Chapter 6: Turbulent Transport and its Modeling

Part 2: Lagrangian Analysis of Turbulent Transport

Gradient transport law requires mixing length (l) $\ll$ region over which mean velocity can be assumed linear. For turbulent transport, l determined by eddy size/action $\gg$ molecular mean free path as per molecular viscosity and viscous shear stress tensor.

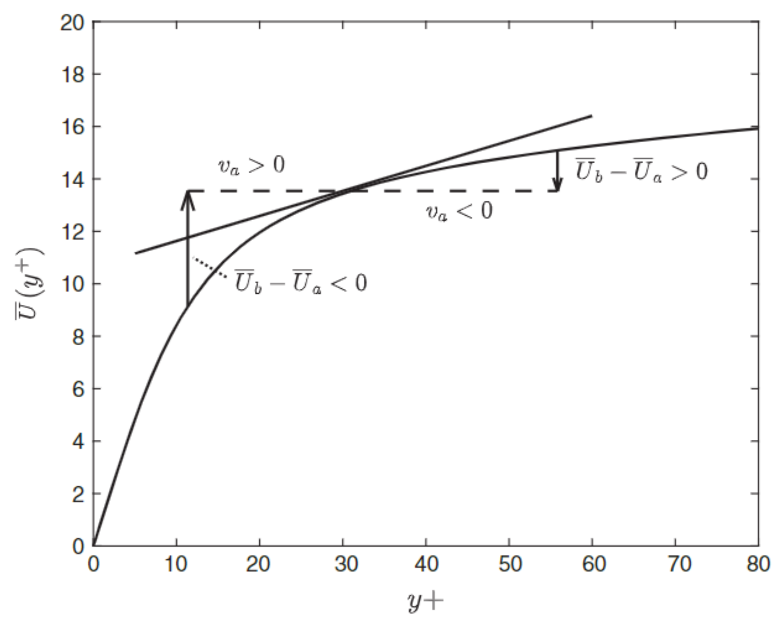


Figure 6.2 A local linear approximation to the mean velocity field $\overline{U}$ at a point **a** in a channel flow is inappropriate for fluid particles traveling significant distances during the mixing time. Fluid particles traveling toward the wall located at $y^+ = 0$ have $v_a < 0$, $\overline{U}_b - \overline{U}_a > 0$, and vice versa for particles traveling away from the wall.

Figure shows linear approximation mean velocity profile is only valid for very small distances.

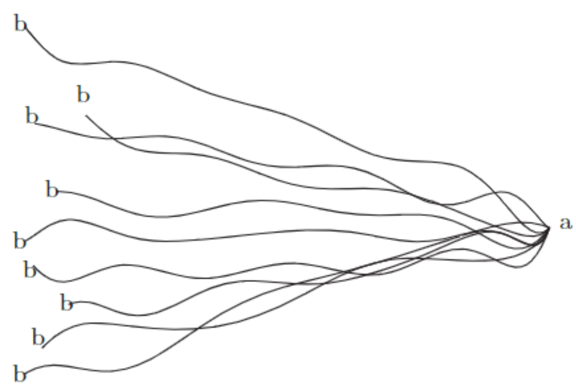
However, concept that turbulent mixing in which fluid particles carry momentum from initial to final points over a mixing time to cause net momentum transport may have some validity.

To analyze the validity of

$$\sigma_{12}^{T} = -\rho\overline{uv} = \mu_T \frac{\partial \overline{U}}{\partial y}$$

Using $\underline{U} = \underline{\overline{U}} + \underline{u}$ nomenclature, the turbulent motions that cause u and v to be correlated are explored using DNS data for channel flow.

Figure 6.3 Ensemble of paths, each with a different initial position **b**, arriving at **a**.



Consider set of particles arriving at $\underline{a}$ at time t, which originated at $\underline{b}$ following various paths $\underline{X}(s)$ such that $\underline{X}(t) = \underline{a}$ and $\underline{X}(t-\tau) = \underline{b}$ where $\underline{X}(s)$ and $\underline{b}$ are a random ensemble of realizations. $\tau > 0 =$ motion at earlier times than t. Note that $s =$ time such that $s < t =$ motion prior arrival at a and $s > t =$ future time, i.e., s at $b = t - \tau$ and at $a = t$.

$$\frac{d\underline{X}(s)}{ds} = \underline{U}\big(\underline{X}(s), s\big) = \underline{\overline{U}}\big(\underline{X}(s), s\big) + \underline{u}\big(\underline{X}(s), s\big) \quad (1)$$

