Chapter 2 Describing Turbulence: Averages, Correlations and Spectra

2.1 Navier-Stokes Equation and Reynolds Number

Spatial and temporal velocity and pressure fields obey the Navier-Stokes equations herewith of interest for incompressible flow:

$$\nabla \cdot \underline{u} = \frac{\partial u_i}{\partial x_i} = 0$$

$$\rho \frac{D\underline{u}}{Dt} = \rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right) = \rho \underline{g} - \nabla p + \mu \nabla^2 \underline{u}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \rho g_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

$$\underline{u}(\underline{x}, t) = (u_1, u_2, u_3) = (u, v, w) \qquad \underline{x} = (x_1, x_2, x_3) = (x, y, z)$$

$$p = p(\underline{x}, t)$$

Inertia = body force due gravity – pressure gradient force + viscous force, which is diffusive. The body force due to gravity may be absorbed into the pressure gradient force via use of piezometric pressure $\hat{p} = p + \gamma z$. The pressure force tends to adapt to whichever of the other terms is dominant and need not be considered explicitly when considering the relative magnitude of the forces.

Inertia includes $\underline{U} \cdot \nabla \underline{U}$ nonlinear mechanism by which perturbations may self-reinforce.

Viscous diffusion derives from molecular momentum diffusion: for non-dense gases due to molecular collisions, whereas for liquids due to local intermolecular cohesion arising from their proximity.

Diffusion smooths flow perturbations, whereas nonlinear inertia may strengthen them such that their effects are often in conflict.

$\underline{x}^* = \underline{x}/L \qquad \underline{U}^* = \underline{U}/U$

$\mu \frac{\partial^2 U_i}{\partial x_i^2} = \mu U/L^2 \qquad \rho U_i \frac{\partial U_i}{\partial x_i} = \rho U^2/L$

$\mu \frac{\partial^2 U_i}{\partial x_i^2} / U_i \frac{\partial U_i}{\partial x_i} = \mu \frac{U}{L^2} / \rho U^2/L$

$= \frac{1}{Re} \qquad Re = UL/\nu$

$U = m/s$

$x = m \qquad \nu = \mu/\rho$

$\mu = \frac{Ns}{m^2} = \frac{kg}{ms}$

$\rho = kg/m^3$

$\nu = m^2/s$

Small Re viscous diffusion dominates, whereas for large Re inertia dominates, i.e., Re most important parameter distinguish stability of transition to turbulence

Recall viscous diffusion layers:

Viscous layers:

Sudden acceleration flat plate: $u_t = \nu u_{yy}$

$$\delta_{99} = 3.64\sqrt{\nu t}$$

$$u(y,0) = 0$$
$$u(0,t) = U$$
$$u(\infty,t) = 0$$

Layer grows in time due viscous diffusion

Oscillating flat plate: $u_t = \upsilon u_{yy}$

$$\delta_{99} = 6.5\sqrt{\nu/\omega}$$

$$u(0,t) = U_0 \cos \omega t$$
$$u(\infty,t) = 0$$

Layer confined constant thickness

Stagnation point flow: $\delta_{99} = 2.4\sqrt{\nu/B}$ layer not a function of x since convection balances diffusion

Flat plate boundary layer:

$$u_x + v_y = 0$$
$$uu_x + vu_y = \upsilon u_{yy}$$
$$u(x,0) = 0$$
$$u(x,\infty) = U$$

$$\delta_{99} = 4.9\sqrt{\nu x/U}$$

Layer grows with x due convection

Ekman Layer on Free Surface: effects due to wind shear

$\underline{\tau} = \tau\hat{i} = .002\rho_{air}\left(v_{wind} - u(0)\right)^2 \hat{i}$

$$\delta = \sqrt{\frac{2\nu}{f}} = \text{Ekman layer thickness}$$

Coriolis force = $2\underline{\Omega} \times \underline{V} = -fv\hat{i} + fu\,\hat{j} - 2\Omega\cos\theta\,u\,\hat{k}$

$f = 2\Omega\sin\theta$ = planetary vorticity = 2*vertical component Ω

$$-fv = \nu u_{ZZ} \quad fu = \nu v_{ZZ}$$

$\mu u_z = \tau$ at z = 0

$v_z = 0$ at z = 0

$(u, v) = 0$ at z = $-\infty$

The Ekman layer thickness is constant in time and space since vortex diffusion balances the Coriolis force.

2.2 Averaging

Averages:

For turbulent flow $\underline{V}(\underline{x}, t)$, $p(\underline{x}, t)$ are random functions of time and must be evaluated statistically using averaging techniques: time, ensemble, phase, or conditional.

Time Averaging

For stationary flow, the mean is not a function of time and we can use time averaging.

$$\bar{u} = \frac{1}{T} \int_{t_0}^{t_0+t} u(t)\, dt$$ T > any significant period of $u' = u - \bar{u}$

(e.g. 1 sec. for wind tunnel and 20 min. for ocean)

Statistically stationary process = the statistical properties, such as mean, variance and autocorrelation, do not change over time.

Ensemble Averaging

For non-stationary flow, the mean is a function of time and ensemble averaging is used

$$\bar{u}(t) = \frac{1}{N} \sum_{i=1}^{N} u^i(t)$$ N is large enough that $\bar{u}$ independent

$u^i(t) =$ collection of experiments performed under identical conditions (also can be phase aligned for same t=o).

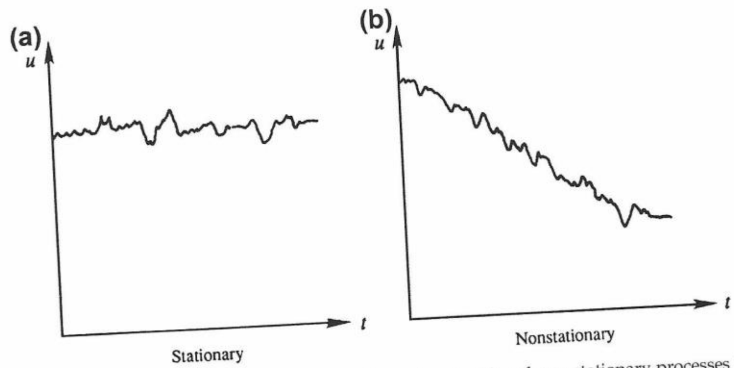

FIGURE 12.2 Sample time series indicating temporally stationary (a) and non-stationary processes (b). The time series in (b) clearly shows that the average value of u decreases with time compared to the time series in (a).

