Chapter 7: Free Shear Flows: Jets, Mixing Layers and Wakes (Pope)

Part 1: Round and 2D Jets

In contrast to wall flows, remote from solid surfaces and turbulence due to mean-velocity differences.

Round jet: EFD

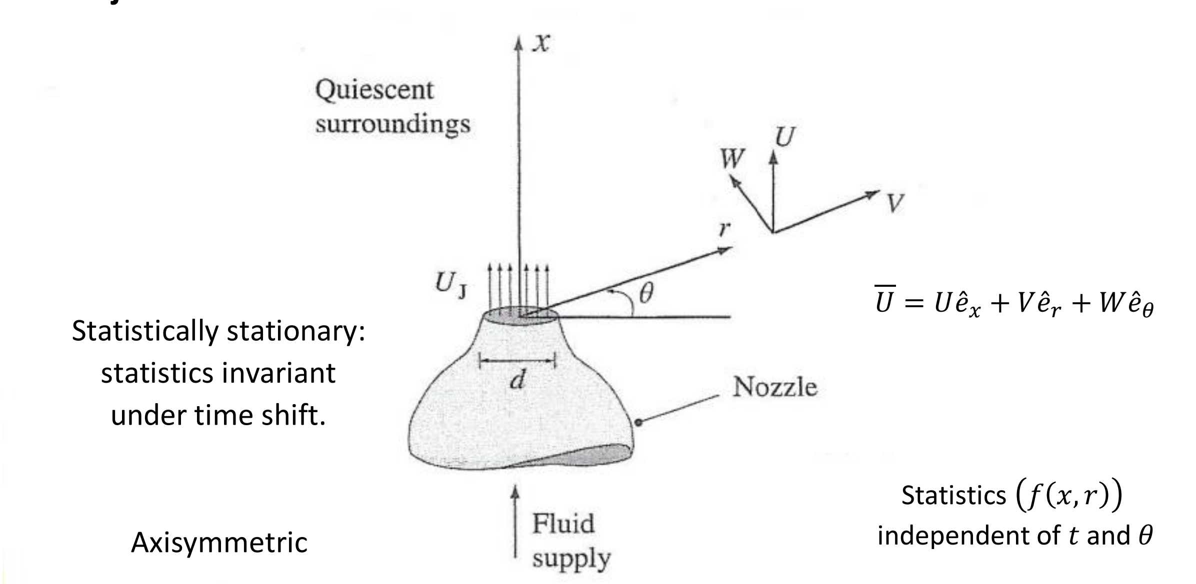


Fig. 5.1. A sketch of a round-jet experiment, showing the polar-cylindrical coordinate system employed.

$Re = \frac{U_J d}{\nu}$ defines the flow, i.e., only non-dimensional parameter.

$$\overline{U}(x, r, \theta) = \overline{U}(x, r)$$

Centerline velocity:

$$U_0(x) = \overline{U}(x, 0)$$

Definition of jet's half-width:

$$\overline{U}\big(x, r_{1/2}(x)\big) = \tfrac{1}{2} U_0(x) \quad \text{defines } r_{1/2}(x)$$

IC dependent details nozzle and U_J: $0 \le x/d \le 25$

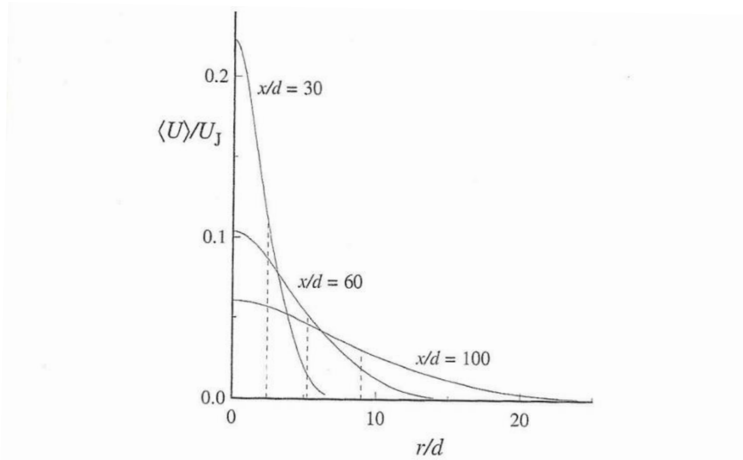


Fig. 5.2. Radial profiles of mean axial velocity in a turbulent round jet, Re = 95,500. The dashed lines indicate the half-width, $r_{1/2}(x)$, of the profiles. (Adapted from the data of Hussein *et al.* (1994).)

For $x \uparrow$: $U_0(x) \downarrow$

$r_{1/2}(x) \uparrow$

i.e., jet decays and spreads, but shape remains same.

Self-similarity

For $x/d > 30$, $\overline{U}/U_0(x)$ vs $r/r_{1/2}(x)$ collapses on a single self-similar curve

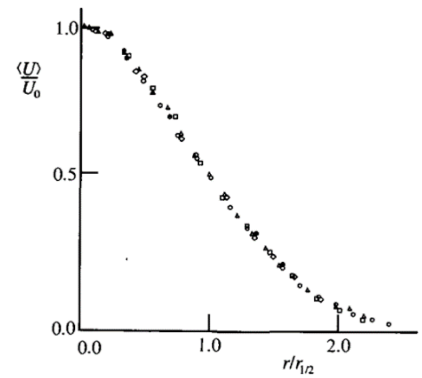


Fig. 5.3: Mean axial velocity against radial distance in a turbulent round jet, Re $\approx 10^5$; measurements of Wygnanski and Fiedler (1969). Symbols: ○, $x/d = 40$; △, $x/d = 50$; □, $x/d = 60$; ◇, $x/d = 75$; •, $x/d = 97.5$.

$$\frac{U_0(x)}{U_J} = \frac{B}{(x - x_0)/d} \sim x^{-1}$$

x_0 = virtual origin

B = experimental constant

$$S \equiv \frac{dr_{1/2}(x)}{dx} = \text{spread rate} = \text{constant}$$

$$r_{1/2}(x) = S(x - x_0) \sim x$$

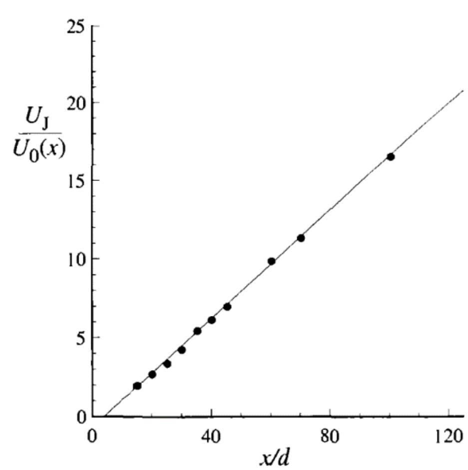


Fig. 5.4. The variation with axial distance of the mean velocity along the centerline in a turbulent round jet, Re = 95,500: symbols, experimental data of Hussein *et al.* (1994); and line, Eq. (5.6) with $x_0/d = 4$ and $B = 5.8$.

$$U = \overline{U} + u \quad V = \overline{V} + v \quad W = w$$

Momentum equation in x −direction × r:

$$\frac{\partial}{\partial x}\left(r\overline{U}^2\right) + \frac{\partial}{\partial r}\left(r\overline{UV} + r\overline{uv}\right) = 0$$

Integrating with respect to r:

$$\frac{d}{dx}\int_0^\infty r\overline{U}^2\, dr = -\left[r\overline{UV} + r\overline{uv}\right]_0^\infty = 0$$

Since, for large r, $\overline{UV}$ and $\overline{uv}$ tend to zero more rapidly than r^{-1}. Therefore, momentum flux of the mean flow is independent of x:

$$\dot{M} = \int_0^\infty 2\pi r\rho\overline{U}^2\, dr = \text{constant} \neq f(x)$$

$$\boxed{\xi = \frac{r}{r_{1/2}(x)}}$$

$$= 2\pi\rho\left(r_{1/2}U_0\right)^2 \int_0^\infty \xi f(\xi)^2 d\xi$$

$$\therefore r_{1/2}(x)U_0(x) \neq f(x)$$

i.e., $r_{1/2}(x) \sim x$ and $U_0(x) \sim x^{-1}$ consistent with momentum flux being constant and

$$Re_0(x) = \frac{r_{1/2}(x)U_0(x)}{\nu} \neq f(x)$$

Table 5.1. *The spreading rate S (Eq. (5.7)) and velocity-decay constant B (Eq. (5.6)) for turbulent round jets (from Panchapakesan and Lumley (1993a))*

