Chapter 3: Overview of Turbulent Flow Physics and Equations

A.5 A summary of Cartesian-tensor suffix notation

The definitions, rules, and operations involved with Cartesian tensors using suffix notation can be summarized thus.

1. In a tensor equation, *tensor expressions* are separated by +, −, or =. For example;

$$b_{ij} + c_{ij} = f_{ijkk}. \qquad \text{(A.75)}$$

2. In a tensor expression, a suffix that appears once is a *free suffix* (e.g., i and j in Eq. (A.75)).
3. In a tensor expression, a suffix that appears twice is a *repeated suffix* or a *dummy suffix* (e.g., the suffixes k in d_{ijkk}). The symbol used for a dummy suffix is immaterial, i.e., $d_{ijkk} = d_{ijpp}$.
4. *Summation convention*: a repeated suffix implies summation, i.e.,

$$d_{ijkk} = \sum_{k=1}^{3} d_{ijkk}. \qquad \text{(A.76)}$$

5. In a tensor expression, a suffix cannot appear more than twice. For example, the expression f_{ijii} is invalid.
6. A tensor expression with N free suffixes is (or, more correctly, represents the components of) an Nth-order tensor. For example, each expression in Eq. (A.75) is a second-order tensor.
7. Each expression in a tensor equation must be a tensor of the same order, with the same free suffixes (not necessarily in the same order). Equation (A.75) is valid, whereas $b_{ij} = d_{ijk}$ and $b_{ij} = c_{ik}$ are both invalid.
8. The *Kronecker delta* δ_{ij} is defined by

$$\begin{aligned} \delta_{ij} &= 1, \quad \text{for } i = j, \\ &= 0, \quad \text{for } i \neq j. \end{aligned} \qquad \text{(A.77)}$$

   It is a second-order tensor. Note that $\delta_{ii} = 3$.
9. The *alternation symbol* ε_{ijk} in Eq. (A.56) is *NOT* a tensor.
10. *Addition*, e.g., $b_{ijk} = c_{ijk} + d_{ikj}$. Each tensor must be of the same order with the same free suffixes.
11. The *tensor product* of an Nth-order tensor and an Mth-order tensor is an $(N + M)$th-order tensor, e.g., $b_{ijk\ell m} = c_{ij} d_{k\ell m}$.
12. An Nth-order tensor ($N \geq 2$) can be *contracted* by changing two free suffixes into repeated suffixes. The result is a tensor of order $N - 2$. Different contractions of d_{ijk} are d_{iik}, d_{iji}, and d_{ijj}.
13. The *inner product* of an Nth-order tensor and an Mth-order tensor ($N \geq 1, M \geq 1$) is a tensor of order $N + M - 2$: e.g., $f_{ik\ell} = c_{ij} d_{jk\ell}$.
14. The *substitution rule* is that the inner product with the Kronecker delta is, for example,

$$\delta_{ij} c_{jk} = c_{ik}. \qquad \text{(A.78)}$$

15. There is no tensor operation corresponding to *division*.
16. The *gradient of a tensor* is a tensor of one order higher, e.g., $d_{jk\ell} = \partial c_{k\ell}/\partial x_j$.
17. The *divergence* of an Nth-order tensor ($N \geq 1$) is a tensor of order $(N - 1)$, e.g., $v_k = \partial c_{jk}/\partial x_j$.
18. There are no tensor operations corresponding to the *vector cross product* or to the *curl*.

N = 0
scalar
N = 1
vector
N = 2
2nd order
etc.

row column
1×3 $\quad v = 1^{st}$ order tensor (column matrix)

$\underline{u}\cdot\underline{v} \qquad u_i v_i \qquad v^T v \qquad v^T = [u_1\ u_2\ u_3]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$A = 1\times3$ (m n) $\quad B = 3\times1$ (n p) $\quad C = 1\times1$ (m p) $\qquad 3\times1$

$v^T v = 1\times1 = 0$ order tensor

$c_{ij} = \sum_{k=1}^{n=3} a_{ik} b_{kj} \quad i=1, m=1 \quad j=1, p=1 \quad = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$

$= u_1v_1 + u_2v_2 + u_3v_3$

$\underline{u}\,\underline{v} \qquad u_i v_j \qquad v\,v^T \qquad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}[v_1\ v_2\ v_3]$

$c_{ij} = \sum_{k=1}^{n=1} a_{i1} b_{1j} \quad i=1,3 \quad j=1,3$

$3\times1 \quad 1\times3 \quad = \quad 3\times3$ (m n, n p, m p)

$a_{11}b_{11} \quad a_{11}b_{12} \quad a_{11}b_{13} \qquad u_1v_1 \quad u_1v_2 \quad u_1v_3$

$a_{21}b_{11} \quad a_{21}b_{12} \quad a_{21}b_{13} \qquad u_2v_1 \quad u_2v_2 \quad u_2v_3$

$a_{31}b_{11} \quad a_{31}b_{12} \quad a_{31}b_{13} \qquad u_3v_1 \quad u_3v_2 \quad u_3v_3$

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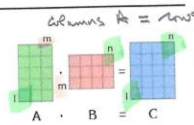
Matrix multiplication

In mathematics, particularly in linear algebra, **matrix multiplication** is a binary operation that produces a matrix from two matrices. For matrix multiplication, the number of columns in the first matrix must be equal to the number of rows in the second matrix. The resulting matrix, known as the **matrix product**, has the number of rows of the first and the number of columns of the second matrix. The product of matrices **A** and **B** is denoted as **AB**.[1]

columns A = rows B



C = rows A & columns B

For matrix multiplication, the number of columns in the first matrix must be equal to the number of rows in the second matrix. The result matrix has the number of rows of the first and the number of columns of the second matrix.