Lagrangian

Eulerian

Reynolds decomposition using ensemble average where both terms are random since $\underline{X}(s)$

At time t:

$$\underline{U_a} = \underline{\overline{U}_a} + \underline{u_a} \quad (2)$$

At time $t - \tau$:

$$\underline{U_b} = \underline{\overline{U}_b} + \underline{u_b} \quad (3)$$

Integration of Eq. (1) between $t-\tau$ and t gives:

$$\int_{t-\tau}^{t} d\underline{X}(s) = \int_{t-\tau}^{t} \underline{U}\big(\underline{X}(s), s\big) ds$$

$$\underline{a} - \underline{b} = \underline{L}$$

Where $\underline{L}$ represents the change in particle position from $\underline{b}$ to $\underline{a}$ in time τ.

Eq. (2) minus Eq. (3) gives:

$$u_a = u_b + \underbrace{\left(\overline{U}_b - \overline{U}_a\right)}_{\boxed{1}} + \underbrace{\left(U_a - U_b\right)}_{\boxed{2}} \qquad (4)$$

Scalar version of Eqs. (2) and (3) for x-component

Where $\overline{U}_b$ represents the ensemble average = sum of all b velocities divided by number of b particles; and similarly, for $\overline{U}_a$.

Eq. (4) expresses u_a in terms of value at earlier time u_b plus factors that led to its change.

1) Change in local mean (ensemble average) velocity field between a and b.
2) Change in instantaneous velocity due to acceleration or deceleration caused by pressure or viscous forces = difference in instantaneous values of velocities.

Thus, even for $U_b = U_a$, i.e., non-accelerating flow $u_a \neq u_b$ due to changes in local mean velocity

Substituting Eq. (4) into $\overline{u_a v_a}$ (time average) yields

$$\overline{u_a v_a} = \underbrace{\overline{u_b v_a}}_{\boxed{1}} + \underbrace{\overline{v_a\left(\overline{U}_b - \overline{U}_a\right)}}_{\boxed{2}} + \underbrace{\overline{v_a\left(U_a - U_b\right)}}_{\boxed{3}} \qquad (5)$$

Note for statistically stationary flow time average = ensemble average.

In Eq. (5) $\overline{u_a v_a}$ represents the Reynolds stress σ_{12}^{T}, such that $u_a v_a$ is time averaged between $t - \tau$ and t.

For small τ, $\overline{u_b v_a}$ converges to $\overline{u_a v_a}$, whereas for large τ, $\overline{u_b v_a}$ goes to zero, which gives an upper limit to the mixing time.

Term 2 is referred to as displacement transport term $= \Phi_{\mathrm{D}} = \overline{v_a\left(\overline{U}_b - \overline{U}_a\right)}$ and represents momentum transport due to eddy mixing over time interval for which $\overline{u_a v_a}$ is correlated. If locally mean velocity is linear this term will yield gradient diffusion/viscosity model, as will be shown later using its Taylor series representation.