	Panchapakesan and Lumley (1993a)	Hussein *et al.* (1994), hot-wire data	Hussein *et al.* (1994), laser-Doppler data
Re	11,000	95,500	95,500
S	0.096	0.102	0.094
B	6.06	5.9	5.8

S and B = constants ≠ f(Re)

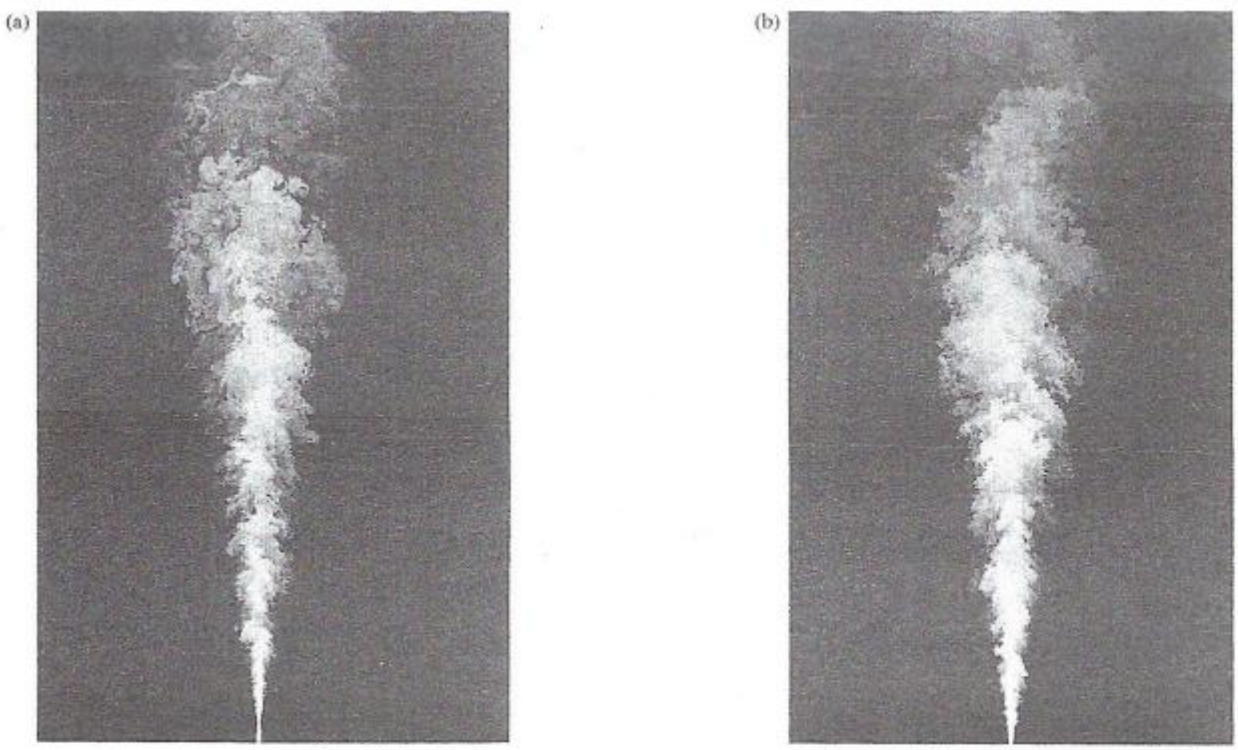


Fig. 1.2. Planar images of concentration in a turbulent jet: (a) Re = 5,000 and (b) Re = 20,000. From Dahm and Dimotakis (1990) .

Re only effects flow via small scale structures.

Cross-stream similarity variable can either be:

$$\xi = r/r_{1/2}$$

or:

$\eta = \frac{r}{x-x_0} = S\xi$ (i..e., ξ and η are linearly realted)

$$S = \frac{dr_{1/2}(x)}{dx} = r_{1/2}/\ x - x_0$$

Self-similar mean velocity profile:

$$f(\eta) = \frac{\overline{U}(x,r)}{U_0(x)}$$

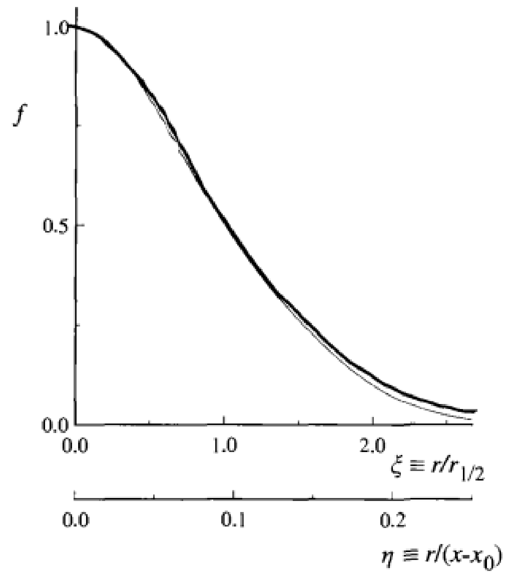
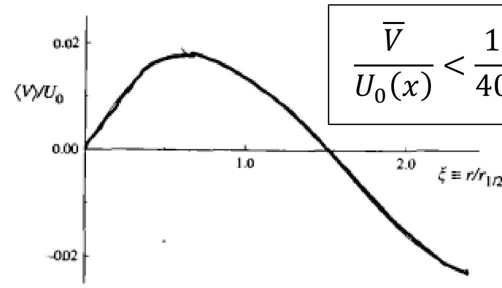


Fig. 5.5. The self-similar profile of the mean axial velocity in the self-similar round jet: curve fit to the LDA data of Hussein *et al.* (1994).

The mean lateral velocity $\overline{V}$ can be determined from $\overline{U}$ via the continuity equation (Pope Ex. 5.4):

$$\frac{\partial \overline{U}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{V}\right) = 0$$



$$\boxed{\frac{\overline{V}}{U_0(x)} < \frac{1}{40} = 0.025}$$

Fig. 5.6. The mean lateral velocity in the self-similar round jet. From the LDA data of Hussein *et al.* (1994).

Such that:

$\overline{V} < 0$ near the edge→ indicating entrainment of the external flow

$$\frac{\overline{V}}{U_0} = h(\eta) \quad \text{where:} \quad \eta(f\eta)' = (h\eta)'$$

Reynolds stresses

$$\overline{u_i u_j} = \begin{bmatrix} \overline{u^2} & \overline{uv} & 0 \\ \overline{uv} & \overline{v^2} & 0 \\ 0 & 0 & \overline{w^2} \end{bmatrix}$$

Due to circumferential symmetry, $\overline{uw} = \overline{vw} = 0$ and normal stresses are even functions of r, while $\overline{uv}$ is an odd function.

Consider the rms axial velocity on the centerline

$$u_0'(x) = \overline{u^2}_{r=0}^{1/2}$$

In the self-similar region:

$$\frac{u_0'(x)}{U_0(x)} \sim 0.25 = \text{constant}$$

$$\therefore u_0'(x) \sim x^{-1} \neq f(Re)$$

$\dfrac{\overline{u_i u_j}}{U_0^2}$ self-similar vs $r/r_{1/2}$ or η.

$\overline{uv} > 0$ where $\overline{U_r} < 0 \rightarrow$ positive turbulent viscosity ν_t:

$$\overline{uv} = -\nu_t \overline{U_r}$$

Since the profiles for $\overline{uv}$ and $\overline{U_r}$ are self-similar $\rightarrow$ the profile of ν_t is also self-similar:

$$\nu_t(x, r) = U_0(x) r_{1/2}(x) \hat{\nu}_t(\eta)$$

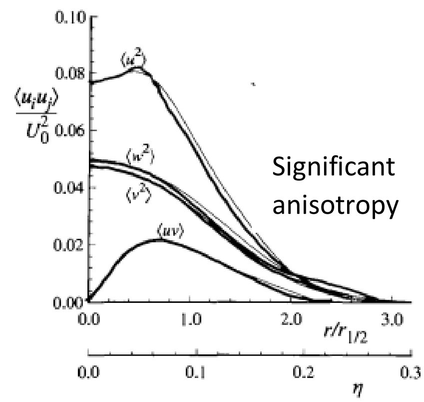


Fig. 5.7. Profiles of Reynolds stresses in the self-similar round jet: curve fit to the LDA data of Hussein *et al.* (1994).