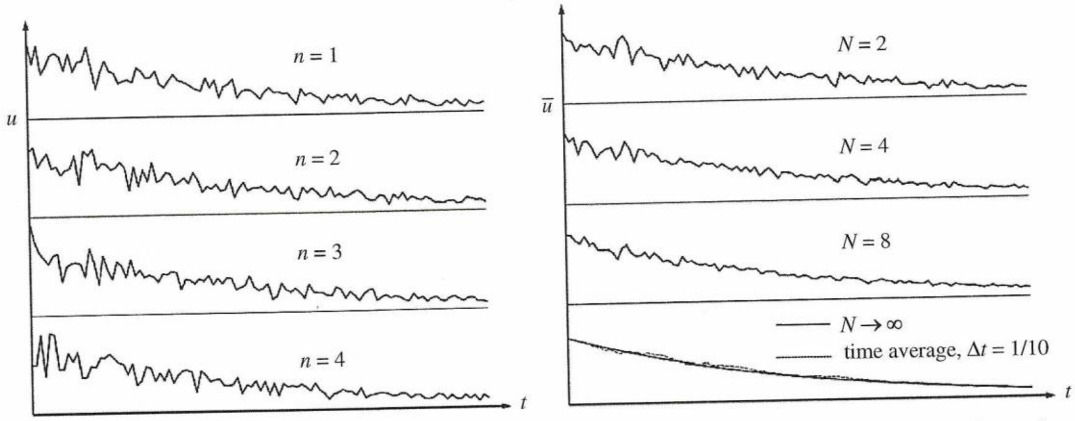

FIGURE 12.3 Illustration of ensemble and temporal averaging. The left panel shows four members of an ensemble of time series for the decaying random variable u. In all four cases, the fluctuations are different but the decreasing trend with increasing t is clearly apparent in each. The right panel shows averages of two, four, and eight members of the ensemble in the upper three plots. As the sample number N increases, fluctuations in the ensemble average decreases. The lowest plot on the right shows the $N \to \infty$ curve – this is the expected value of $u(t)$ – and a simple sliding time average of the $n = 4$ curve where the duration of the time average is one-tenth of the time period shown. In this case, time and ensemble averaging produce nearly the same curve.

Averaging Rules:

$f = \overline{f} + f'$ $\qquad g = \overline{g} + g'$ $\qquad s = x \text{ or } t$

$\overline{f'} = 0$ $\qquad \overline{\overline{f}} = \overline{f}$ $\qquad \overline{\overline{f}g} = \overline{f}\,\overline{g}$ $\qquad \overline{f'\overline{g}} = 0$

$\overline{f + g} = \overline{f} + \overline{g}$ $\qquad \overline{\frac{\partial f}{\partial s}} = \frac{\partial \overline{f}}{\partial s}$ $\qquad \overline{fg} = \overline{f}\,\overline{g} + \overline{f'g'}$

$\overline{\int f\, ds} = \int \overline{f}\, ds$

Phase and Conditional Averaging

Similar to ensemble averaging, but for flows with dominant frequency content or other condition, which is used to align time series for some phase/condition. In this case triple velocity decomposition is used: $u = \bar{u} + u'' + u'$ where u'' is called organized oscillation. Phase/conditional averaging extracts all three components.

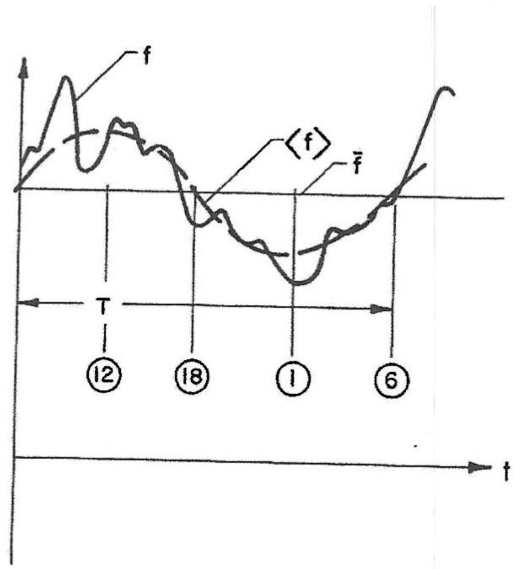

Fig. 6.1 Schematic representation of the triple decomposition: ——, instantaneous signal; ---, conditionally averaged; ——, mean. The numerals in circles characterize the phase for later reference.

$$\langle f' \rangle = 0, \qquad \overline{f''} = 0, \qquad \overline{f'} = 0, \tag{6.2.7}$$

$$\overline{\langle f \rangle} = \bar{f}, \qquad \langle \bar{f} \rangle = \bar{f}, \tag{6.2.8}$$

$$\overline{\bar{f} g} = \bar{f}\bar{g}, \qquad \langle f'' g \rangle = f'' \langle g \rangle, \qquad \langle \bar{f} g \rangle = \bar{f} \langle g \rangle, \tag{6.2.9}$$

$$\overline{f'' g'} = \overline{\langle f'' g' \rangle} = 0. \tag{6.2.10}$$

Acharya, M., and Reynolds, W. C. 1975. "Measurements and Predictions of a Fully Developed Turbulent Channel Flow with Imposed Controlled Oscillations," Stanford University Technical Report Number TF-8.

In some situations, particularly in numerical work, it is convenient to use spatial averaging to get flow statistics. For a given volume of fluid, $\mathcal{V}$, this is defined in an obvious way via

$$\overline{\mathbf{U}}(\mathbf{x}, t) = \frac{1}{\mathcal{V}} \int_{\mathcal{V}} \mathbf{U}(\mathbf{x}', t) d\mathbf{x}', \tag{2.12}$$

where $\mathcal{V}$ is generally taken to be centered around the point $\mathbf{x}$. Volume averaging is most appropriate if the turbulent flow is *homogeneous* over the averaging region in the sense that mean statistics do not vary over this domain. In some cases, particularly flows with geometrical symmetries, averaging may be taken along particular lines or within surfaces in the flow field over which the velocity field has a uniform mean behavior. For example, Eq. (2.12) is appropriate to use in a channel flow with $\mathcal{V}$ taken to be planes parallel to the bounding surfaces since mean statistics do not vary over such regions. Averaging over planes in this case is equivalent to averaging over many experiments and provides a convenient means of getting rapidly converged statistics. Similar to the case of time averaging, spatial averaging is not generally appropriate for flows with non-uniform turbulence properties, that is, *non-homogeneous turbulence.* In particular, the average of terms containing spatial derivatives in the directions over which averaging is implemented cannot be computed accurately. In the example of a channel flow spatial averaging over planes parallel to the boundaries is legitimate because the non-homogeneity of the mean flow in this instance is in the direction normal to the averaging planes.