Matrix multiplication was first described by the French mathematician Jacques Philippe Marie Binet in 1812,[2] to represent the composition of linear maps that are represented by matrices. Matrix multiplication is thus a basic tool of linear algebra, and as such has numerous applications in many areas of mathematics, as well as in applied mathematics, statistics, physics, economics, and engineering.[3][4] Computing matrix products is a central operation in all computational applications of linear algebra.

Notation

This article will use the following notational conventions: matrices are represented by capital letters in bold, e.g. **A**; vectors in lowercase bold, e.g. **a**; and entries of vectors and matrices are italic (they are numbers from a field), e.g. A and a. Index notation is often the clearest way to express definitions, and is used as standard in the literature. The entry in row i, column j of matrix **A** is indicated by $(\mathbf{A})_{ij}$, A_{ij} or a_{ij}. In contrast, a single subscript, e.g. $\mathbf{A}_1, \mathbf{A}_2$, is used to select a matrix (not a matrix entry) from a collection of matrices.

Definition

If **A** is an $m \times n$ matrix and **B** is an $n \times p$ matrix,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

the *matrix product* $\mathbf{C} = \mathbf{AB}$ (denoted without multiplication signs or dots) is defined to be the $m \times p$ matrix[5][6][7][8]

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

such that

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj},$$

for $i = 1, ..., m$ and $j = 1, ..., p$.

That is, the entry c_{ij} of the product is obtained by multiplying term-by-term the entries of the ith row of **A** and the jth column of **B**, and summing these n products. In other words, c_{ij} is the dot product of the ith row of **A** and the jth column of **B**.

Therefore, **AB** can also be written as

$$\mathbf{C} = \begin{pmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + \cdots + a_{1n}b_{n2} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + \cdots + a_{2n}b_{n2} & \cdots & a_{21}b_{1p} + \cdots + a_{2n}b_{np} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + \cdots + a_{mn}b_{n2} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{pmatrix}$$

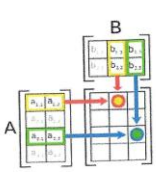
Thus the product **AB** is defined if and only if the number of columns in **A** equals the number of rows in **B**,[1] in this case n.

In most scenarios, the entries are numbers, but they may be any kind of mathematical objects for which an addition and a multiplication are defined, that are associative, and such that the addition is commutative, and the multiplication is distributive with respect to the addition. In particular, the entries may be matrices themselves (see block matrix).

Illustration

The figure to the right illustrates diagrammatically the product of two matrices **A** and **B**, showing how each intersection in the product matrix corresponds to a row of **A** and a column of **B**.

$$\overset{4\times2\text{ matrix}}{\begin{bmatrix} a_{11} & a_{12} \\ \cdot & \cdot \\ a_{31} & a_{32} \\ \cdot & \cdot \end{bmatrix}} \overset{2\times3\text{ matrix}}{\begin{bmatrix} \cdot & b_{12} & b_{13} \\ \cdot & b_{22} & b_{23} \end{bmatrix}} = \overset{4\times3\text{ matrix}}{\begin{bmatrix} \cdot & c_{12} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & c_{33} \\ \cdot & \cdot & \cdot \end{bmatrix}}$$



The values at the intersections, marked with circles in figure to the right, are:

$c_{12} = a_{11}b_{12} + a_{12}b_{22}$

$c_{33} = a_{31}b_{13} + a_{32}b_{23}$

1.8.3 The Dyad (the tensor product)

The vector *dot product* and vector *cross product* have been considered in previous sections. A third vector product, the **tensor product** (or **dyadic product**), is important in the analysis of tensors of order 2 or more. The tensor product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is written as[4]

$$\boxed{\mathbf{u} \otimes \mathbf{v}} \qquad \textbf{Tensor Product} \qquad (1.8.2)$$

This tensor product is itself a tensor of order two, and is called **dyad**:

$\mathbf{u} \cdot \mathbf{v}$	is a scalar	(a zeroth order tensor)
$\mathbf{u} \times \mathbf{v}$	is a vector	(a first order tensor)
$\mathbf{u} \otimes \mathbf{v}$	is a dyad	(a second order tensor)

It is best to *define* this dyad by what it *does*: it transforms a vector $\mathbf{w}$ into another vector with the direction of $\mathbf{u}$ according to the rule[5]

$$\boxed{(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = \mathbf{u}(\mathbf{v} \cdot \mathbf{w})} \qquad \textbf{The Dyad Transformation} \qquad (1.8.3)$$

This relation defines the symbol "$\otimes$".

The length of the new vector is $|\mathbf{u}|$ times $\mathbf{v} \cdot \mathbf{w}$, and the new vector has the same direction as $\mathbf{u}$, Fig. 1.8.4. It can be seen that the dyad is a second order tensor, because it operates linearly on a vector to give another vector {▲Problem 2}.

Note that the dyad is not commutative, $\mathbf{u} \otimes \mathbf{v} \neq \mathbf{v} \otimes \mathbf{u}$. Indeed it can be seen clearly from the figure that $(\mathbf{u} \otimes \mathbf{v})\mathbf{w} \neq (\mathbf{v} \otimes \mathbf{u})\mathbf{w}$.