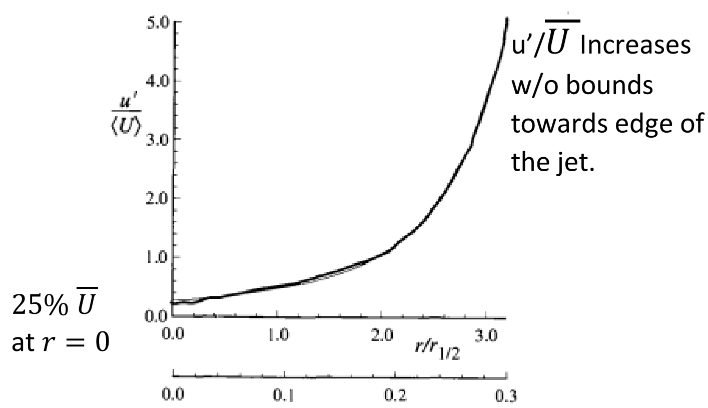


Fig. 5.8. The profile of the local turbulence intensity – $\langle u^2 \rangle^{1/2} / \langle U \rangle$ – in the self-similar round jet. From the curve fit to the experimental data of Hussein *et al.* (1994).

$\rho_{uv} = \frac{\overline{uv}}{\overline{u}}\overline{v} = \overline{u} - \ \overline{v}$ correlation coefficient

Both curves same shape

0.40

0.27

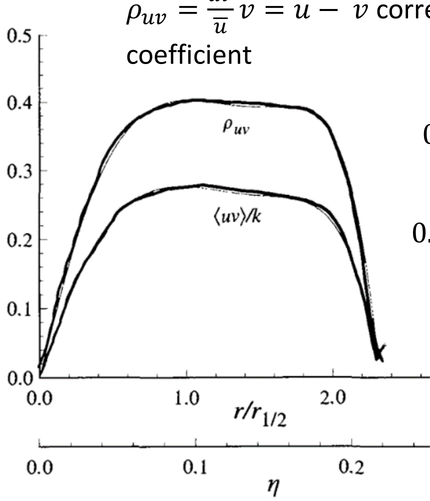


Fig. 5.9. Profiles of $\langle uv\rangle/k$ and the u–v correlation coefficient ρ_{uv} in the self-similar round jet. From the curve fit to the experimental data of Hussein *et al.* (1994).

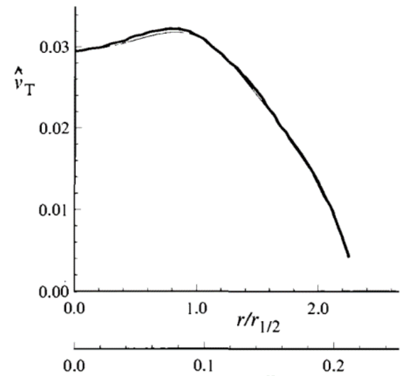


Fig. 5.10. The normalized turbulent diffusivity $\hat{\nu}_T$ (Eq. (5.34)) in the self-similar round jet. From the curve fit to the experimental data of Hussein *et al.* (1994).

$\hat{\nu}_t(\eta)$ fairly uniform over bulk of the jet, within 15% of 0.028 for $0.1 < r/r_{1/2} < 1.5$, afterwards decreases towards zero at the jet edge.

$$\nu_t = \frac{m}{s} \times m \rightarrow \nu_t = u'l$$

Where $u' = \overline{u^2}^{1/2}$.

$l =$ local length scale $l(x, r)$ = self-similar and $l/r_{1/2}$ within 15% of 0.12 for most of the jet ($0.1 < r/r_{1/2} < 2.1$).

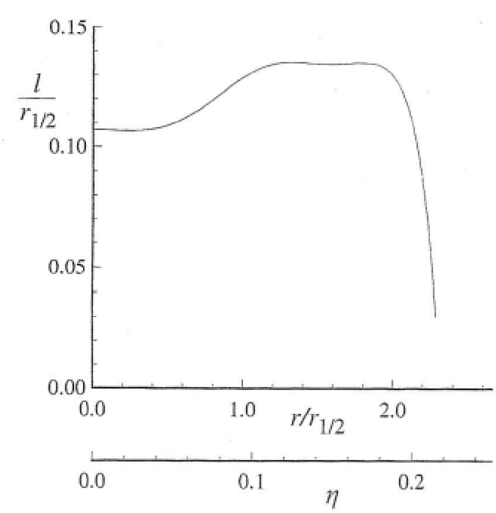


Fig. 5.11. The profile of the lengthscale defined by Eq. (5.35) in the self-similar round jet. From the curve fit to the experimental data of Hussein *et al.* (1994).

longitudinal $\bar{R}_1(x,r,s) \equiv \dfrac{\langle u(x+\frac{1}{2}s,r,\theta)u(x-\frac{1}{2}s,r,\theta)\rangle}{[\langle u(x+\frac{1}{2}s,r,\theta)^2\rangle\langle u(x-\frac{1}{2}s,r,\theta)^2\rangle]^{1/2}},$

transverse $\bar{R}_2(x,r,s) \equiv \dfrac{\langle u(x,r+\frac{1}{2}s,\theta)u(x,r-\frac{1}{2}s,\theta)\rangle}{[\langle u(x,r+\frac{1}{2}s,\theta)^2\rangle\langle u(x,r-\frac{1}{2}s,\theta)^2\rangle]^{1/2}},$

and then the corresponding integral lengthscales are

$$L_{11}(x,r) \equiv \int_0^\infty \bar{R}_1(x,r,s)\,\mathrm{d}s,$$

$$L_{22}(x,r) \equiv \int_0^\infty \bar{R}_2(x,r,s)\,\mathrm{d}s.$$

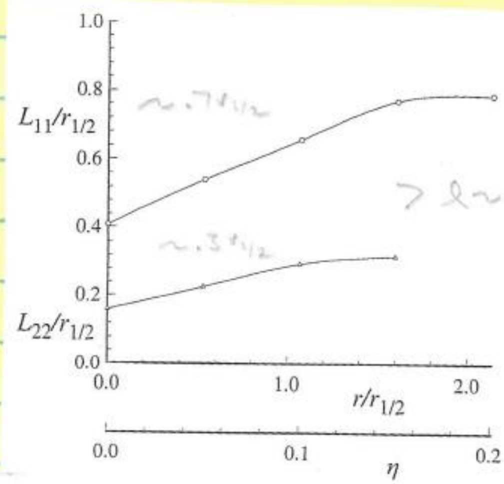


Fig. 5.12. Self-similar profiles of the integral lengthscales in the turbulent round jet. From Wygnanski and Fiedler (1969).

Significant correlation for s = separation distance > L_{11}

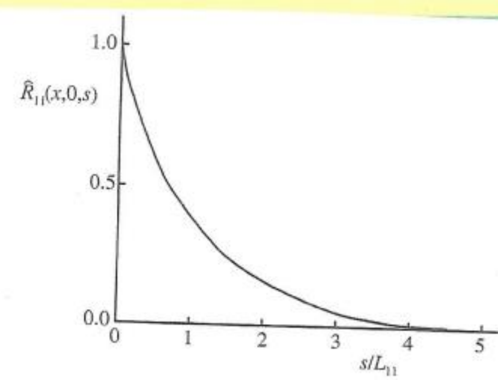


Fig. 5.13. The longitudinal autocorrelation of the axial velocity in the self-similar round jet. From Wygnanski and Fiedler (1969).

Longitudinal and transverse 2-point velcoity correlation.

L_{11} and L_{22} characterize distance over which the fluctuating velocities are correlated.

Mean momentum: Boundary-layer equations

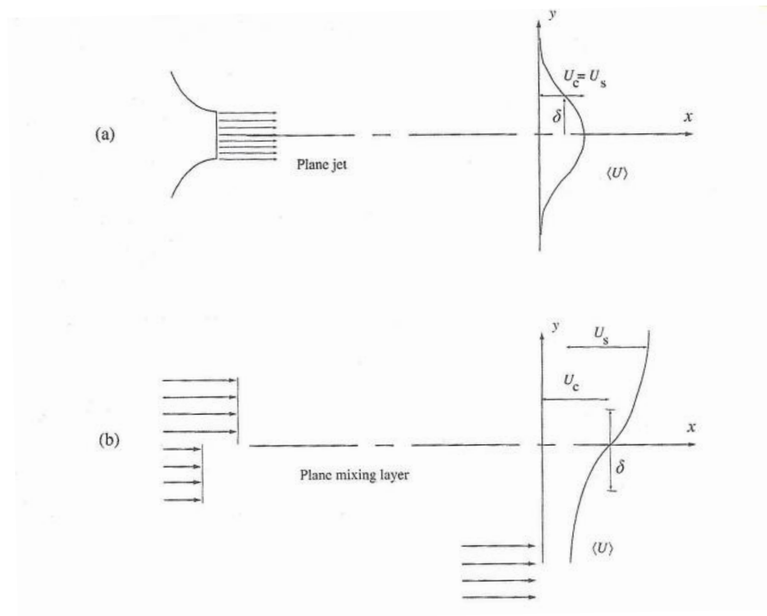
Dominant flow direction: x

$\overline{V} \approx 0.03|\overline{U}|$ and the flow spreads gradually $(dr_{1/2}/dx = S \approx 0.1)$

$\therefore \dfrac{\partial}{\partial x} \ll \dfrac{\partial}{\partial r}$

Consider statistically stationary 2D flows, with velocity components U, V, and W, with $\overline{W} = 0$.