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Impulsive plunging wave breaking downstream of a bump in a shallow water flume—Part I: Experimental observations

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ABSTRACT

The plunging wave-breaking process for impulsive flow over a bump in a shallow water flume is described, which is relevant to ship hydrodynamics albeit for an idealized geometry since it includes the effects of wave–body interactions and the wave breaking direction is opposite to the mean flow. This paper consists of two parts, which deal with experimental measurements and numerical simulations, respectively. In Part I, ensemble-averaged measurements are conducted, including the overall flume flow, 2-D particle image velocimetry (PIV) center-plane velocities, turbulence inside the breaking wave, and bottom pressures under the breaking wave. A series of individual plunging wave-breaking tests were conducted, which all followed a similar time line consisting of startup, steep wave formation, plunging wave breaking, and chaotic wave breaking swept downstream time phases. The plunging wave breaking process consists of four repeated plunging events each with three [jet impact (plunge), oblique splash and vertical jet] sub-events, which were identified first using a complementary computational fluid dynamics (CFD) study. Video images with red dye display the plunging wave breaking events and sub-events. The wave profile at maximum height, first plunge, bump and wave breaking vortex and entrapped air tube trajectories, entrapped air tube diameters, kinetic, potential, and total energy are analyzed. Similarities and differences are discussed with the previous deep water or sloping beach experimental and computational studies. The numerical simulations using the exact experimental initial and boundary conditions are presented in Part II of this paper.

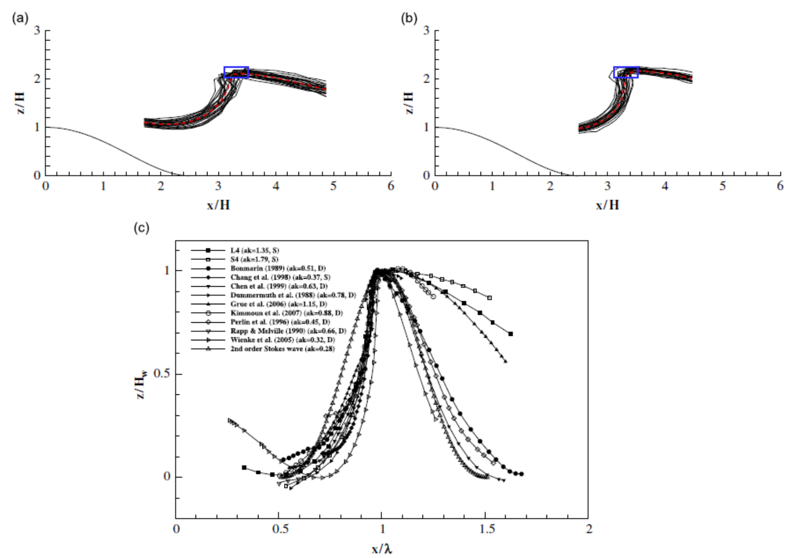

Fig. 5. Wave breaking profiles: (a) L4; (b) S4; and (c) non-dimensional with wave length (λ) and wave height (H_w) at t_b in x and z directions, respectively.

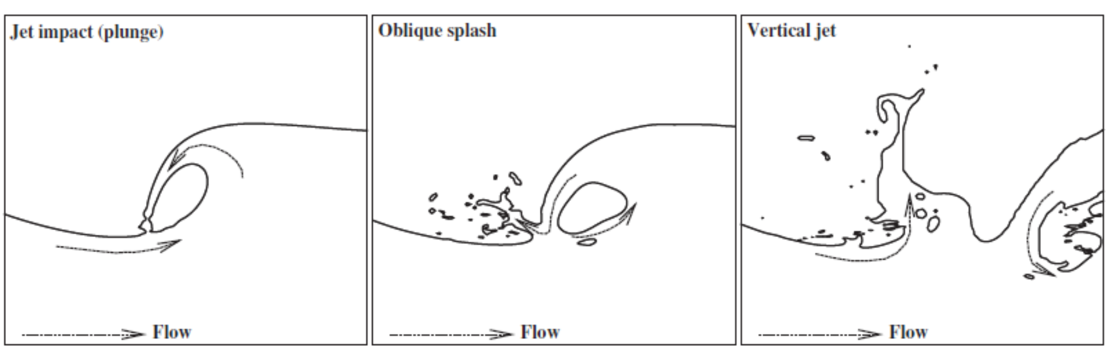

Fig. 1. Major events of the plunging wave breaking over a submerged bump: (a) jet impact (plunge); (b) oblique splash; and (c) vertical jet.

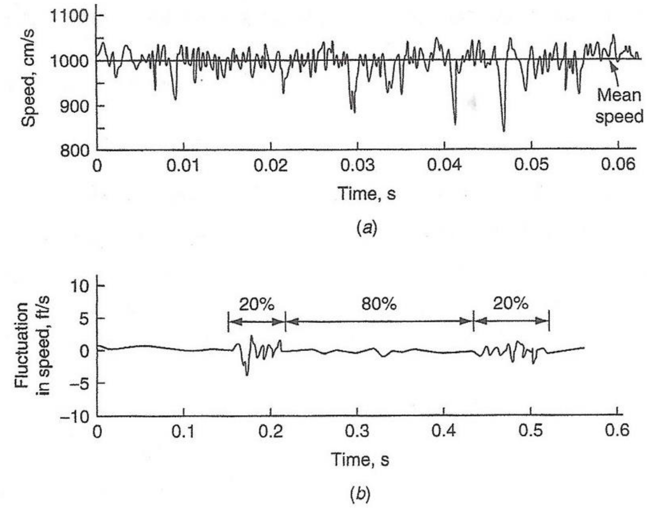

FIGURE 6-3
Hot-wire measurements showing turbulent velocity fluctuations: (*a*) typical trace of a single velocity component in a turbulent flow; (*b*) trace showing intermittent turbulence at the edge of a jet.