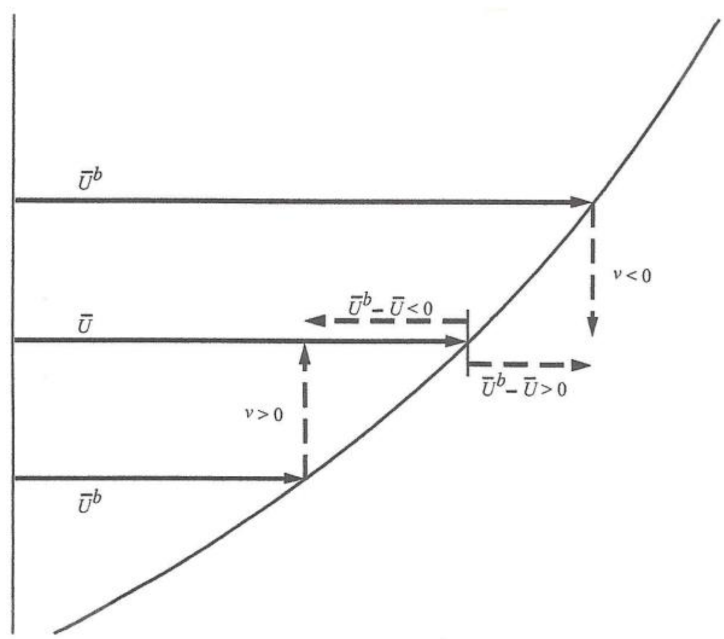


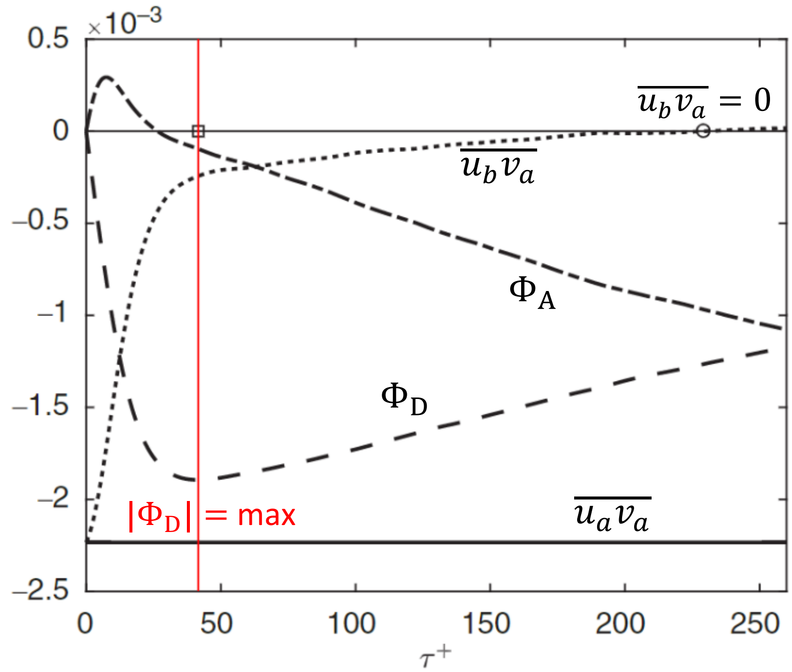
Fig. 6.7 Source of displacement transport correlation.

In this figure
$\overline{U} = \overline{U}_a$

$\Phi_{\mathrm{D}} < 0$ since for $v_a < 0$, $\overline{U}_b - \overline{U}_a > 0$ and for $v_a > 0$, $\overline{U}_b - \overline{U}_a < 0$.

Term 3 is referred as Φ_{A} and is absent in molecular model (and gradient model) since molecules are assumed to retain their momentum over the mixing time. In fact, this term represents the changes in momentum of fluid particles due to viscous and pressure forces.

Channel Flow DNS



$$y^+ = \frac{yU_\tau}{\nu}$$

$$U_\tau = \left[\nu \frac{d\overline{U}}{dy}(0)\right]^{1/2} = \sqrt{\frac{\tau_w}{\rho}}$$

U_τ = friction velocity

For all τ^+ values, the sum of $\overline{u_b v_a}$, Φ_A and Φ_D must equal $\overline{u_a v_a}$, but magnitude of each term varies with τ^+.

$\overline{u_b v_a}$ goes to zero for large $\tau^+ = \frac{\tau U_\tau}{y}$, whereas for $\tau^+ = 0$, $\overline{u_a v_a} = \overline{u_b v_a}$.

Φ_A trend for short-term $(\tau^+ < 100)$ strongly depends on y^+, although $\Phi_A(\tau^+ = 0) = 0$. For large τ^+, independent of y^+, its value tends to $\overline{u_a v_a}$.

$$\Phi_A = \overline{v_a(U_a - U_b)} = \overline{v_a\left(\overline{U_a} + u_a\right)} - \overline{v_a\left(\overline{U_b} + u_b\right)}$$

$$= \underbrace{\overline{v_a \overline{U_a}}}_{= 0 \text{ for all } \tau^+} + \overline{v_a u_a} - \underbrace{\overline{v_a \overline{U_b}}}_{\text{Goes to 0 as } \tau^+ \to \infty} - \underbrace{\overline{v_a u_b}}_{\text{Goes to 0 as } \tau^+ \to \infty}$$

No correlation between a and b as their distance increases

$$\boxed{\overline{f'g} = 0}$$

Φ_D decreases towards a minimum close in value to $\overline{u_a v_a}$ before rising back towards zero. For large τ^+, $\Phi_D \to 0$

$$\Phi_D = \overline{v_a\left(\overline{U_b} - \overline{U_a}\right)} = \underbrace{\overline{v_a \overline{U}_{b_\infty}}}_{\text{Goes to 0 as } \tau^+ \to \infty} - \underbrace{\overline{v_a \overline{U_a}}}_{= 0 \text{ for all } \tau^+}$$

Φ_D minimum same order of magnitude as $\overline{u_a v_a}$ at time τ_D = mixing time as reflects most closely idea of gradient hypothesis.