As $y \rightarrow \infty$ no flow or uniform stream. It is possible to define $\delta(x)$ as the characteristic flow width, $U_c(x)$ the characteristic convective velocity, and $U_s(x)$ as the characteristic velocity difference.



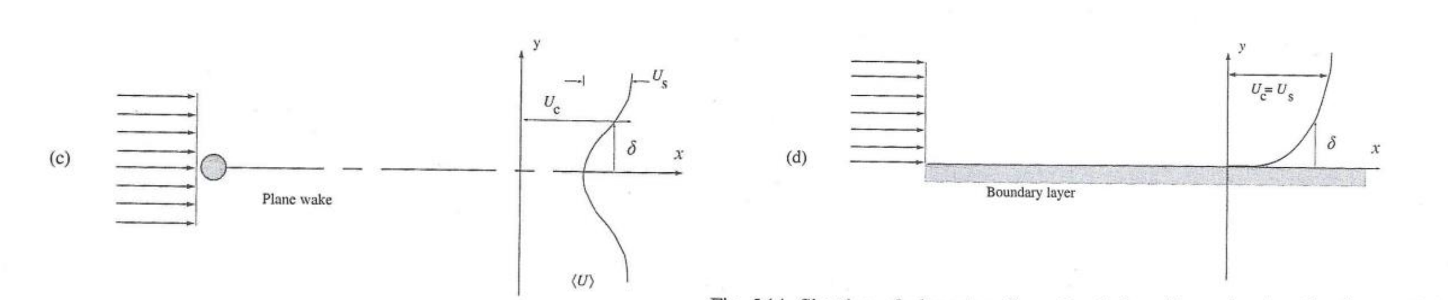


Fig. 5.14. Sketches of plane two-dimensional shear flows showing the characteristic flow width $\delta(x)$, the characteristic convective velocity U_c, and the characteristic velocity difference U_s.

Mean flow continuity and momentum equations:

$$\frac{\partial \overline{U}}{\partial x} + \frac{\partial \overline{V}}{\partial y} = 0$$

$$\overline{U}\frac{\partial \overline{U}}{\partial x} + \overline{V}\frac{\partial \overline{U}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu\frac{\partial^2 \overline{U}}{\partial x^2} + \nu\frac{\partial^2 \overline{U}}{\partial y^2} - \frac{\partial \overline{u^2}}{\partial x} - \frac{\partial \overline{uv}}{\partial y}$$

$$\overline{U}\frac{\partial \overline{V}}{\partial x} + \overline{V}\frac{\partial \overline{V}}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + \nu\frac{\partial^2 \overline{V}}{\partial x^2} + \nu\frac{\partial^2 \overline{V}}{\partial y^2} - \frac{\partial \overline{uv}}{\partial x} - \frac{\partial \overline{v^2}}{\partial y}$$

Turbulent y-momentum BL equation neglects convection and viscosity terms, and axial derivatives of RS:

$$\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y}+\frac{\partial \overline{v^2}}{\partial y}=0$$

Integrating between 0 and y, with $y \to \infty$, such that $\overline{p}(\infty)=p_0$ and $\overline{v^2}(\infty)=0$:

$$\frac{\overline{p}}{\rho}=\frac{p_0}{\rho}-\overline{v^2}$$

And the axial pressure gradient is:

$$\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}=\frac{1}{\rho}\frac{dp_0}{dx}-\frac{\partial \overline{v^2}}{\partial x}$$

For flows with quiescent or uniform free streams, dp_0/dx is zero. In general, it can be obtained in terms of the free-stream velocity by Bernoulli's equation.

The axial momentum equation becomes:

$$\overline{U}\frac{\partial \overline{U}}{\partial x}+\overline{V}\frac{\partial \overline{U}}{\partial y}=\nu\frac{\partial^2 \overline{U}}{\partial y^2}-\frac{1}{\rho}\frac{dp_0}{dx}-\frac{\partial \overline{uv}}{\partial y}-\frac{\partial}{\partial x}\left(\overline{u^2}-\overline{v^2}\right)$$

In turbulent free shear flows, $\nu\frac{\partial^2 \overline{U}}{\partial y^2}\sim \nu U_s^2/\delta^2 \sim Re^{-1}$ and is negligible, which is not the case for BL flows.

In laminar BL, $\nu\frac{\partial^2 U}{\partial x^2}\sim Re^{-1}$ and negligible. Comparable term in turbulent BL flows is $\frac{\partial}{\partial x}\left(\overline{u^2}-\overline{v^2}\right)$ → it can be neglected, but is $\sim 10\%$ dominant terms, i.e., not insignificant approximation.

Therefore, axial momentum equation becomes:

$$\overline{U}\frac{\partial \overline{U}}{\partial x} + \overline{V}\frac{\partial \overline{U}}{\partial y} = \nu\frac{\partial^2 \overline{U}}{\partial y^2} - \frac{\partial \overline{uv}}{\partial y}$$

For statistically axisymmetric, stationary non-swirling flows, the corresponding BL equations are:

$$\frac{\partial \overline{U}}{\partial x} + \frac{1}{r}\frac{\partial\left(r\overline{V}\right)}{\partial r} = 0$$

$$\overline{U}\frac{\partial \overline{U}}{\partial x} + \overline{V}\frac{\partial \overline{U}}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \overline{U}}{\partial r}\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(r\overline{uv}\right) \quad (1)$$

The mean pressure distribution is

$$\frac{\overline{p}}{\rho} = \frac{p_0}{\rho} - \overline{v^2} + \int_r^\infty \frac{\overline{v^2} - \overline{w^2}}{r'}dr'$$

Axisymmetric $W = 0$ equations.

Mass, momentum and energy fluxes

Neglecting viscous term and multiplying by r, Eq. (1) becomes:

$$\frac{\partial}{\partial x}\left(r\overline{U}^2\right) + \frac{\partial}{\partial r}\left(r\overline{U}\,\overline{V} + r\overline{uv}\right) = 0$$

Integrating with respect to r:

$$\frac{d}{dx}\int_0^\infty r\overline{U}^2 dr = -\left[\, r\overline{U}\,\overline{V} + r\overline{uv}\right]_0^\infty = 0$$

Since, for large r, $\overline{UV}$ and $\overline{uv}$ tend to zero more rapidly than r^{-1}.

The momentum flow rate of the mean flow is:

$$\dot{M} = \int_0^\infty 2\pi r\rho\,\overline{U}^2 dr \neq f(x) \quad (2)$$

And is conserved.

The mean velocity profile can be written as:

$$\overline{U}(x,r,0) = U_0(x)f(\xi)$$

Where

$$\xi = \frac{r}{r_{1/2}(x)}$$

Eq. (2) can be rewritten as:

$$\dot{M}(x) = 2\pi\rho\left(r_{1/2}U_0\right)^2 \int_0^\infty \xi f(\xi)^2 d\xi$$

Where the integral is a non-dimensional constant determined by the shape of the profile, but independet of x.
For the self-similar round jet, the mass flow rate is:

$$\dot{m}(x) = \int_0^\infty 2\pi r\rho\overline{U}dr = 2\pi r_{1/2}\rho(r_{1/2}U_0)\int_0^\infty \xi f(\xi)d\xi$$

The kinetic energy flow rate is:

$$\dot{E}(x) = \int_0^\infty \pi r\rho\overline{U}^3 dr = \frac{\pi\rho}{r_{1/2}}\left(r_{1/2}U_0\right)^3 \int_0^\infty \xi f(\xi)^3 d\xi$$

The integrals and $r_{1/2}U_0 \neq f(x)$, $\dot{m}(x) \propto x \propto r_{1/2}$ and $\dot{E}(x) \propto x^{-1} \propto r_{1/2}^{-1}$.