Other useful statistical definitions

$$\bar{u} = \frac{1}{T}\int_{t_0}^{t_0+T} u\,dt$$

$u' = u - \bar{u}$ Reynolds decomposition

$\overline{u'} = 0$

$$\overline{u'^2} = \underbrace{\frac{1}{T}\int_{t_0}^{t_0+T} u'^2\,dt}_{\text{mean square}} = \underbrace{\frac{1}{T}\int_{t_0}^{t_0+T} (u-\bar{u})^2\,dt}_{\text{variance}}$$

$\sqrt{\overline{u'^2}}$ = RMS u' = Standard deviation SD u

SD % Mean = coefficient of variation

If integrals independent to t_0 u' is statistically stationary

2.3 One-Point Statistics

Statistical analysis needed for $\underline{U}$, $\underline{\Omega} = \nabla \times \underline{U}$, p, and other variables of interest. Reynolds decomposition

$\underline{u} = \underline{U} - \overline{\underline{U}}$ $\underline{u}$ = turbulent fluctuation

or $\underline{U} = \overline{\underline{U}} + \underline{u}$ where $\overline{\underline{u}} = 0$ and $\overline{\overline{\underline{U}}} = \overline{\underline{U}}$

The TKE $= k = \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2}) = \overline{u_i u_i}/2$

$= \overline{\underline{u} \cdot \underline{u}}/2$

Is of central importance as it characterizes the strength of the turbulence where $\overline{u_i^2}$ are the variances in the x_i coordinate directions, e.g., the inflow $\sqrt{\overline{u^2}}/U$ characterizes the free stream turbulence in a wind tunnel experiment.

If $\overline{u^2} = \overline{v^2} = \overline{w^2}$ the turbulence is isotropic, whereas more generally $\overline{u^2} \neq \overline{v^2} \neq \overline{w^2}$ and the turbulence is non-isotropic, e.g., for near wall turbulence $\overline{v^2} < \overline{u^2}, \overline{w^2}$.

Homogeneous turbulence: the time-averaged properties of the flow are uniform and independent of position. For example, whereas $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$ may differ from each other, each must be constant throughout the system.

Isotropic turbulence: Turbulence in which the products and squares of the velocity components and their derivatives are independent of direction, or, more precisely, invariant with respect to rotation and reflection of the coordinate axes in a coordinate system moving with the mean motion of the fluid. Then all the normal stresses are equal, and the tangential stresses are zero.
$\overline{u^2} = \overline{v^2} = \overline{w^2},\ \overline{uv} = \overline{uw} = \overline{vw} = 0$

Homogeneous and isotropic turbulence play a fundamental role in the physics and modeling of turbulence.

The normal variances in combination with the off-diagonal quadratic products form the Reynolds stress tensor = symmetric second order tensor

$$R_{ij} = \overline{u_i u_j} = \begin{matrix} \overline{u^2} & \overline{uv} & \overline{uw} \\ \overline{vu} & \overline{v^2} & \overline{vw} \\ \overline{wu} & \overline{wv} & \overline{w^2} \end{matrix} = \text{turbulent velocity correlations / variances and covariances}$$

$$\sigma_t = -\rho R_{ij}$$

$$\overline{\sigma} = -p\delta_{ij} + \mu\left(v_{i,j} + v_{j,i}\right) - \rho\,\overline{u_i u_j}$$

$R_{12} = \overline{u_1 u_2} = \overline{uv}$ is of central importance for turbulence modeling for canonical flows such as shear layers and wall flows.

ρR_{ij} represents turbulent momentum fluxes

$$M_{ij} = \int \rho\, v_i v_j n_j \, dA \qquad v_i = \overline{v}_i + u_i = (\overline{v}+u, v, w)$$

$$\frac{d\overline{M_{xy}}}{dA} = \rho\,\overline{(\overline{U}+u)v} = \rho\,\overline{uv} \qquad \overline{Uv} = 0$$

Flux of x momentum in the y direction due to turbulent u_2 = v fluctuation. Turbulent transport of momentum and other quantities is one of the primary physical turbulent flow processes.

Higher-order statistics:

S = skewness = $\overline{u_1^3} / \overline{u_1^2}^{3/2}$

F = flatness or kurtosis = $\overline{u_1^4} / \overline{u_1^2}^{2}$

S = bias or symmetry between + and − values

F = congregation at large or small values

Probability density functions provide information about the range of values u_i or $u_i u_j$ take and their likelihood to assume specific values in that range.

$p(u)$ = probability density function u_1

$p(u)\,du$ = probability u_1 is between u and $u+du$

$$\int_{-\infty}^{\infty} p(u)\,du = 1$$ since u must take on some value

$$\overline{u_1} = \int_{-\infty}^{\infty} u\,p(u)\,du = 0 \qquad \overline{u_1^2} = \int_{-\infty}^{\infty} u^2 p(u)\,du$$

$$\overline{u_1 u_2} = \iint uv\,p(u,v)\,du\,dv$$

= joint probability function for $\overline{uv}$

2.4 Two-Point Velocity Correlations[1]

To reveal the structural characteristics of turbulence such as vortical features (eddy size) two-point velocity correlations are required. Dependence f(t) implies ensemble averaging.

velocity correlation

$$R_{ij}(\underline{x},\underline{y},t) = \overline{u_i(\underline{x},t)\,u_j(\underline{y},t)}$$

2 point
2 velocity

$$S_{ijk}(\underline{x},\underline{y},t) = \overline{u_i(\underline{x},t)\,u_j(\underline{x},t)\,u_k(\underline{y},t)}$$

2 point
3 velocity

for $\underline{x}=\underline{y}$ $\quad R_{ij}(\underline{x},t) = \overline{u_i u_j}$ = Reynolds stress tensor Eq. (2.19)

Letting $\underline{r} = \underline{y} - \underline{x}$ which is appropriate for homogeneous turbulence where $R_{ij}(\underline{r},t)$ is the same for all $\underline{x}$ (R_{ij} also important for inhomogeneous flows where f($\underline{x}$) dependence is implied).

$$R_{ij}(\underline{r},t) = \overline{u_i(\underline{x},t)\,u_j(\underline{x}+\underline{r},t)}$$

$$S_{ijk}(\underline{r},t) = \overline{u_i(\underline{x},t)\,u_j(\underline{x},t)\,u_k(\underline{x}+\underline{r},t)}$$

$R_{ij}(\underline{r},t)$ and $S_{ijk}(\underline{r},t)$ are of central importance for isotropic turbulence. Note that:

$$R_{ij}(0,t) = \overline{u_i u_j}$$

$$R_{ij}(\underline{r} \to \underset{\text{value}}{\text{large}}, t) = 0$$

Velocities are uncorrelated for distances greater than the largest eddy size.