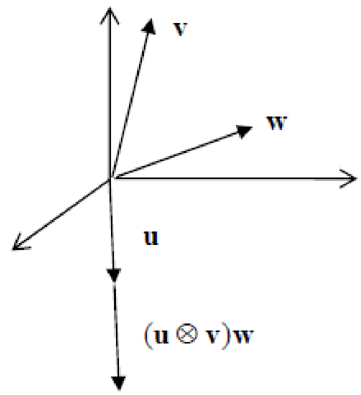


Figure 1.8.4: the dyad transformation

The following important relations follow from the above definition {▲Problem 4},

$$\begin{aligned}(\mathbf{u} \otimes \mathbf{v})(\mathbf{w} \otimes \mathbf{x}) &= (\mathbf{v} \cdot \mathbf{w})(\mathbf{u} \otimes \mathbf{x}) \\ \mathbf{u}(\mathbf{v} \otimes \mathbf{w}) &= (\mathbf{u} \cdot \mathbf{v})\mathbf{w}\end{aligned} \tag{1.8.4}$$

It can be seen from these that the operation of the dyad on a vector is not commutative:

$$\mathbf{u}(\mathbf{v} \otimes \mathbf{w}) \neq (\mathbf{v} \otimes \mathbf{w})\mathbf{u} \tag{1.8.5}$$

Example (The Projection Tensor)

Consider the dyad $\mathbf{e} \otimes \mathbf{e}$. From the definition 1.8.3, $(\mathbf{e} \otimes \mathbf{e})\mathbf{u} = (\mathbf{e} \cdot \mathbf{u})\mathbf{e}$. But $\mathbf{e} \cdot \mathbf{u}$ is the projection of **u** onto a line through the unit vector **e**. Thus $(\mathbf{e} \cdot \mathbf{u})\mathbf{e}$ is the vector projection of **u** on **e**. For this reason $\mathbf{e} \otimes \mathbf{e}$ is called the **projection tensor**. It is usually denoted by **P**.

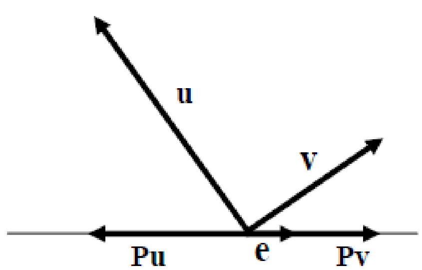


Figure 1.8.5: the projection tensor

$$(u \otimes v)\, w = u v^{T} w$$

$$= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} (v_1, v_2, v_3) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$= \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$= \begin{bmatrix} u_1 (v_1 w_1 + v_2 w_2 + v_3 w_3) \\ u_2 (v_1 w_1 + v_2 w_2 + v_3 w_3) \\ u_3 (v_1 w_1 + v_2 w_2 + v_3 w_3) \end{bmatrix}$$

$$= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \left(v_1 w_1 + v_2 w_2 + v_3 w_3 \right) = u (w \cdot v)$$

$\underline{U} = U_i$

Part 1: Instantaneous Equations: Focus DNS

Einstein summation convention: if an index occurs twice in a term a summation over the repeated index is implied

$$\sigma_{ij} = -\left(p + \frac{2}{3}\mu\nabla\cdot\underline{U}\right)\delta_{ij} + 2\mu\varepsilon_{ij}$$

$$\frac{\partial U_i}{\partial x_j} = e_{ij}{}^{1} = \underbrace{\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)}_{\varepsilon_{ij} = \varepsilon_{ji}} + \underbrace{\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)}_{\omega_{ij} = -\omega_{ji}}$$

Deformation | Strain | Rotation | Rate tensors

$$\underline{\omega} = \nabla\times\underline{U} = \underbrace{(w_y - v_z)}_{2\omega_{32}}\hat{\imath} + \underbrace{(u_z - w_x)}_{2\omega_{13}}\hat{\jmath} + \underbrace{(v_x - u_y)}_{2\omega_{21}}\hat{k}$$

Continuity

$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{U})$ in index notation $\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho U_i)$ (conservative form)

Using material derviative operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{U}\cdot\nabla$

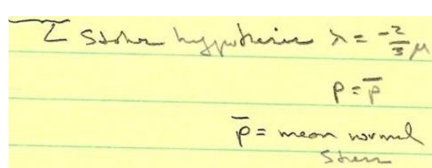
$$\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{U} = 0$$

Momentum

$$\rho\frac{D\underline{U}}{Dt} = \rho(\frac{\partial\underline{U}}{\partial t} + \underline{U}\cdot\nabla\underline{U}) = -\rho g\hat{k} + \nabla\cdot\sigma_{ij}$$

$$\rho\frac{DU_i}{Dt} = \rho(\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j})$$

$$= -\rho g\hat{k} - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu\varepsilon_{ij} - \frac{2}{3}\mu\frac{\partial U_m}{\partial x_m}\delta_{ij}\right)$$



[1] Deformation rate tensor is derived from the analysis of the relative motion between two neighboring fluid particles.