Self-similarity

$$\overline{U}(x,r) = U_0(x) f(\xi)$$

$$\overline{uv}(x,r) = U_0(x)^2 g(\xi)$$

$$U_0(x) \sim x^{-1}$$

$$\boxed{\begin{gathered} \xi = \frac{r}{r_{1/2}(x)} \\ \frac{dr_{1/2}}{dx} = S \rightarrow r_{1/2} \propto x \\ U_0(x) \propto x^{-1} \end{gathered}}$$

Assuming self-similar flow, and neglecting viscous term, Eq. (1) can be rewritten as (Pope Ex. 5.12):

$$[\xi f^2]\left\{\frac{r_{1/2}}{U_0}\frac{dU_0}{dx}\right\} - \left[\frac{df}{d\xi}\int_0^\infty \xi f(\xi) d\xi\right]\left\{\frac{r_{1/2}}{U_0}\frac{dU_0}{dx} + 2\frac{dr_{1/2}}{dx}\right\} = -\left[\frac{d}{d\xi}(\xi g)\right]$$

The terms in [ ] depend only on ξ, while those in { } depend only on x.

RHS = $f(\xi)$ ∴ LHS $\neq f(x)$, i.e,

$$\frac{r_{1/2}}{U_0}\frac{dU_0}{dx} = C \quad (3)$$

$$\frac{r_{1/2}}{U_0}\frac{dU_0}{dx} + 2\frac{dr_{1/2}}{dx} = C + 2S$$

Assuming { } $\neq 0$. Eliminating C from the above two equations:

$$\frac{dr_{1/2}}{dx} = S \quad r_{1/2}(x) = S(x - x_0)$$

Showing that the linear spreading of the jet is a consequence of self-similarity. Eq. (3) implies that $U_0(x) \sim x^n$, where $n = -1$, i.e., $\frac{dU_0}{dx} \sim x^{-2}$. Thus,

$$C = \frac{r_{1/2}}{U_0}\frac{dU_0}{dx} = -S$$

Uniform turbulent viscosity

Closure problem → ν_t is defined using eddy viscosity concept:

$$\overline{uv} = -\nu_t \overline{U_r}$$

Where for the self-similar round jet:

$$\nu_t(x,r) = r_{1/2}(x) U_0(x) \hat{\nu}_t(\eta)$$

And $\hat{\nu}_t(\eta)$ is within 15% of 0.028 for $0.1 < r/r_{1/2} < 1.5$ → assume $\hat{\nu}_t$ is constant, i.e., $\neq f(\eta)$ such that BL momentum equation becomes:

$$\overline{U}\frac{\partial \overline{U}}{\partial x} + \overline{V}\frac{\partial \overline{U}}{\partial r} = \frac{\nu_t}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \overline{U}}{\partial r}\right) \quad (4)$$

Where the viscous term has been neglected, although it could be retained by replacing ν_t with ν_{eff}.

$$\boxed{\begin{array}{c} T = 20°C \\ \nu_{water} = 10^{-6}\ m^2/s \\ \nu_{air} = 1.5 \cdot 10^{-5}\ m^2/s \end{array}}$$

Similarity solution round jet for uniform turbulent viscosity:

$$\overline{U} = \frac{1}{r}\frac{\partial \psi}{\partial r} \quad \overline{V} = -\frac{1}{r}\frac{\partial \psi}{\partial x}$$

This choice automatically satisfies the continuity equation. With x measured from the virtual origin (x_0) based $U_J/U_0(x)$ vs. x/d so that $\eta = r/x$:

$$\psi = \nu_t x F(\eta)$$

Where F is non-dimensional. Consequently,

$$\overline{U} = \frac{\nu_t}{x}\frac{F'}{\eta}$$

$$\overline{V} = \frac{\nu_t}{x}\left(F' - \frac{F}{\eta}\right)$$

$$\boxed{F' = \frac{dF}{d\eta}}$$

To satisfy the condition that $\overline{V} = 0$ on the axis $\rightarrow F(0) = F'(0) = 0$.

All the terms in Eq. (4) can be expresed as a function of F and its derivatives:

$$\frac{FF'}{\eta^2} - \frac{F'^2}{\eta} - \frac{FF''}{\eta} = \frac{d}{d\eta}\left(F'' - \frac{F'}{\eta}\right)$$

The LHS is $(-FF'/\eta)'$, so that the equation can be integrated to yield

$$FF' = F' - \eta F'' \quad (5)$$

And the constant of integration is zero due to BCs. Eq. (5) can be rewritte as:

$$\left(\frac{1}{2}F^2\right)' = 2F' - (\eta F')'$$

And integrated a second time, with integration constant equal to zero:

$$\frac{1}{2}F^2 = 2F - \eta F'$$

or

$$\frac{1}{2F - \frac{1}{2}F^2}\frac{dF}{d\eta} = \frac{1}{\eta}$$

Integrating a third time:

$$\frac{1}{2}\ln\left(\frac{F}{4 - F}\right) = \ln \eta + c$$

Setting $a = e^{2c}$, the solution is:

$$F(\eta) = \frac{4a\eta^2}{1 + a\eta^2}$$

By differentiating the solution, the mean velocity profile is obtained:

$$\overline{U} = \frac{8a\nu_t}{x}\frac{1}{(1+a\eta^2)^2}$$

And the centerline velocity is:

$$U_0(x) = \frac{8a\nu_t}{x} \qquad (6)$$

And the self-similar profile:

$$f(\eta) = \frac{1}{(1+a\eta^2)^2}$$

The constant a and ν_t can be related to $S = r_{1/2}/x$ (see Pope Ex. 5.3). Noting that $r = r_{1/2}$ corresponds to $\eta = S$:

$$\hat{\nu}_t = \frac{\nu_t}{r_{1/2}U_0} \qquad S = \frac{r_{1/2}}{x} = \frac{dr_{1/2}}{dx} = \text{constant}$$
$$\eta = \frac{x}{r} \rightarrow x = \frac{r}{\eta} \rightarrow S = \frac{r_{1/2}\eta}{r}$$
$$\text{If } r_{1/2} = r \rightarrow S = \eta$$

from the definition of $r_{1/2} \rightarrow \overline{U}\left(x, r_{1/2}(x)\right) = \frac{1}{2}U_0(x)$, it is required that

$$f(S) = 1/2$$

This leads to

$$f(S) = \frac{1}{2} = \frac{1}{(1+aS^2)^2} \rightarrow 1 + aS^2 = \sqrt{2}$$

$$a = \frac{\sqrt{2}-1}{S^2}$$

And from Eq. (6),

$$U_0(x) = \frac{8a\nu_t}{x} \rightarrow \nu_t = \frac{U_0 x S^2}{8(\sqrt{2}-1)} \rightarrow \hat{\nu}_t = \frac{\nu_t}{r_{1/2}U_0} = \frac{U_0 x S^2}{8(\sqrt{2}-1) r_{1/2} U_0}$$

$$\hat{\nu}_t = \frac{S^2}{8(\sqrt{2}-1)S} = \frac{S}{8(\sqrt{2}-1)}$$

Using the constant value $\hat{\nu}_t = 0.028$, the corresponding S is given by:

$$S = 8\left(\sqrt{2}-1\right)\hat{\nu}_t = 0.094$$

Which agrees with the profile shown in Fig. 5.15.

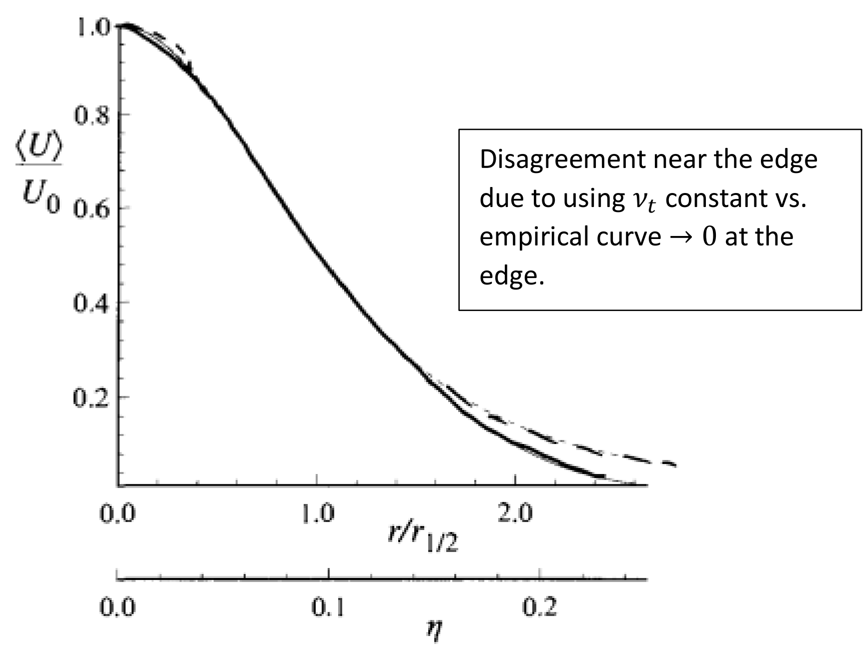


Fig. 5.15. The mean velocity profile in the self-similar round jet: solid line, curve fit to the experimental data of Hussein *et al.* (1994); dashed line, uniform turbulent viscosity solution (Eq. 5.82).