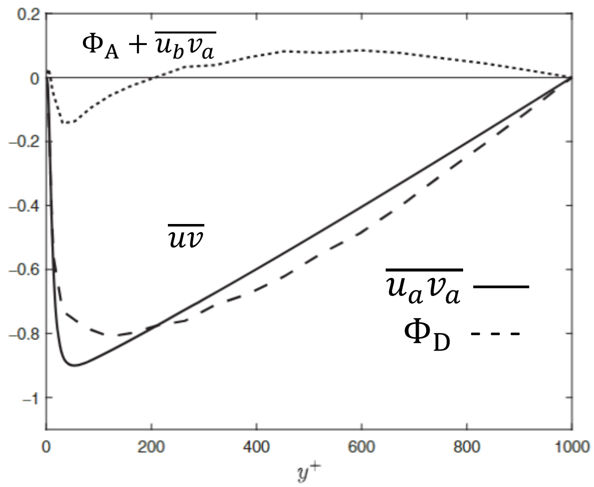


Figure 6.5 Evaluation of Eq. (6.24) at τ_D computed across the channel. —, $\overline{u_a v_a}$; – –, $\overline{(\overline{U}_b - \overline{U}_a) v_a}$; · · · , $\overline{(U_a - U_b) v_a} + \overline{u_b v_a}$.

Lower half of channel

$0 < y^+ < 1000$

Time averaging from t-τ_D to t. Showing that gradient model works for this flow and conditions.

$\overline{uv} < 0$ = transport u towards wall

Shows $\overline{uv} \approx \Phi_\mathrm{D}$ and Term 1 + Term 3 only small effect at time τ_D.

Conclusion:

For averaging $0 \le \tau^+ \le \tau_D$: $\Phi_\mathrm{D} \approx \overline{u_a v_a}$ and Φ_A relatively small

Whereas for averaging $0 \le \tau^+ \le \infty$: Φ_A=$\overline{u_a v_a}$ and $\Phi_\mathrm{D} = 0$.

Since τ represents the time/spatial difference between $\underline{a}$ and $\underline{b}$, for τ=τ_D, Φ_D defines the mixing time and can be used to provide a model for $\overline{uv}$, which is related to mean flow gradient transport.