LHS in conservative form:

$$\rho\frac{\partial U_i}{\partial t} + \rho U_j\frac{\partial U_i}{\partial x_j} + U_i\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j)\right] = \frac{\partial(\rho U_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_i U_j) = U_i\frac{\partial \rho}{\partial t} + \rho\frac{\partial U_i}{\partial t} + \frac{\partial \rho}{\partial x_j}U_i U_j + \rho\frac{\partial U_i}{\partial x_j}U_j + \rho U_i\frac{\partial U_j}{\partial x_j}$$

Highlighted terms sum to zero, as per continuity equation.

In the context of fluid dynamics, the "conservative" form of the Navier-Stokes equations represents a mathematical formulation that explicitly expresses the conservation of mass and momentum within a control volume, while the "non-conservative" form does not, with the key difference lying in how the time derivative is calculated, where the conservative form uses a local derivative (fixed control volume) and the non-conservative form uses a substantial derivative (moving control volume).

Incompressible flow

$$\frac{\partial U_m}{\partial x_m} = \nabla\cdot\underline{U} = 0$$

$$\rho\frac{D\underline{U}}{Dt} = -\rho g\hat{k} - \nabla \mathrm{p} + \mu\nabla^2\underline{U} = -\nabla\hat{\mathrm{p}} + \mu\nabla^2\underline{U}$$

Where

$$2\frac{\partial}{\partial x_j}\varepsilon_{ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) = \frac{\partial^2 U_i}{\partial x_j \partial x_j} = \nabla^2 U_i = \nabla^2\underline{U}$$

$\hat{\mathrm{p}} = \mathrm{p} + \rho g z$ =piezometric pressure

Mechanical energy equation

$$U_i\left[\rho\frac{\mathrm{D}U_i}{Dt} = \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j}\right]$$

$$\rho\frac{D\left(\frac{1}{2}\mathrm{U}_i{}^2\right)}{Dt} = \rho \mathrm{U}_i g_i + \mathrm{U}_i\frac{\partial \sigma_{ij}}{\partial x_j}$$

Rate of increase KE	Rate of work done by body force	Rate of work done by net surface force $\nabla\cdot\sigma_{ij}$

$U_i^2 = \mathrm{U}_i\mathrm{U}_i = U_1^2 + U_2^2 + U_3^2$

Consider:

$$\frac{\partial}{\partial x_j}\left(U_i\sigma_{ij}\right) \quad = \quad \sigma_{ij}\frac{\partial U_i}{\partial x_j} \quad + \quad U_i\frac{\partial \sigma_{ij}}{\partial x_j}$$

Total work of surface force	Deformation work w/o $\underline{a}$ and lost to internal energy	Increase of KE since contributes fluid $\underline{a}$

$$\sigma_{ij}\frac{\partial U_i}{\partial x_j} = \sigma_{ij}\left(\varepsilon_{ij}+\omega_{ij}\right) = \sigma_{ij}\varepsilon_{ij}$$

$\sigma_{ij}\omega_{ij} = 0$ since it is the product of a symmetric and an anti-symmetric tensor.

$$\sigma_{ij}\frac{\partial U_i}{\partial x_j} = \left[-\left(p+\frac{2}{3}\mu\nabla\cdot\underline{U}\right)\delta_{ij}+2\mu\varepsilon_{ij}\right]\varepsilon_{ij}$$

$$\sigma_{ij}\frac{\partial U_i}{\partial x_j} = -p\nabla\cdot\underline{U} + \underbrace{2\mu\varepsilon_{ij}\varepsilon_{ij} - \frac{2}{3}\mu\left(\nabla\cdot\underline{U}\right)^2}_{\varphi}$$

Since $\varepsilon_{ij}\delta_{ij} = \varepsilon_{ii} = \nabla\cdot\underline{U}$

$$\sigma_{ij}\frac{\partial U_i}{\partial x_j} = -p\nabla\cdot\underline{U} + \varphi$$

$$\rho\frac{D\left(\frac{1}{2}\mathrm{U}_i{}^2\right)}{Dt} = \rho\mathrm{U}_i g_i + \frac{\partial\left(\mathrm{U}_i\sigma_{ij}\right)}{\partial x_j} - \sigma_{ij}\varepsilon_{ij}$$

$$\rho\frac{D\left(\frac{1}{2}\mathrm{U}_i{}^2\right)}{Dt} \quad = \quad \rho\underline{g}\cdot\underline{U} \quad + \quad \frac{\partial\left(\mathrm{U}_i\sigma_{ij}\right)}{\partial x_j} \quad + \quad p\nabla\cdot\underline{U} \quad - \quad \varphi$$

$\rho\frac{D}{Dt}\left(\frac{1}{2}\mathrm{U}_i{}^2\right) = \rho[\frac{\partial\left(\frac{1}{2}\mathrm{U}_i{}^2\right)}{\partial t} + U_j\frac{\partial\left(\frac{1}{2}\mathrm{U}_i{}^2\right)}{\partial x_j}]$	Rate of work done by body force $\underline{g}$	Total rate of work done σ_{ij}	Rate of work due to volume expansion; converts mechanical energy to internal energy and vice-versa	Rate of viscous dissipation

$\varphi \geq 0$ = loss of mechanical energy = gain of internal energy due to the deformation of the fluid particle

$-\sigma_{ij}\varepsilon_{ij} = p\nabla \cdot \underline{U} - \varphi$ =total rate of deformation work

$p\nabla \cdot \underline{U}$= reversible part

φ = irreversbile part = rate of viscous dissipation of KE per unit volume. φ α μ and φ α $(velocity\ gradients)^2$ and important in regions of high shear with outcomes, e.g. , hot lubrificant in bearings and burning surfaces on re-entry of the atmosphere for spacecraft.