Turbulent Reynolds number:

$$R_T = \frac{U_0(x) r_{1/2}(x)}{\nu_t} = \frac{1}{\hat{\nu}_t} \approx 35$$

i.e., mean velocity in the turbulent round jet is the same as the velocity field in a laminar jet with $Re = 35$.

Kinetic Energy

$$E(\underline{x},t) = \frac{1}{2}\underline{U}(\underline{x},t)\cdot\underline{U}(\underline{x},t)$$

The ensemble averaged mean of E can be decomposed into two parts:

$$\langle E(\underline{x},t)\rangle = \overline{E}(\underline{x},t) + k(\underline{x},t)$$

Where $\overline{E}(\underline{x},t)$ is the kinetic energy of the mean flow

$$\overline{E}(\underline{x},t) = \frac{1}{2}\underline{\overline{U}}(\underline{x},t)\cdot\underline{\overline{U}}(\underline{x},t)$$

And $k(\underline{x},t)$ is the TKE:

$$k(\underline{x},t) = \frac{1}{2}\overline{u_i u_j}$$

The anisotropic tensor is:

$$a_{ij} = \overline{u_i u_j} - \frac{2}{3}k\delta_{ij}$$

And scales with k.

For turbulent jet the anisotropic part also scales with k: $\overline{uv} \approx 0.27k$ and bounded by $\overline{uv} < k$.

The equation for the evolution of the instantaneous kinetic energy is:

$$\frac{DE}{Dt} + \nabla\cdot\underline{T} = -2\nu S_{ij}S_{ij} \quad (7)$$

Where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

And

$$T_{ij} = \frac{U_i p}{\rho} - 2\nu U_j S_{ij}$$

Is the flux of energy.

Integrating Eq. (7) over a fixed control volume gives:

$$\underbrace{\frac{d}{dt}\iiint_V E dV}_{\boxed{1}} + \underbrace{\iint_A \left(\underline{U}E + \underline{T}\right) \cdot \underline{n} dA}_{\boxed{2}} = \underbrace{-\iiint_V 2\nu S_{ij} S_{ij} dV}_{\boxed{3}}$$

2) accounts for inflow, outflow, and work done on the control surface, i.;e., energy transfer.

3) ≥ 0, i.e., energy sink due to viscous dissipation: conversion of mechanical energy into heat.

Conclusion: no source energy in the flow.

The equation for the mean kinetic energy $\langle E\left(\underline{x}, t\right)\rangle$ is obtained by taking the mean of Eq. (7):

$$\frac{\overline{D}\langle E\rangle}{\overline{D}t} + \nabla \cdot \left(\langle \underline{u}E\rangle + \langle \underline{T}\rangle\right) = -\overline{\varepsilon} - \varepsilon$$

Where

$$\overline{\varepsilon} = 2\nu \overline{S}_{ij}\overline{S}_{ij} \qquad \varepsilon = 2\nu\overline{s_{ij}s_{ij}}$$

And

$$\overline{S}_{ij} = \langle S_{ij}\rangle = \frac{1}{2}\left(\frac{\partial \overline{U_i}}{\partial x_j} + \frac{\partial \overline{U_j}}{\partial x_i}\right)$$

$$s_{ij} = S_{ij} - \overline{S}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$\overline{\varepsilon} \sim Re^{-1}$ and $\ll \varepsilon \rightarrow$ negligible.

The equations for $\overline{E}$ and k can be written as:

$$\frac{\overline{D}\,\overline{E}}{\overline{D}t} + \nabla \cdot \underline{\overline{T}} = -P - \overline{\varepsilon}$$

$$\frac{\overline{D}k}{\overline{D}t} + \nabla \cdot \underline{T}' = -P - \varepsilon$$

Where

$$\overline{T_i} = \frac{\langle U_j \rangle}{\langle u_i u_j \rangle} + \frac{\langle U_i \rangle \langle p \rangle}{\rho} - 2\nu \langle U_j \rangle \overline{S}_{ij}$$

$$T_i{}' = \frac{1}{2} \langle u_i u_j u_j \rangle + \frac{\langle u_i p' \rangle}{\rho} - 2\nu \langle u_j s_{ij} \rangle$$

$$p' = p - \langle p \rangle$$

And

$$P = -\langle u_i u_j \rangle \frac{\partial \overline{U_i}}{\partial x_j}$$

Represents production, i.e., source of energy = action of the mean velocity gradient working against RS: removes energy from $\overline{E}$ and transfers it to k.

Production

1) Only the symmetric part of the velocity gradient affects production, i.e.,

$$P = -\langle u_i u_j \rangle \overline{S}_{ij}$$

Since product of symmetric (RS) and antisymmetric tensor is zero.

2) Only the anisotropic part of RS affects production, i.e.,

$$P = -a_{ij}\overline{S}_{ij}$$

Where: $a_{ij} = \langle u_i u_j \rangle - \frac{2}{3}k\delta_{ij}$.

$$-\langle u_i u_j \rangle \overline{S}_{ij} = -(a_{ij} + \frac{2}{3}k\delta_{ij})\,\overline{S}_{ij}$$

$$\frac{2}{3}k\delta_{ij}\overline{S}_{ij} = \frac{2}{3}k\frac{1}{2}\left(\frac{\partial \overline{U_i}}{\partial x_i} + \frac{\partial \overline{U_i}}{\partial x_i}\right) = 0$$

3) According to the turbulent viscosity hypothesis: $a_{ij} = -2\nu_t\overline{S}_{ij}$ the production term is:

$$P = 2\nu_t\overline{S}_{ij}\overline{S}_{ij} = \overline{\varepsilon}\nu_t/\nu$$

4) For BL flow, only mean velocity gradient given by $\overline{U}_y$ or $\overline{U}_r$:

$$P = -\overline{uv}\frac{\partial \overline{U}}{\partial y}$$

5) Using both BL and turbulent viscosity hypothesis:

$$P = \nu_t\left(\frac{\partial \overline{U}}{\partial y}\right)^2$$

Dissipation

$$\varepsilon = 2\nu\langle s_{ij}s_{ij}\rangle$$

Fluctuating velocity gradients work against fluctuating rate of strain, transform KE into internal energy.

$$s_{ij} = S_{ij} - \langle S_{ij}\rangle = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

For self-similar jet $\overline{U}/U_0$ and $\overline{u_i u_j}/U_0^2$ are function of $\xi = r/r_{1/2}$ and independent of Re

Consequently,

$$\hat{P} = \frac{P}{U_0^3/r_{1/2}} \approx -\frac{\overline{uv}}{U_0^2}\frac{r_{1/2}}{U_0}\frac{\partial\overline{U}}{\partial r}$$

Also self-similar and independent of Re.

Dk/Dt and P scale with $U_0^3/r_{1/2} \to \hat{\varepsilon} = \varepsilon/(U_0^3/r_{1/2})$ also self-similar and independent from Re.

Suppose have two jets with same U_J and d, but different ν_a and ν_b, e.g., air and water. At same x, $U_0(x)$ and $r_{1/2}(x)$ same since

$$\frac{U_0(x)}{U_J} = \frac{B}{(x - x_0)/d}$$

$$r_{1/2}(x) = S(x - x_0)$$

$$\therefore \varepsilon_a(x,r) = \varepsilon_b(x,r) = \hat{\varepsilon}\left(\frac{r}{r_{1/2}(x)}\right)\frac{U_0^3}{r_{1/2}}$$

However, $\varepsilon = 2\nu\langle s_{ij}s_{ij}\rangle \propto \nu$ which is different. Explanation is s_{ij} are different: higher Re finer scale of small structure $\to$ steeper gradients $\to$ larger s_{ij}.

Kolmogorov: universal equilibrium range small scale motions only depend on ε and ν.