Transport Producing Motions

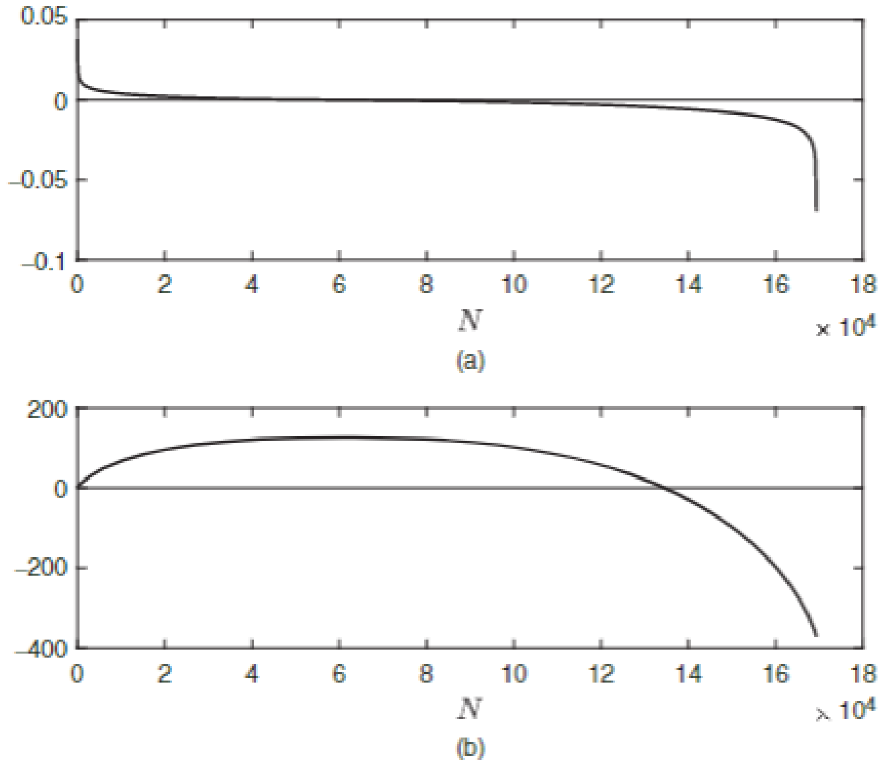


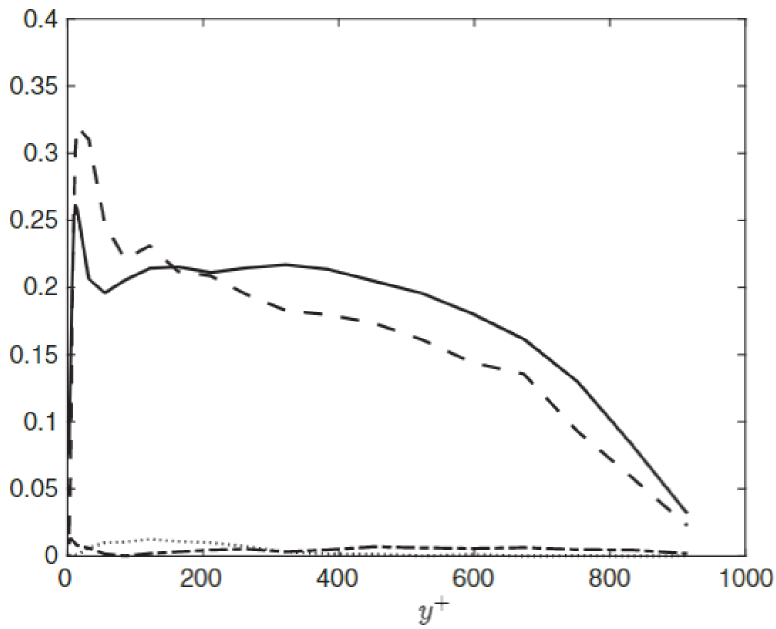
Figure 6.6 Contributions to $\overline{uv}$ at $y^+ = 84.8$ from a data set consisting of 169,344 points in the lower channel half. (a) Individual contributions ranked from largest to smallest. (b) Cumulative sum of contributions in (a) showing zero crossing at $N_0 = 134{,}543$.

N paths that lead to $\overline{u_a v_a} < 0$, i.e., u_a and v_a opposite sign must be greater than events same sign.

(a) Ranks from most + to most –

(b) partial sums $\sum u^i v^i$, a point is reached (N_0) where sign change from $+$ to $-$ $\therefore$ $n > N_0$ responsible $\overline{uv} < 0$ since other contributions cancel out between $+$ and $-$. Fraction $(N - N_0)/N$ reveals useful information on how $\overline{uv}$ is created.

% of events,
i.e., 0.3=30%



Shows $\overline{u_a v_a}$ and $\overline{v_a\left(\overline{U}_b - \overline{U}_a\right)}$ are relatively rare events <1%

Figure 6.7 Fraction of points in the data ensembles that account for the local computed values of the terms in Eq. (6.24). —, $\overline{u_a v_a}$; – –, $\overline{(\overline{U}_b - \overline{U}_a)v_a}$; – · –, $\overline{(U_a - U_b)v_a}$; · · · , $\overline{u_b v_a}$.

$$\overline{u_a v_a} = \underbrace{\overline{u_b v_a}}_{\boxed{1}} + \underbrace{\overline{v_a\left(\overline{U}_b - \overline{U}_a\right)}}_{\boxed{2}} + \underbrace{\overline{v_a(U_a - U_b)}}_{\boxed{3}} \qquad (5)$$

$\overline{u_a v_a}$ terms 1,2 and 3 for $(N - N_0)/N$ fraction of events time interval from $t - \tau_D$ to t. For large portion of the channel $\overline{u_a v_a}$ and Φ_{D} follow same trend. Fraction is generally 20% and rises to 30% at $y^+ = 30$. Towards the center of the channel, i.e., large $y+$, all the terms go to 0 due to $+$ and – cancellation, as it would be expected in a symmetric flow.