Energy equation

$\frac{\dot{Q}}{\forall} = \underline{q}$ = heat (conduction/radiation) added to MV

$\frac{\dot{W}}{\forall} = \dot{w}$ = work done by MV

$$\rho\frac{De}{Dt} = \dot{q} - \dot{w} = \dot{Q}/\forall - \dot{W}/\forall$$

$$\mathrm{e} = \hat{u} + \frac{1}{2}U^2 + gz = \hat{u} + \frac{1}{2}\underline{U}\cdot\underline{U} - \underline{g}\cdot\underline{r}$$

$$\dot{q} = -\nabla\cdot\underline{q} = -\nabla\cdot(-\mathrm{k}\nabla\mathrm{T}) = \nabla\cdot(\mathrm{k}\nabla\mathrm{T})$$

$$\dot{w} = -\nabla\cdot(\underline{U}\cdot\sigma_{ij}) = \underbrace{-\underline{U}\cdot(\nabla\cdot\sigma_{ij})}_{-\rho\left(\underline{U}\cdot\frac{D\underline{U}}{Dt} - \underline{U}\cdot\underline{g}\right)} - \sigma_{ij}\frac{\partial U_i}{\partial x_j}$$

$\underline{g} = -g\hat{k}$

$\underline{q} = -k\nabla T$ heat flux vector

$-\nabla\cdot\underline{q}$ since + sign for heat added

$$\frac{D(-\underline{g}\cdot\underline{r})}{Dt} = -\underline{g}\cdot\frac{D\underline{r}}{Dt} = -\underline{g}\cdot\underline{U}$$

$$\underline{U}\cdot\frac{D\underline{U}}{Dt} = \underline{U}\cdot\left(\frac{\partial\underline{U}}{\partial t} + \underline{U}\cdot\nabla\underline{U}\right) = \frac{1}{2}\frac{\partial U^2}{\partial t} + \frac{1}{2}\underline{U}\cdot\nabla U^2 = \frac{1}{2}\frac{DU^2}{Dt} = U\frac{DU}{Dt}$$

$$\rho\left[\frac{D\hat{\mathrm{u}}}{Dt} + \cancel{\underline{U}\cdot\frac{D\underline{U}}{Dt}} - \cancel{\underline{U}\cdot\underline{g}}\right] = \nabla\cdot(\mathrm{k}\nabla\mathrm{T}) + \rho\left(\cancel{\underline{U}\cdot\frac{D\underline{U}}{Dt}} - \cancel{\underline{U}\cdot\underline{g}}\right) + \sigma_{ij}\frac{\partial U_i}{\partial x_j}$$

$$\rho\frac{D\hat{\mathrm{u}}}{Dt} = \nabla\cdot(\mathrm{k}\nabla\mathrm{T}) + \sigma_{ij}\frac{\partial U_i}{\partial x_j}$$

$$\rho\frac{D\hat{\mathrm{u}}}{Dt} = \nabla\cdot(\mathrm{k}\nabla\mathrm{T}) \underbrace{-p\nabla\cdot\underline{U} + \varphi}$$

Same terms mechanical energy equation with change of sign!

Notice that:

$$-p\nabla\cdot\underline{U} = \frac{\mathrm{p}}{\rho}\frac{D\rho}{Dt} = \frac{D\mathrm{p}}{Dt} - \rho\frac{D}{Dt}\left(\frac{\mathrm{p}}{\rho}\right)$$

So an alternative formulation is:

$$\rho\frac{D}{Dt}\Bigl(\underbrace{\hat{\mathrm{u}} + \frac{\mathrm{p}}{\rho}}_{\mathrm{h}}\Bigr) = \nabla\cdot(\mathrm{k}\nabla\mathrm{T}) + \frac{D\mathrm{p}}{Dt} + \varphi$$

Vorticity equation

Start from NS:

$$\frac{\partial \underline{U}}{\partial t} + \underline{U} \cdot \nabla \underline{U} = -\frac{\nabla \hat{\mathrm{p}}}{\rho} + \nu \nabla^2 \underline{U}$$

Using the identity: $\underline{U} \cdot \nabla \underline{U} = \nabla \left(\frac{1}{2} \underline{U} \cdot \underline{U}\right) - \underline{U} \times \underline{\omega}$ and taking the curl of NS:

$$\frac{D\underline{\omega}}{Dt} = \quad \underline{\omega} \cdot \nabla \underline{U} \quad + \quad \nu \nabla^2 \underline{\omega}$$

Rate of deforming vortex lines

Rate of viscous diffusion

Enstrophy equation

Enstrophy definition:

$$\frac{\omega^2}{2} = \frac{\underline{\omega} \cdot \underline{\omega}}{2}$$

Intensification $\underline{\omega}$ by stretching with similarity mechanical enragy equation as can be destroyed by viscosity μ. Multiply vorticity equation by $\underline{\omega}$:

$$\underline{\omega} \cdot \left[\frac{D\underline{\omega}}{Dt} = \underline{\omega} \cdot \nabla \underline{U} + \nu \nabla^2 \underline{\omega}\right]$$

$$\frac{D}{Dt}\left(\frac{\omega^2}{2}\right) = \underbrace{\omega_i \omega_j \varepsilon_{ij}} - \underbrace{\nu \left(\nabla \times \underline{\omega}\right)^2} + \nu \nabla \cdot \left[\underline{\omega} \times \left(\nabla \times \underline{\omega}\right)\right]$$