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \quad \tau_\eta = \left(\frac{\nu}{\varepsilon}\right)^{1/2} \quad u_\eta = (\nu\varepsilon)^{1/4}$$

Kolmogorov scales vary with $Re_0 = U_0 r_{1/2}/\nu = SBU_j d/\nu$ ($S{\sim}0.1, B{\sim}6$), whereas $U_0(x)$ and $r_{1/2}(x)$ do not.

$$\frac{\eta}{r_{1/2}} = \frac{\left(\frac{\nu^3}{\varepsilon}\right)^{1/4}}{r_{1/2}} = \frac{\nu^{3/4}}{\varepsilon^{1/4} r_{1/2}} = \frac{\nu^{3/4} {r_{1/2}}^{1/4}}{\hat{\varepsilon}^{1/4} U_0^{3/4} r_{1/2}} = Re_0^{-3/4} \hat{\varepsilon}^{-1/4}$$

Similarly,

$$\tau_\eta/(r_{1/2}/U_0) = Re_0^{-1/2} \hat{\varepsilon}^{-1/2}$$
$$\frac{u_\eta}{U_0} = Re_0^{-1/4} \hat{\varepsilon}^{1/4}$$

i.e., smallest motions decrease in size and timescale as Re increases.
Note that

$$\frac{\eta u_\eta}{\nu} = 1$$

i.e., however large Re_0, Re of smallest scales is unity and motions at these small scales are strongly dependent on ν.

$$\nu\left(\frac{u_\eta}{\eta}\right)^2 = \frac{\nu}{\tau_\eta^2} = \frac{\nu}{\nu/\varepsilon} = \varepsilon, \text{ i.e.}, \left(\frac{u_\eta}{\eta}\right)^2 = \varepsilon/\nu$$

i.e., velocity gradients $\propto$ to the inverse of the turnover time such that ε is independent of ν.

$\langle s_{ij}s_{ij}\rangle$ scales as τ_η^{-2}, i.e., inversely proportional to ν, so that

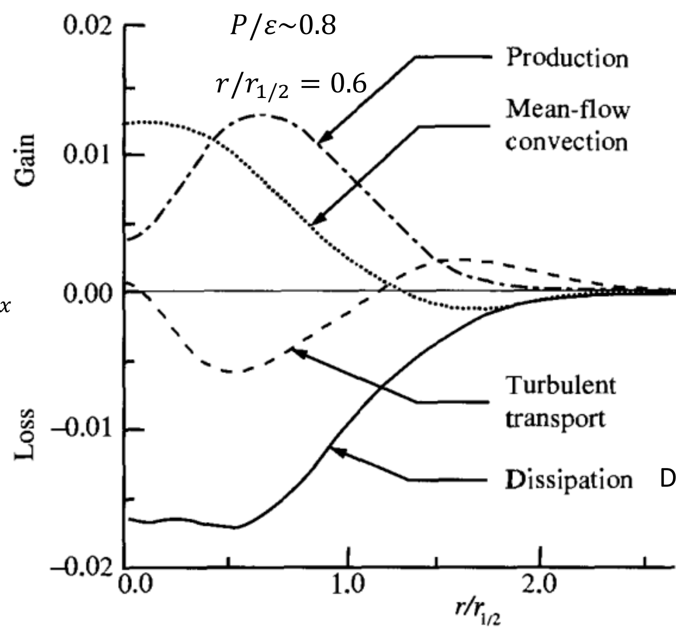
$$\varepsilon_a = \nu_a \underbrace{\langle s_{ij}s_{ij}\rangle_a}_{\boxed{\text{scales } \nu_a^{-1}}} = \varepsilon_b = \nu_b \underbrace{\langle s_{ij}s_{ij}\rangle_b}_{\boxed{\text{scales } \nu_b^{-1}}}$$

TKE Budget

$$\frac{\overline{D}k}{\overline{D}t} + \nabla \cdot \underline{T}' = -P - \varepsilon \quad (8)$$

Fig. 5.16 shows the terms of Eq. (8) divided by $U_0^3/r_{1/2}$.

On centerline predicted $P = 0$ since $\overline{uv}\overline{U}_r = 0$ and $\left(\overline{u^2} - \overline{v^2}\right)\overline{U}_x$ neglected in BL approximation



Dominant

Fig. 5.16. The turbulent-kinetic-energy budget in the self-similar round jet. Quantities are normalized by U_0 and $r_{1/2}$. (From Panchapakesan and Lumley (1993a).)

P and $\overline{D}k/\overline{D}t$ $\pm 20\%$ EFD accuracy, while other terms have large uncerstainty and the results differs by a factor two or more in different experiments.
At the edge $P/\varepsilon = 0$ such that:

$$\nabla \cdot \underline{T}' = -\varepsilon - \langle \underline{u} \rangle \cdot \nabla k$$

Comparison of scales

$\tau =$ time to dissipate k at rate ε.
$\tau_p =$ time to produce k at rate $P =$ flight time from τ_J (virtual origin) of a particle moving on the centerline at speed $U_0(x) \approx 3\tau_s$ time scale imposed shear $S^{-1} \rightarrow$ turbulence is long-lived.
L_{11} and L_{22} have physical significance, while $l' = \nu_t/u'$ and $L = k^{3/2}/\varepsilon$ do not.

Pseudo-dissipation

$$\tilde{\varepsilon} = \nu \langle \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \rangle = \varepsilon - \underbrace{\nu \frac{\partial^2 \langle u_i u_j \rangle}{\partial x_i \partial x_j}}_{\boxed{\text{usually small}}}$$

This gives an alternative form of the TKE equation:

$$\frac{\overline{D}k}{\overline{D}t} + \frac{\partial}{\partial x_i}\left[\frac{1}{2}\langle u_i u_j u_j \rangle + \frac{\langle u_i p' \rangle}{\rho}\right] = \nu \nabla^2 k + P - \tilde{\varepsilon}$$

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_1}\left[\frac{1}{2}\langle u_i u_j u_j \rangle + \frac{\langle u_i p' \rangle}{\rho}\right] = \nu \nabla^2 k + P - \varepsilon$$

vs.

$$\frac{\overline{D}k}{\overline{D}t} + \nabla \cdot \underline{T}' = P - \varepsilon$$

$$T_i' = \frac{\partial}{\partial x_i}\left[\frac{1}{2}\langle u_i u_j u_j \rangle + \frac{\langle u_i p' \rangle}{\rho} - 2\nu \langle u_j s_{ij} \rangle\right]$$

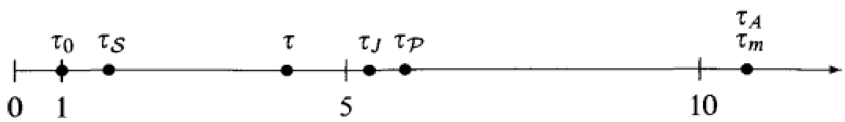


Fig. 5.17. Timescales in the self-similar round jet in units of τ_0. See Table 5.2 for definitions.

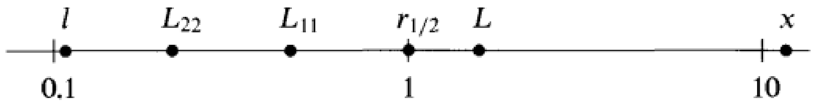


Fig. 5.18. Lengthscales in the self-similar round jet in units of $r_{1/2}$. L_{11} and L_{12} are the longitudinal and lateral integral scales; $L \equiv k^{3/2}/\varepsilon$; $l = \nu_T/u'$; evaluated at $r/r_{1/2} \approx 0.7$. (Note the logarithmic scale.)

Table 5.2. *Timescales, rates, and ratios in the self-similar round jet: the first four entries are evaluated from $U_0(x)$, $r_{1/2}(x)$ and the spreading rate S; the remaining entries are estimated from experimental data at $r/r_{1/2} \approx 0.7$, where $\langle uv \rangle$ and $|\partial\langle U\rangle/\partial r|$ peak*

Definition	Description	Timescale	Value in self-similar round jet, normalized by τ_0
$\tau_0 = r_{1/2}/U_0$	Reference timescale used for normalization	τ_0	1
$\tau_J = \frac{1}{2}x/U_0$	Mean flight time from virtual origin	τ_J	5.3
$\Omega_m = \dfrac{U_0}{\dot{m}}\dfrac{d\dot{m}}{dx}$	Entrainment rate	$\tau_m = \Omega_m^{-1}$	10.6
$\Omega_A = \left\|\dfrac{dU_0}{dx}\right\|$	Axial strain rate	$\tau_A = \Omega_A^{-1}$	10.6
$\mathcal{S} = (2\bar{S}_{ij}\bar{S}_{ij})^{1/2} \approx \left\|\dfrac{\partial\langle U\rangle}{\partial r}\right\|$	Strain rate	$\tau_{\mathcal{S}} = \mathcal{S}^{-1}$	1.7
$\omega = \varepsilon/k$	Turbulence decay rate	$\tau = \omega^{-1} = k/\varepsilon$	4.5
$\Omega_{\mathcal{P}} = \mathcal{P}/k$	Turbulence-production rate	$\tau_{\mathcal{P}} = \Omega_{\mathcal{P}}^{-1}$	5.7
$\mathcal{P}/\varepsilon$	Ratio of production to dissipation		0.8
$\mathcal{S}/\omega = \mathcal{S}k/\varepsilon = \tau/\tau_{\mathcal{S}}$	Ratio of strain rate to decay rate		2.6

Plane jet

Statistically 2D. In EFD, rectangular nozzle with height d (y) and width w (z) and flows in x direction.