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[1] Related and originally used by Kolmogorov for his hypotheses are the 2nd order velocity structure functions, which is the co-variance of the difference in velocity between two points **x** + **r** and **x** $D_{ij}(r,x,t) \equiv \langle [U_i(x+r,t) - U_i(x,t)][U_j(x+r,t) - U_j(x,t)] \rangle$.

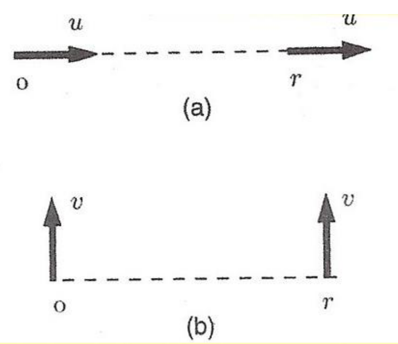
Of special importance are the two-point longitudinal and transverse correlation coefficients f(r) and g(r), respectively:

$$\overline{u^2}\, f(r) = R_{11}(r\hat{e}_1)$$

$$\overline{v^2}\, g(r) = R_{22}(r\hat{e}_1)$$

Where both are f(t) for non-stationary flows. For stationary flows time averaging implied.

Figure 2.1 (a) Longitudinal and (b) transverse velocities appearing in the definitions of $f(r)$ and $g(r)$, respectively.



$$f(r) = \overline{u(x)\, u(x+r)} \,/\, \overline{u^2(x)} \qquad f(0) = 1$$

$$g(r) = \overline{v(x)\, v(x+r)} \,/\, \overline{v^2(x)} \qquad g(0) = 1$$

TS $\quad f(r) = 1 + r f_r(0) + \frac{r^2}{2!} f_{rr}(0) + \cdots$

$$\overline{u^2}\, f_r(r) = \overline{u(x)\, u(x+r)_{,r}} = \overline{u(x)\, u_x(x+r)}$$

$$\overline{u^2}\, f_r(0) = \frac{1}{2} \frac{\partial \overline{u^2}}{\partial x}$$

$\therefore$ for homogeneous turbulence where $\overline{u^2}$ = constant

$$f(r) = 1 + \frac{r^2}{2} f_{rr}(0)$$

ie $f(r)$ is parabolic

$$\overline{u^2}\,\frac{df}{dr}(r) = \overline{u(x)\,\frac{\partial}{\partial r}\,u(x+r)} \qquad x' = x + r \qquad \frac{\partial x'}{\partial r} = 1$$

$$= \overline{u(x)\,\frac{\partial u}{\partial x'}(x+r)\,\frac{\partial x'}{\partial r}}$$

lim $r \to 0$ $\quad = \overline{u(x)\,\frac{\partial u}{\partial x}(x)}$

$$\overline{u^2}\,\frac{df}{dr}(0) = \frac{1}{2}\,\frac{\partial \overline{u^2}}{\partial x} \qquad \text{assuming} \quad \overline{\frac{\partial u^2}{\partial x}} = \frac{\partial \overline{u^2}}{\partial x}$$

= 0 for homogeneous turbulence where $\overline{u^2}$ = constant

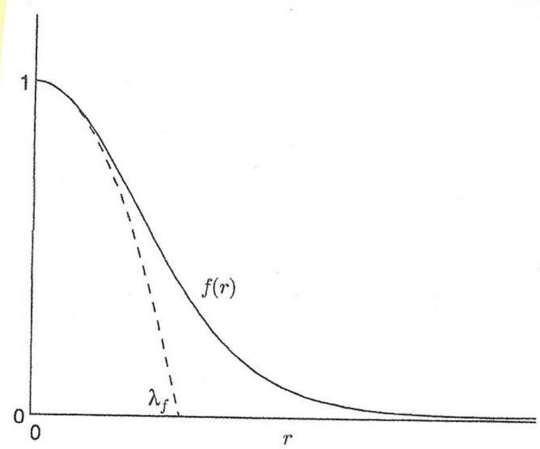


Figure 2.2 Definition of the microscale λ_f.

$f(r) = 0 = 1 + \frac{\lambda_f^2}{2} f_{rr}(0)$ = Taylor microscale

= distance from $r=0$ to r at which $f(r)=0$

$$\lambda_f = \left[-2/f_{rr}(0)\right]^{1/2}$$

$$\lambda_g = \left[-2/g_{rr}(0)\right]^{1/2}$$

The Taylor micro scales λ_f and λ_g are measures of the size of the flow features where viscous effects are imporant, i.e., measures of the smaller scales of turbulence.

Whereas a measure of the largest "energetic" scales is given by the integral length scales, i.e., the Taylor macro scales:

$$\Lambda_f = \int_0^\infty f(r)\,dr$$

$$\Lambda_g = \int_0^\infty g(r)\,dr$$

Like the longitudinal correlation coefficient f(r) is the temporal auto-correlation coefficient $R_E(\tau) = r_{11}(\tau)$

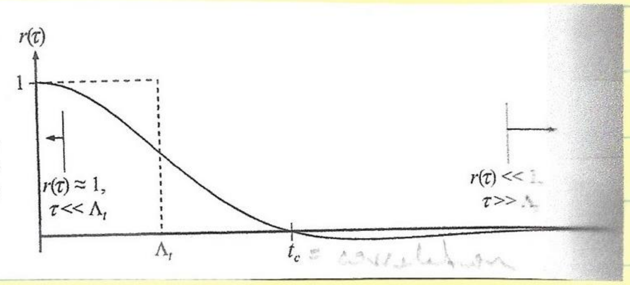
$$r_{11}(\tau) = R_{11}(\tau)/R_{11}(0) = \overline{u(t)u(t+\tau)}/\overline{u^2} \quad R_E(\tau)$$