Generation/ reduction due to vortex stretching

Destruction/ dissipation due to μ

$\int = 0$ for localized disturbance and often not important

Derivation of Eqs. 3.35 to 3.36 in Davidson's book

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{u} + \nu\nabla^2\vec{\omega} \Rightarrow$$

$$\vec{\omega}\cdot\frac{D\vec{\omega}}{Dt} = \vec{\omega}\cdot[(\vec{\omega}\cdot\nabla)\vec{u} + \nu\nabla^2\vec{\omega}] \Rightarrow$$

$$\frac{D}{Dt}\left(\frac{\overrightarrow{\omega^2}}{2}\right) = \underbrace{\vec{\omega}\cdot(\vec{\omega}\cdot\nabla)\vec{u}}_{(1)} + \underbrace{\vec{\omega}\cdot\nu\nabla^2\vec{\omega}}_{(2)}$$

First term (1): $\omega_i\cdot\left(\omega_j\frac{\partial u_i}{\partial x_j}\right) = \omega_i\omega_j\frac{\partial u_i}{\partial x_j} = \omega_i\omega_j(S_{ij} - \frac{1}{2}\epsilon_{ijk}\omega_k) = \omega_i\omega_j S_{ij}$

Second term (2): $\nu\vec{\omega}\cdot\nabla^2\vec{\omega}$

Using the identity for the curl of curl:

$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} \Rightarrow \nabla^2\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla\times(\nabla\times\vec{A})$$

$$\nu\vec{\omega}\cdot\nabla^2\vec{\omega} = \nu\vec{\omega}\cdot\left[\underbrace{\nabla(\nabla\cdot\vec{\omega})}_{=0} - \nabla\times(\nabla\times\vec{\omega})\right] = -\nu\underbrace{\vec{\omega}}_{a}\cdot\left[\nabla\times\underbrace{(\nabla\times\vec{\omega})}_{b}\right]$$

Using the vector identity:

$$\nabla\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{b}) \Rightarrow \vec{a}\cdot(\nabla\times\vec{b}) = \vec{b}\cdot(\nabla\times\vec{a}) - \nabla\cdot(\vec{a}\times\vec{b})$$

$$-\nu\underbrace{\vec{\omega}}_{a}\cdot\left[\nabla\times\underbrace{(\nabla\times\vec{\omega})}_{b}\right] = -\nu(\nabla\times\vec{\omega})^2 + \nu\nabla\cdot[\vec{\omega}\times(\nabla\times\vec{\omega})]$$

Final expression:

$$\frac{D}{Dt}\left(\frac{\overrightarrow{\omega^2}}{2}\right) = \omega_i\omega_j S_{ij} - \nu(\nabla\times\vec{\omega})^2 + \nu\nabla\cdot[\vec{\omega}\times(\nabla\times\vec{\omega})]$$

Exe. 2. 10 of Pope's Book:

$$\frac{D\omega^2}{Dt} = \underbrace{\nu\nabla^2\omega^2}_{(1)} + 2\omega_i\omega_j\frac{\partial u_i}{\partial x_j} \underbrace{- 2\nu\frac{\partial\omega_i}{\partial x_j}\frac{\partial\omega_i}{\partial x_j}}_{(2)}$$

$$\frac{Terms\ (1)+(2)}{2} = \frac{1}{2}\nu\nabla^2\omega^2 - \nu\frac{\partial\omega_i}{\partial x_j}\frac{\partial\omega_i}{\partial x_j} = \frac{1}{2}\nu\nabla\cdot(\nabla\omega^2) - \nu(\nabla\omega)\cdot(\nabla\omega)$$
$$= \nu\nabla\cdot(\omega\cdot\nabla\omega) - \nu(\nabla\omega)\cdot(\nabla\omega) = \nu\nabla(\omega)\cdot(\nabla\omega) + \nu\omega\cdot\nabla^2\omega - \nu(\nabla\omega)\cdot(\nabla\omega)$$
$$= \nu\omega\cdot\nabla^2$$

Turbulent Flows
Stephen B. Pope
Cambridge University Press (2000)

Solution to Exercise 2.10

Prepared by: Mark Fogleman *Date:* 1/27/03

The enstrophy equation can be found by dotting the vorticity equation, Eq.(2.60), written in index notation with vorticity:

$$\omega_i \frac{\partial \omega_i}{\partial t} + U_j \omega_i \frac{\partial \omega_i}{\partial x_j} = \nu \omega_i \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} + \omega_i \omega_j \frac{\partial U_i}{\partial x_j}. \qquad (1)$$

Integrating the first three terms by parts gives

$$\frac{\partial \frac{1}{2}\omega_i \omega_i}{\partial t} + U_j \frac{\partial \frac{1}{2}\omega_i \omega_i}{\partial x_j} = \nu \frac{\partial}{\partial x_j}(\omega_i \frac{\partial \omega_i}{\partial x_j}) - \nu \frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j} + \omega_i \omega_j \frac{\partial U_i}{\partial x_j}. \qquad (2)$$

The third term can be integrated by parts once more to yield

$$\frac{\partial \frac{1}{2}\omega_i \omega_i}{\partial t} + U_j \frac{\partial \frac{1}{2}\omega_i \omega_i}{\partial x_j} = \frac{1}{2}\nu \frac{\partial^2}{\partial x_j \partial x_j}(\omega_i \omega_i) - \nu \frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j} + \omega_i \omega_j \frac{\partial U_i}{\partial x_j}. \qquad (3)$$