$w/d \gg 1 \approx 50$ such that for $z = 0$ the flow is statistically 2D and free of end effects, for x/w not large.

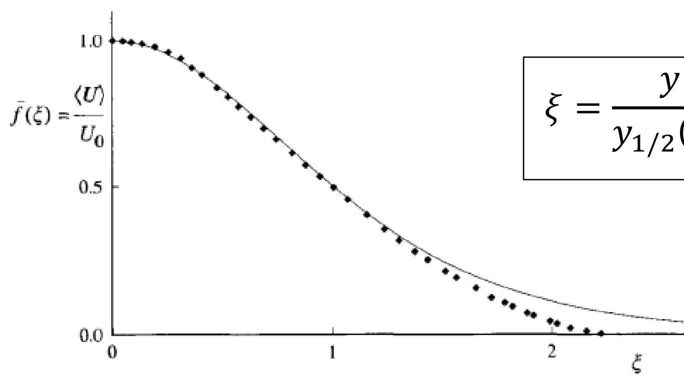
Centerline velocity:

$$U_0(x) = \langle U(x, 0, 0) \rangle$$

Half-width:

$$\frac{1}{2} U_0(x) = \langle U\big(x, y_{1/2}(x), 0\big) \rangle$$

Mean velocity and RS self-similar for $x/d > 40$, when scaled with $U_0(x)$ and $y_{1/2}(x)$.



$$\boxed{\xi = \frac{y}{y_{1/2}(x)}}$$

Fig. 5.19. The mean velocity profile in the self-similar plane jet. Symbols, experimental data of Heskestad (1965); line, uniform turbulent-viscosity solution, Eq. (5.187) (with permission of ASME).

Profile shapes and scales RS comparable with round jet.

$$\frac{dy_{1/2}}{dx} = S \approx 0.1$$

$U_0(x) \approx x^{-1/2}$ vs. x^{-1} round jet due differences similarity transformation.

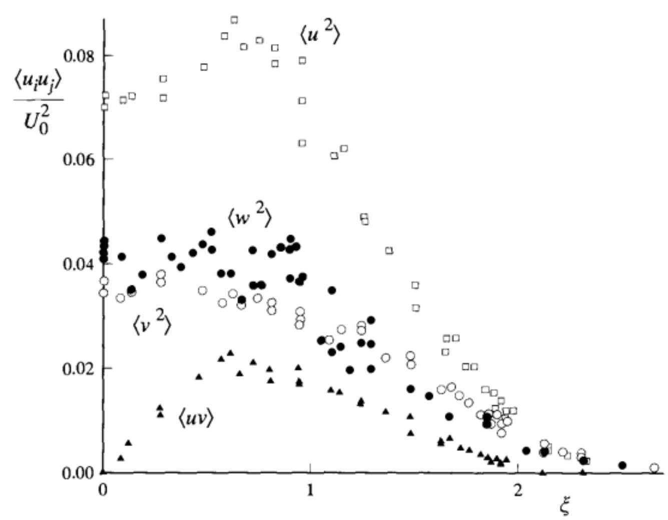


Fig. 5.20. Reynolds-stress profiles in the self-similar plane jet. From the measurements of Heskestad (1965) (with permission of ASME).

Conservative form BL equation neglecting viscous term:

$$\frac{\partial}{\partial x}\langle U\rangle^2+\frac{\partial}{\partial y}(\langle U\rangle\langle V\rangle)=-\frac{\partial}{\partial y}\langle uv\rangle$$

Integrating with respect to y, gives:

$$\frac{d}{dx}\int_{-\infty}^{\infty}\langle U\rangle^2 dy=0$$

Since $\langle U\rangle$ and $\langle uv\rangle$ are zero for $y\to\pm\infty$. Hence, momentum flow rate per unit span:

$$\dot{M}=\int_{-\infty}^{\infty}\rho\langle U\rangle^2 dy=\text{constant}\neq f(x)$$

In the self-similar region:

$$\boxed{\xi=\frac{y}{y_{1/2}(x)}}$$

1) $\langle U\rangle=U_0(x)f(\xi)$

And the momentum flow rate is:

$$\dot{M}=\rho U_0(x)^2 y_{1/2}(x)\int_{-\infty}^{\infty}f(\xi)^2 d\xi$$

$$U_0(x)^2 y_{1/2}(x)\neq f(x)$$

i.e.,

$$2U_0\frac{dU_0}{dx}y_{1/2}+U_0^2\frac{dy_{1/2}}{dx}$$

$$\frac{y_{1/2}}{U_0}\frac{dU_0}{dx}=-\frac{1}{2}\frac{dy_{1/2}}{dx}$$

2) $\langle uv\rangle=U_0^2 g(\xi)$

Plugging in 1) and 2) into BL equation, gives:

$$\frac{1}{2}\frac{dy_{1/2}}{dx}\underbrace{\left(f^2 + f'\int_0^{\xi} f d\xi\right)}_{\boxed{\neq f(x)}} = g' \quad (9)$$

$\therefore dy_{1/2}/dx \neq f(x)$, i.e., S is constant and $U_0 \sim x^{-1/2}$.

3) $\nu_t = U_0(x) y_{1/2}(x) \widehat{\nu_t}(\xi)$

$$\nu_t \sim x^{1/2}$$

$$Re_0 = \frac{U_0(x) y_{1/2}(x)}{\nu} \sim x^{1/2}$$

$$R_T = \frac{U_0(x) y_{1/2}(x)}{\nu_t(x, y_{1/2})} \neq f(x)$$

For $\widehat{\nu_t} = \text{constant}$, Eq. (9) becomes:

$$\frac{1}{2}S\left(f^2 + f'\underbrace{\int_0^{\xi} f d\xi}_{\boxed{F(\xi)}}\right) = -\widehat{\nu_t} f'' \quad (10)$$

Since $f(\xi)$ is an even function, $F(\xi)$ is odd:
Even:

$$\begin{gathered} f(x) = f(-x) \\ \rightarrow z = -x \rightarrow f(x) = f(z) \\ \rightarrow f'(x) = f'(z)\frac{dz}{dx} = -f'(z) \\ = -f'(-x) \end{gathered}$$

i.e., odd → F odd since f even.

$$F(0) = F''(0) = 0$$

Eq. (10) becomes:

$$\frac{1}{2}S[F'^2 + F''F] = -\hat{\nu}_t F''' \quad (11)$$

Noting that:

$$F'^2 + F''F = (FF')' = \frac{1}{2}(F^2)''$$

And integrating Eq. (11) twice:

$$\frac{1}{4}SF^2 = -\hat{\nu}_t F' + a + b\xi \quad (12)$$

F^2 and F' even $\rightarrow b = 0$

$$F'(0) = 1 \rightarrow a = \hat{\nu}_t$$

Defining:

$$\alpha = \sqrt{\frac{S}{4\hat{\nu}_t}} \quad (13)$$

Eq. (12) then becomes:

$$F' = 1 - (\alpha F)^2$$

Integrating:

$$F = \frac{1}{\alpha}\tanh(\alpha\xi)$$
$$f = F' = \mathrm{sech}^2(\alpha\xi)$$

$$\langle U \rangle = U_0 f(\xi) \rightarrow \frac{\langle U \rangle}{U_0} = \frac{1}{2} = f(1) = \operatorname{sech}^2(\alpha)$$
$$\alpha = \frac{1}{2} \ln\left(1 + \sqrt{2}\right)^2 \approx 0.88$$

This, together with Eq. (13), relates S to $\widehat{\nu_t}$:

$$S = \left[\ln\left(1 + \sqrt{2}\right)^2\right]^2 \widehat{\nu_t}$$

Or

$$R_T = \frac{1}{\widehat{\nu_t}} = \frac{\left[\ln\left(1 + \sqrt{2}\right)^2\right]^2}{S} \approx 31$$

Using $S = 0.1$.