Which can be obtained at a fixed point in space. Similarly, for r_{22} and r_{33}. Taylor micro and integral macro "time duration" scales can be computed similarly as done previously for the corresponding spatial length scales.

$$\Lambda_t = \int_0^\infty r_{11}(\tau)\,d\tau = \frac{1}{u^2}\int_0^\infty R_{11}(\tau)\,d\tau$$

= measure of the memory of the turbulence

FIGURE 12.5 Sample plot of an autocorrelation coefficient showing the integral time scale Λ_t, and the correlation time t_c. The normalization requires $r(0) = 1$. In the limit $\tau \to \infty$, $r(\tau) \to 0$ and thereby indicates that the random process used to construct r becomes uncorrelated with itself when the time shift τ is large enough.



time when

$r_{11}(\tau)$ just = 0

$$\lambda_t^2 = -2/r_{11_{tt}}(0) \qquad \tau_E$$

τ_E is obtained from the curvature of the $r_{11}(\tau)$ peak and indicates where $u_1(t)$ is well correlated with itself.

More generally for two random variables $u_i(\underline{x_1}, t_1)$ and $u_j(\underline{x_2}, t_2)$

$$R_{ij}(\underline{x}_1, t_1, \underline{x}_2, t_2) = \overline{u_i(\underline{x}_1, t_1)\,u_j(\underline{x}_2, t_2)}$$

R_{ij} specifies how similar $u_i(\underline{x_1}, t_1)$ and $u_j(\underline{x_2}, t_2)$ are to each other.

$R_{ij} = 0$ $\quad u_i > 0$ & u_j equally likely + or −
ie uncorrelated
or weakly correlated $R_{ij} > 0$ small

$R_{ij} \gg 0$ $\quad u_i < 0$ & $u_j < 0$ ie strongly
or $u_i > 0$ & $u_j > 0$ correlated

$R_{ij} \ll 0$ $\quad u_i > 0$ & $u_j < 0$ ie anti correlated
or $u_i < 0$ & $u_j > 0$

$i \neq j$ cross correlation
$i = j$ auto correlation

$$R_{12}(\underline{x}_1, t_1, \underline{x}_2, t_2) = \overline{u_1(\underline{x}_1, t_1)\, u_2(\underline{x}_2, t_2)}$$

$$r_{12} = R_{12} / \sqrt{R_{11} R_{22}} = \overline{u_1 u_2} / \sqrt{\overline{u_1^2}\ \overline{u_2^2}}$$

$$R_{11}(\underline{x}_1, t_1, \underline{x}_2, t_2) = \overline{u_1(\underline{x}_1, t_1)\, u_1(\underline{x}_2, t_2)}$$

$$r_{11} = R_{11} / \sqrt{R_{11} R_{11}} = \overline{u_1 u_1} / \sqrt{\overline{u_1^2}\ \overline{u_1^2}}$$

-1 (anti correlation) $\leq r_{ij} \leq 1$ (correlation) $\quad r_{11}(\underline{x}_1, t_1, \underline{x}_1, t_1) = 1$

$$\overline{u(\underline{x}_1, t_1)\, u(\underline{x}_2, t_2)} \leq \left[\overline{u^2(\underline{x}_1, t_1)}\right]^{1/2} \left[\overline{u^2(\underline{x}_2, t_2)}\right]^{1/2}$$

Schwartz inequality
analogous $\underline{a} \cdot \underline{b} = ab \cos\alpha$

For temporally stationary flows with u_i sampled at the same point in space $\underline{x} = \underline{x_1} = \underline{x_2}$, the listing of $\underline{x}$ is not required and the statistics are independent of the time origin such that $\tau = t_2 - t_1$ = time lag and

Time

$R_{ij}(\tau) = \overline{u_i(t)\,u_j(t+\tau)}$ correlation

$R_{11}(\tau) = \overline{u_1(t)\,u_1(t+\tau)}$ auto correlation

note. $R_{11}(\tau) = \overline{u_1(t)\,u_1(t+\tau)}$

$= \overline{u_1(t-\tau)\,u_1(t)}$

$= \overline{u_1(t)\,u_1(t-\tau)} = R_{11}(-\tau)$

$R_{ij}(\tau) \neq R_{ij}(-\tau) \quad i \neq j$ cross correlation

$R_{ij}(0) = \overline{u_i u_j}$

Time shift for $\overline{u(t)u(t+\tau)} = 0$ is for equal width eddies all through $u(t)$

R_{11} is maximum when $\tau = 0$; or when $t_o = \tau$ and in this case R_{11} is symmetric about t_o

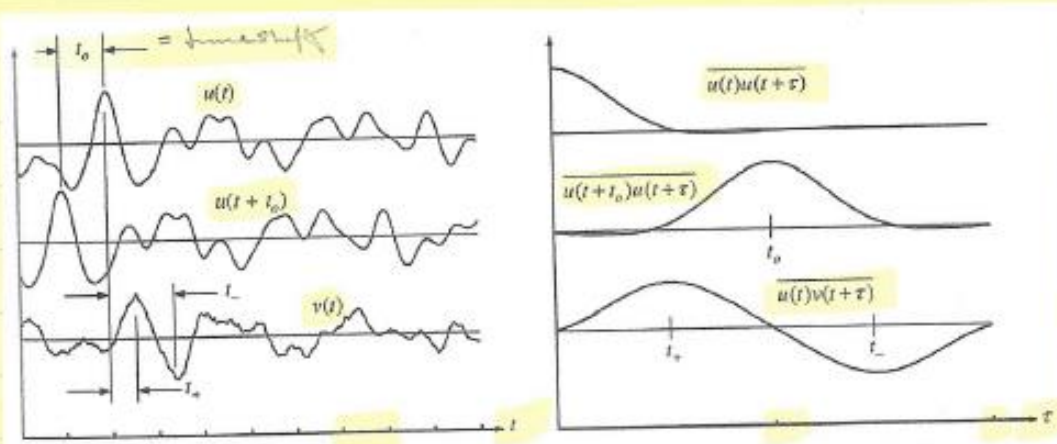


FIGURE 12.4 Sample results for auto- and cross-correlation functions of $u(t)$, $u(t+t_o)$, and $v(t)$. These three time series are shown on the left. The upper curve on the right is the autocorrelation function, $\overline{u(t)u(t+\tau)}$, of the upper time series on the left. The tic marks on the axes represent the same time interval so the width of a peak of $u(t)$ is about equal to the correlation time determined from $\overline{u(t)u(t+\tau)}$. The correlation osf $u(t+t_o)$ and $u(t)$ is shown as the middle curve on the right, and it is just a shifted replica of $\overline{u(t)u(t+\tau)}$. The cross-correlation of $u(t)$ and $v(t)$ is the lower curve on the right. Here the maximum cross-correlation occurs when $\tau = t_+$ and the peaks of u and v coincide. Similarly, u and v are most anti-correlated when peaks in u align with valleys in v at $\tau = t_-$