Multipyling through and converting some terms back to vector notation gives the final result:

$$\frac{D\omega^2}{Dt} = \nu \nabla^2 \omega^2 + 2\omega_i \omega_j \frac{\partial U_i}{\partial x_j} - 2\nu \frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j}. \qquad (4)$$

Pressure equation

Obtained by taking the divergence of NS equation:

$$\nabla^2\left(\frac{\mathrm{p}}{\rho}\right) = -\nabla\cdot\left(\underline{U}\cdot\nabla\underline{U}\right) = -\frac{\partial U_k}{\partial x_j}\frac{\partial U_j}{\partial x_k}$$

Pressure Poisson equation

$g \downarrow \uparrow z$

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + g_i + \nu\frac{\partial^2 u_i}{\partial x_j^2} \qquad g_i = \underline{g} = -g\hat{k}$$

$$\frac{\partial}{\partial x_i}(NS): \quad \cancel{\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_i}\right)} + \frac{\partial}{\partial x_i}\left(u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2} + \frac{\partial g_i}{\partial x_i} + \nu\frac{\partial^2}{\partial x_j^2}\cancel{\left(\frac{\partial u_i}{\partial x_i}\right)}$$

$$\cancel{u_j\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_i}} + \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i^2} + \frac{\partial g_i}{\partial x_i}$$

$$\frac{\partial^2 p}{\partial x_i^2} = -\rho\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} + \rho\frac{\partial g_i}{\partial x_i}$$

Interestingly, μ not in equation; however, first term RHS is $f(\mu)$

Additional considerations ME equation

Other forms are obtained by adding continuity equation

$$\rho \frac{D}{Dt}\left(\frac{1}{2}u_i^2\right) = \rho u_i g_i + u_i \frac{\partial}{\partial x_j}\sigma_{ij} \qquad u_i^2 = u_i u_i = u_1^2 + u_2^2 + u_3^2$$

$$\rho\left[\frac{\partial}{\partial t}\left(\frac{1}{2}u_i^2\right) + u_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}u_i^2\right)\right]$$

$$+ \underbrace{\frac{1}{2}\rho u_i^2\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right]}_{\text{from continuity} = 0} = \rho u_i g_i + u_i \frac{\partial}{\partial x_j}\sigma_{ij}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_i^2\right) + \frac{\partial}{\partial x_j}\left[u_j \frac{1}{2}\rho u_i^2\right] = \rho u_i g_i + u_i \frac{\partial}{\partial x_j}\sigma_{ij}$$

Weak derivation

$E = KE/\forall = \frac{1}{2}\rho u_i^2$ KE per unit volume

$$\frac{\partial E}{\partial t} + \nabla\cdot(\underline{u}E) = \rho\underline{u}\cdot\underline{g} + \underline{u}\cdot(\nabla\cdot\sigma_{ij})$$

Weak derivation — divergence of KE flux $\underline{u}E$

Flux divergence terms often arise energy balances

Net loss due divergence flux, e.g., RHS source terms = 0 then E_t ↑ if $\nabla\cdot(\underline{u}E) < 0$. Also called transport terms since

$$\int_\forall \nabla\cdot(\underline{u}E)\,d\forall = \int_A E\underline{u}\cdot d\underline{A}$$

transfer E w/o net contribution

$= 0$ if $\underline{u}E = 0$ at boundary

Equation for a fixed region: integral equation

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}[u_j E] = \rho\underline{g}\cdot\underline{u} + \frac{\partial}{\partial x_j}(u_i\sigma_{ij}) + p\nabla\cdot\underline{u} - \Phi$$

Weak derivation $i = j$ LHS?

flux divergent form — $\int_\forall \to \int_A$ via Gauss theorem

$\int\forall$

$$\frac{d}{dt}\int E\,d\forall + \int E\underline{u}\cdot d\underline{A} = \int \rho\underline{g}\cdot\underline{u}\,d\forall + \int u_i\sigma_{ij}\,dA_j + \int p(\nabla\cdot\underline{u})\,d\forall - \int \Phi\,d\forall$$

rate of:	change KE	outflow across A	work body force	work surface force @ A	work volume expansion	viscous dissipation KE

$\sigma_{ij}\,dA_j$ = force i direction

$u_i\sigma_{ij}\,dA_j$ = scalar product force & u_i

ME equation in terms of potential energy

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u_i^2\right) = \rho \underline{g} \cdot \underline{v} + \frac{\partial}{\partial x_j}(u_i \sigma_{ij}) + p \nabla \cdot \underline{v} - \epsilon$$

can be moved LHS as change in PE or RHS = work body force $\underline{g}$ on fluid particle

$$u_i g_i = -u_i \frac{\partial}{\partial x_i}(gz) = -\frac{D}{Dt}(gz) \qquad \frac{\partial}{\partial t}(gz) = 0$$

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u_i^2 + gz\right) = \frac{\partial}{\partial x_j}(u_i \sigma_{ij}) + p(\nabla \cdot \underline{v}) - \epsilon$$

$\Pi = gz = PE$ per unit mass

Davidson 2.1.4.

