \underline{u}_t = -\rho\, \underline{u} \cdot \nabla \underline{u} - \nabla p + \mu \nabla^2 \underline{u} \qquad (1)$$

$$\underline{u}_t = F_1(\underline{u}, p)$$

$$\nabla \cdot \underline{u} = 0$$

$$\nabla \cdot (1) \Rightarrow \nabla^2 (p/\rho) = -\nabla \cdot (\underline{u}\, \nabla \underline{u})$$

$$p(\underline{x}) = \frac{\rho}{2\pi} \int \frac{[\nabla \cdot (\underline{u} \cdot \nabla \underline{u})]'}{|x - x'|}\, dx'$$ Biot-Savart Law

ie $\underline{u}_t = F_2(\underline{u})$ for IC integrate in time "deterministically" find $u(\underline{x}, t)$

DNS!

However, impracticable for industrial use and often statistical variables of greatest interest such as $\bar{u}$, $\overline{u'^2}$ etc. [$\underline{u}(\underline{x},t) = \bar{\underline{u}}(x) + \underline{u}'(\underline{x},t)$]

Need equations $\bar{u}$, $\overline{u'^2}$ etc. However turns out impossible to derive equations for statistical variables as leads to hierarchy of statistical nondeterministic equations each depending on higher order statistical variables

$\frac{\partial}{\partial t}$(statistical property $\underline{u}$) = F(order higher order statistical properties $\underline{u}$)

"Closure problem"

Arrow of Time

## Arrow of Time

The arrow of time, also called time's arrow, is the concept positing the "one-way direction" or "asymmetry" of time. It was developed in 1927 by the British astrophysicist Arthur Eddington and is an unsolved general physics question. This direction, according to Eddington, could be determined by studying the organization of atoms, molecules, and bodies, and might be drawn upon a four-dimensional relativistic map of the world ("a solid block of paper").

Euler equation $\underline{u}_t + \underline{u}\cdot\nabla\underline{u} = -\nabla(p/\rho)$ is time reversible, ie, replace $t$ and $\underline{u}$ by $-t$ and $-u$ and can integrate backwards to obtain IC. However due to effects $\mu$ and entropy NS equations are not time reversible!