Correlation is a mathematical shape-comparison indicator sensitive to space or time alignment

2.5 Spatial Spectra

$$\mathcal{R}_{ij}(\underline{x}, \underline{r}, t) = \overline{u_i(\underline{x}, t) u_j(\underline{x} + \underline{r}, t)} \quad [\mathrm{m^2/s^2}]$$

where $\underline{r} = \underline{y} - \underline{x}$. For homogeneous turbulence $R_{ij} \neq$ f($\underline{x}$) $\therefore$ often the dependence is not shown, but implied for inhomogeneous flows.

The wave number space vector is defined by $\underline{k}$ = (k_1,k_2,k_3). The purpose of the Fourier transform is to map various functions from physical space $\underline{x}$ = (x_1,x_2,x_3) to $\underline{k}$ space, which is useful for structural analysis such as distributions of eddy sizes etc.

$$\mathcal{E}_{ij}(\underline{\kappa}, t) = \frac{1}{(2\pi)^3} \int_{\forall} e^{i\underline{\kappa}\cdot\underline{r}} \mathcal{R}_{ij}(\underline{r}, t)\, d\underline{r}$$

$$\mathcal{R}_{ij}(\underline{r}, t) = \int_{\forall} e^{-i\underline{\kappa}\cdot\underline{r}} \mathcal{E}_{ij}(\underline{\kappa}, t)\, d\underline{\kappa}$$

where:

$e^{i\underline{\kappa}\cdot\underline{r}} = cos(\underline{\kappa} \cdot \underline{r}) + isin(\underline{\kappa} \cdot \underline{r})$ = spatial Fourier mode, which varies sinusoidal with wavelength $\lambda = \frac{2\pi}{|\underline{\kappa}|}$

$d\underline{r} = dr_1 dr_2 dr_2$ = differencial volume in physical space

$d\underline{\kappa} = d\kappa_1 d\kappa_2 d\kappa_2$ = differencial volume in wave number space. Admittedly poor nomenclature as $d\underline{r}$ and $d\underline{\kappa}$ are not and should not be confused as vectors.

$\mathcal{E}_{ij}(\underline{\kappa}, t) =$ energy spectrum tensor/velocity spectrum tensor [m^5/s^2], which along with $\mathcal{R}_{ij}(\underline{r}, t)$ decompose turbulence into Fourier modes.

For $\underline{r}$ = 0:

$$\mathcal{R}_{ij}(0, t) = \int_{\forall} \mathcal{E}_{ij}(\underline{\kappa}, t)\ d\underline{\kappa} = \langle u_i u_j \rangle$$

$\mathcal{E}_{ij}(\underline{\kappa}, t)$ represents the variance and covariance $\langle u_i u_j \rangle$ of velocity modes with wave number $\underline{\kappa}$.

$$k(\underline{x}, t) = \frac{1}{2}\overline{u_i u_i} = \frac{1}{2}\left(\overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}\right) \quad [m^2/s^2]$$

TKE per unit mass where integration is from -∞ to ∞.

R_{ij} and ε_{ij} contain two different kinds of information. The dependence of R_{ij} on $\underline{r}$ and $\mathcal{E}_{ij}$ on $\underline{\kappa}$ give information about directional dependence of the correlations, while their components give information on the direction of the velocities. Their implicit dependence on $\underline{x}$ provides the ability of decomposing the turbulence correlations into a range of scales as represented by the Fourier components $e^{i\underline{r}\cdot\underline{k}}$.

$$k(t) = \frac{1}{2}\int_{\forall} tr\, \mathcal{E}_{ij}\left(\underline{\kappa}, t\right) d\underline{\kappa} = \frac{1}{2}\int_{\forall} \mathcal{E}_{ii}\left(\underline{\kappa}, t\right)\ d\underline{\kappa}$$

$\frac{1}{2}\mathrm{tr}\mathcal{E}_{ij} = \frac{1}{2}\mathcal{E}_{ii}\left(\underline{\kappa}, t\right)$

$\mathcal{E}_{ii}\left(\underline{\kappa}, t\right)$ = density energy in $\underline{\kappa}$ space

It is useful to collect the energy onto shells of fixed distance $\kappa = \left|\underline{\kappa}\right|$ from the origin, which energy spectrum

$$k(t) = \int_0^{\infty} [\frac{1}{2}\int \mathcal{E}_{ii}\left(\underline{\kappa}, t\right)\kappa^2 d\Omega]\, d\kappa$$

$d\underline{\kappa} = \kappa^2 d\Omega\, d\kappa$
$d\Omega =$ elemental solid angle

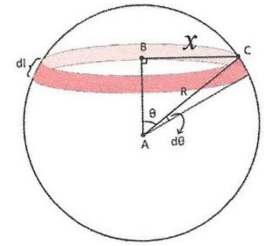
$\frac{1}{2}\int \mathcal{E}_{ii}\left(\underline{\kappa}, t\right)\kappa^2 d\Omega = \mathrm{E}(\kappa, t)$ = energy spectrum = TKE per unit κ and shows how the kinetic energy is distributed across the different scales of the flow, which has units [m^3/s^2].

$$k(t) = \int_0^{\infty} \mathrm{E}(\kappa, t)\ d\kappa$$

$\mathrm{E}(\kappa, t)\ d\kappa$ represents the contribution to the TKE = $\frac{1}{2}\overline{u_i u_i}$ from all modes with $\left|\underline{\kappa}\right|$ in the range $\kappa \leq \left|\underline{\kappa}\right| < \kappa +\ d\kappa$.