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u_i^2\right) = \rho u_i g_i + u_i \frac{\partial}{\partial x_j}\sigma_{ij}$$

$$\sigma_{ij} = -p\delta_{ij} + 2\mu \varepsilon_{ij} \qquad \rho, \mu = \text{constant property flow}$$

$$= -p\delta_{ij} + \tau_{ij} \qquad \tau_{ij} = 2\mu\varepsilon_{ij} = \text{viscous stress}$$

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u_i^2\right) = \rho u_i g_i - u_i\frac{\partial p}{\partial x_i} + u_i \frac{\partial \tau_{ij}}{\partial x_j} = u_i\left(\rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}\right)$$

$$u_i \frac{\partial \tau_{ij}}{\partial x_j} = u_i \frac{\partial}{\partial x_j}\left[\mu(u_{i,j} + u_{j,i})\right] = \mu u_i\left[\frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right) + \frac{\partial^2 u_i}{\partial x_j^2}\right]$$

$$= \mu u_i \frac{\partial^2 u_i}{\partial x_j^2}$$

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u_i^2\right) = \rho u_i g_i - u_i\frac{\partial p}{\partial x_i} + u_i \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

$$= \underline{u}\cdot(\rho \underline{g} - \nabla p + \mu \nabla^2 \underline{u})$$

$$\underline{u}\cdot\nabla^2\underline{u} = \underline{u}\cdot(-\nabla\times(\nabla\times\underline{u})) = -\underline{u}\cdot\nabla\times\underline{\omega}$$

$$= \nabla\cdot(\underline{u}\times\underline{\omega}) - \underline{\omega}\cdot(\nabla\times\underline{u})$$

$$= -(\nabla\times\underline{u})^2 + \nabla\cdot[\underline{u}\times(\nabla\times\underline{u})]$$

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2}u_i^2\right) = -u_i\frac{\partial \hat{p}}{\partial x_i} + \mu\left[-(\nabla\times\underline{u})^2 + \nabla\cdot(\underline{u}\times\underline{\omega})\right]$$

$$\underline{\omega}\cdot\underline{\omega} = (\nabla\times\underline{u})^2 = \omega^2 \Rightarrow \omega^2/2 = \text{enstrophy}$$

Needs derivation or

$$\frac{\partial}{\partial t}\left(\frac{u_i^2}{2}\right) = -\nabla\cdot\left[\left(u_i^2/2 + \hat{p}/\rho\right)\underline{u} + \nu(\underline{\omega}\times\underline{u})\right] - \nu(\nabla\times\underline{u})^2$$

For closed domain + with stationary boundaries, ie, no flux across boundary

$$\frac{d}{dt}\int\left(\frac{u_i^2}{2}\right)dV = -\nu\int(\nabla\times\underline{u})^2\,dV$$

Total rate of dissipation ME

ie $\int \varepsilon\, dV = 2\nu\int \varepsilon_{ij}\varepsilon_{ij}\, dV = \int(\nabla\times\underline{u})^2\, dV$

Alternate derivation vorticity transport equation

$$\frac{\partial \underline{U}}{\partial t} + \underline{U} \cdot \nabla \underline{U} = -\frac{\nabla \hat{p}}{\rho} + \nu \nabla^2 \underline{U}$$

Using the identities:

$$\underline{U} \cdot \nabla \underline{U} = \nabla \left(\frac{1}{2} \underline{U} \cdot \underline{U}\right) - \underline{U} \times \underline{\omega}$$

$\mathrm{K} = \frac{1}{2} \underline{U} \cdot \underline{U}$

$$\nabla^2 \underline{U} = \nabla(\nabla \cdot \underline{U}) - \nabla \times (\nabla \times \underline{U}) = -\nabla \times \underline{\omega}$$

Viscous force directly related to $\underline{\omega}$, i.e., existance $\underline{\omega}$ implies viscous forces

Therefore:

$$\frac{\partial \underline{U}}{\partial t} + \nabla \mathrm{K} - \underline{U} \times \underline{\omega} = -\frac{\nabla \hat{p}}{\rho} - \nu \nabla \times \underline{\omega}$$

$$\frac{\partial \underline{U}}{\partial t} - \underline{U} \times \underline{\omega} + \nabla \left(\underbrace{\mathrm{K} + \frac{\hat{p}}{\rho}}\right) = -\nu \nabla \times \underline{\omega}$$

Stokes form NS

Bernoulli equation for steady inviscid irrotational flow

Curl Stokes form NS: **Helmholtz vorticity equation.**

$$\nabla \times (\underline{U} \times \underline{\omega}) = \underline{U}(\cancel{\nabla \cdot \underline{\omega}}) + \underline{\omega} \cdot \nabla \underline{U} - \underline{\omega}(\cancel{\nabla \cdot \underline{U}}) - \underline{U} \cdot \nabla \underline{\omega} = \underline{\omega} \cdot \nabla \underline{U} - \underline{U} \cdot \nabla \underline{\omega}$$

$$\nabla \times (\nabla \times \underline{\omega}) = \nabla(\cancel{\nabla \cdot \underline{\omega}}) - \nabla^2 \underline{\omega}$$

$$\frac{\partial \underline{\omega}}{\partial t} + \underline{U} \cdot \nabla \underline{\omega} = \underbrace{\underline{\omega} \cdot \nabla \underline{U}} + \underbrace{\nu \nabla^2 \underline{\omega}} = \frac{D\underline{\omega}}{Dt} =$$

Rate of change following fluid particle

Vortex stretching /turning

Viscous diffusion