A.5 A summary of Cartesian-tensor suffix notation

The definitions, rules, and operations involved with Cartesian tensors using suffix notation can be summarized thus.

1. In a tensor equation, *tensor expressions* are separated by +, −, or =. For example;
$$b_{ij} + c_{ij} = f_{ijkk}. \tag{A.75}$$
2. In a tensor expression, a suffix that appears once is a *free suffix* (e.g., i and j in Eq. (A.75)).
3. In a tensor expression, a suffix that appears twice is a *repeated suffix* or a *dummy suffix* (e.g., the suffixes k in d_{ijkk}). The symbol used for a dummy suffix is immaterial, i.e., $d_{ijkk} = d_{ijpp}$.
4. *Summation convention*: a repeated suffix implies summation, i.e.,
$$d_{ijkk} = \sum_{k=1}^{3} d_{ijkk}. \tag{A.76}$$
5. In a tensor expression, a suffix cannot appear more than twice. For example, the expression f_{ijii} is invalid.
6. A tensor expression with N free suffixes is (or, more correctly, represents the components of) an Nth-order tensor. For example, each expression in Eq. (A.75) is a second-order tensor.
7. Each expression in a tensor equation must be a tensor of the same order, with the same free suffixes (not necessarily in the same order). Equation (A.75) is valid, whereas $b_{ij} = d_{ijk}$ and $b_{ij} = c_{ik}$ are both invalid.
8. The *Kronecker delta* δ_{ij} is defined by
$$\begin{aligned} \delta_{ij} &= 1, \quad \text{for } i = j, \\ &= 0, \quad \text{for } i \neq j. \end{aligned} \tag{A.77}$$
It is a second-order tensor. Note that $\delta_{ii} = 3$.
9. The *alternation symbol* ε_{ijk} in Eq. (A.56) is *NOT* a tensor.
10. *Addition*, e.g., $b_{ijk} = c_{ijk} + d_{ikj}$. Each tensor must be of the same order with the same free suffixes.
11. The *tensor product* of an Nth-order tensor and an Mth-order tensor is an $(N+M)$th-order tensor, e.g., $b_{ijk\ell m} = c_{ij}d_{k\ell m}$.
12. An Nth-order tensor ($N \geq 2$) can be *contracted* by changing two free suffixes into repeated suffixes. The result is a tensor of order $N-2$. Different contractions of d_{ijk} are d_{iik}, d_{iji}, and d_{ijj}.
13. The *inner product* of an Nth-order tensor and an Mth-order tensor ($N \geq 1, M \geq 1$) is a tensor of order $N+M-2$: e.g., $f_{ik\ell} = c_{ij}d_{jk\ell}$.
14. The *substitution rule* is that the inner product with the Kronecker delta is, for example,
$$\delta_{ij}c_{jk} = c_{ik}. \tag{A.78}$$
15. There is no tensor operation corresponding to *division*.
16. The *gradient of a tensor* is a tensor of one order higher, e.g., $d_{jk\ell} = \partial c_{k\ell}/\partial x_j$.
17. The *divergence* of an Nth-order tensor ($N \geq 1$) is a tensor of order $(N-1)$, e.g., $v_k = \partial c_{jk}/\partial x_j$.
18. There are no tensor operations corresponding to the *vector cross product* or to the *curl*.

N = 0
scalar
N = 1
vector
N = 2
2nd order
etc.