Calculate the surface area of a sphere with a radius of R:

$dS = Cdl$
where $dl = Rd\theta$; *C is the perimeter (circumference) of the red circle, and*
$C = 2\pi x = 2\pi R sin(\theta)$
$dS(R) = 2\pi R sin(\theta) R d\theta$



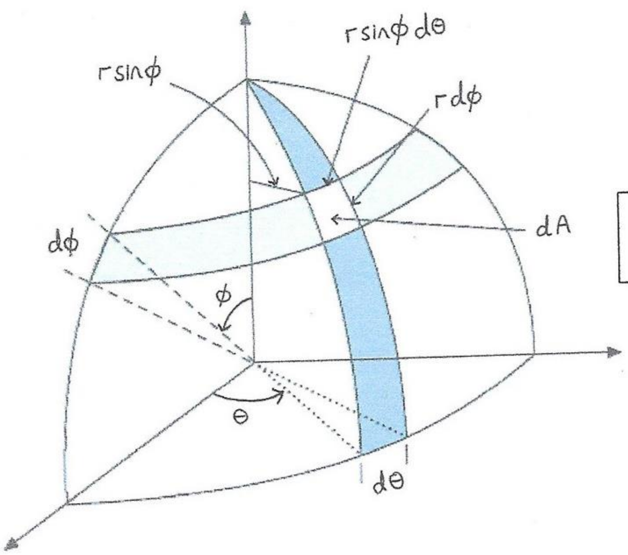
$$\oint dS(k) = \int_0^{\pi} 2\pi k sin(\theta) k d\theta$$

$$= -2\pi k^2 cos(\theta)|_0^{\pi}$$

$$= 2\pi k^2 - (-2\pi k^2) = 4\pi k^2$$

$$\oint dS(k) = 4\pi k^2 \qquad (6.194)$$

Alternately, the area element can be written as $dS(k) = k^2 sin\varphi d\theta d\varphi$ *according to the following figure.*



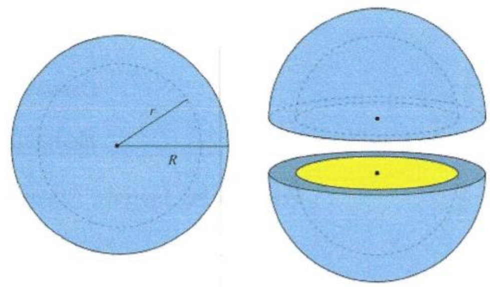
$$dA = r sin\phi d\theta \, r d\phi = r^2 sin\phi d\phi d\theta$$

$\underline{d\kappa}$ = $d\kappa_1 d\kappa_2 d\kappa_3$=d∀ (not vector)
$=\kappa^2 d\Omega d\kappa$
$d\Omega = sin\phi d\phi d\theta$
$dS(\kappa) = \kappa^2 d\Omega$

$$\oint d\,S(k) = \int_0^{\pi}\int_0^{2\pi} k^2 sin\varphi d\varphi d\theta = \int_0^{\pi} 2\pi k^2 sin\varphi d\varphi = -2\pi k^2 cos\varphi|_0^{\pi} = 4\pi k^2$$

Spherical shell

In geometry, a **spherical shell** is a generalization of an annulus to three dimensions. It is the region of a ball between two concentric spheres of differing radii.[1]



spherical shell, right: two halves

Volume

The volume of a spherical shell is the difference between the enclosed volume of the outer sphere and the enclosed volume of the inner sphere:

$$V = \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$$
$$V = \frac{4}{3}\pi (R^3 - r^3)$$
$$V = \frac{4}{3}\pi (R - r)(R^2 + Rr + r^2)$$

where r is the radius of the inner sphere and R is the radius of the outer sphere.

An approximation for the volume of a thin spherical shell is the surface area of the inner sphere multiplied by the thickness t of the shell:[2]

$$V \approx 4\pi r^2 t,$$

when t is very small compared to r ($t \ll r$).

Total surface area of spherical shell is *4(pi)r*^{2}

2.6 Time Spectra

Fourier Transform pair

$$\hat{R}_E(\omega') = \int_{-\infty}^{\infty} e^{-i\tau\omega'} R_E(\tau)\, d\tau \quad [s] \qquad R_E(\tau) = \frac{\overline{u(t)u(t+\tau)}}{\overline{u^2}}$$

$$R_E(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau\omega'} \hat{R}_E(\omega')\, d\omega' \qquad \omega' = 2\pi\omega \ \frac{rad}{s}$$

ω hz

$$R_E(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{R}_E(\omega')\, d\omega' = 1 = \int_{-\infty}^{\infty} \underbrace{\hat{R}_E(2\pi\omega)}_{= \hat{E}_{11}(\omega)/2\overline{u^2}}\, d\omega$$

$$\int_{-\infty}^{\infty} \frac{\hat{E}_{11}(\omega)}{2\overline{u^2}}\, d\omega = 1$$

define 1D energy spectrum

$$\overline{u^2} = \frac{1}{2}\int_{-\infty}^{\infty} \hat{E}_{11}(\omega)\, d\omega \qquad \widehat{E_{11}}(\omega) = 2\overline{u^2}\widehat{R_E}(\omega')$$

$\widehat{E_{11}}$ $[m^2/s]$

For stationary flow: $\overline{u(t)u(t+\tau)} = \overline{u(t-\tau)u(t)}$

$$R_E(\tau) = R_E(-\tau)$$

$$\Rightarrow \hat{R}_E(\omega') = 2\int_{0}^{\infty} \cos\tau\omega'\, R_E(\tau)\, d\tau$$

$$\hat{R}_E(-\omega') = \hat{R}_E(\omega')$$

$$R_E(\tau) = \frac{1}{\pi}\int_{0}^{\infty} \cos\tau\omega'\, \hat{R}_E(\omega')\, d\omega'$$

$$\overline{u^2} = \int_{0}^{\infty} \hat{E}_{11}(\omega)\, d\omega$$

Note: $R_E(\tau)$ (temporal auto-correlation coefficient) and thus $\widehat{E_{11}}(\omega)$ can be obtained from single point measurements vs. E(k,t) which requires volume or line measurements and if Taylors frozen turbulence hypothesis is used it can be transformed from time to space as approximation for spatial spectra, e.g., $\lambda_f = \overline{U}\tau_E$ and $E_{11}(k_1) = \frac{\overline{U}}{2\pi}\widehat{E_{11}}(\omega)$ where $\overline{U}$ is the mean velocity.