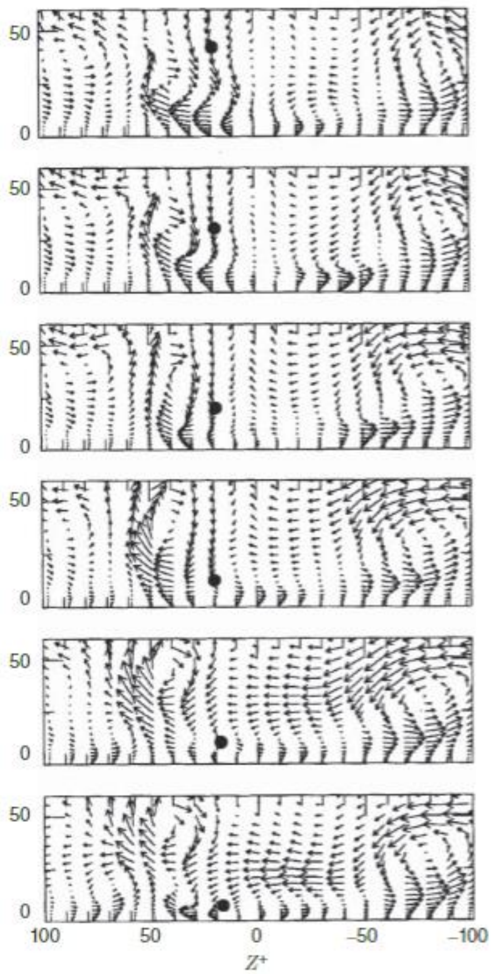


Figure 6.8 Fluid particle arriving at $y^+ = 7.3$ [7] due to a sweep event. Time increases moving from top to bottom image. Reprinted with the permission of Cambridge University Press.

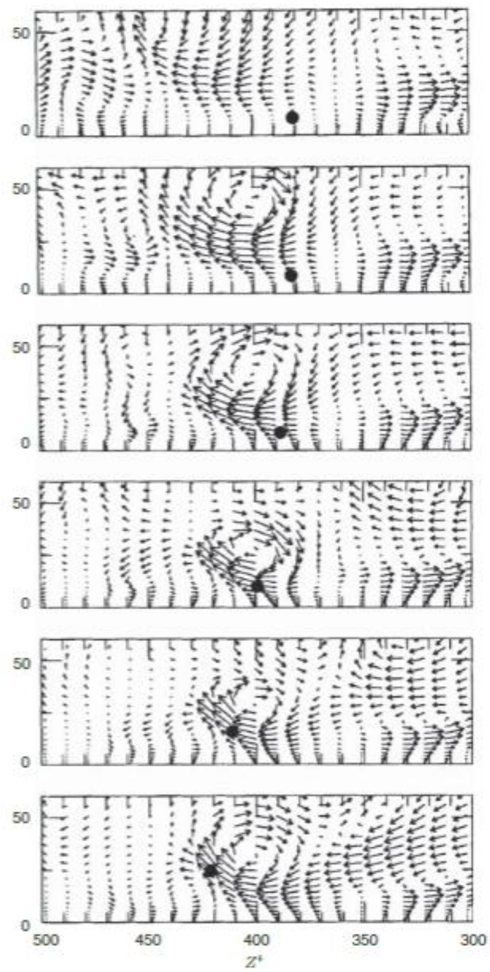


Figure 6.9 Fluid particle arriving at $y^+ = 24.6$ [7] due to an ejection event. Time increases moving from top to bottom image. Reprinted with the permission of Cambridge University Press.

Events that make significant contributions to $\overline{uv}$ → vortical eddies with streamwise orientation.

Sweep event: high speed flow towards wall, dominant contribution in buffer layer.

Ejection event: low speed flow ejected outward, occurs outside buffer layer.

Mixing time = time over which coherent vortices exert influence over motions of fluid particles.

Gradient Transport

If gradient transport is valid, it should be due to Φ_{D} under further hypothesis that change in local mean velocity along particle paths is linear such that a Taylor series in the y direction can be used:

$$\underline{b} = \underline{a} - \underline{L}$$

$$\overline{U_b} = \overline{U}_{\underline{a}-\underline{L}} = \overline{U}_{\underline{a}} - L_2 \frac{d\overline{U}}{dy} + \cdots$$

$$\Phi_{\mathrm{D}} = \overline{v_a(\overline{U}_b - \overline{U}_a)} = -\overline{vL_2}\frac{d\overline{U}}{dy} + \cdots$$

which shows Φ_{D} equivalent gradient transport model; thus,

$$\boxed{-\overline{uv} = \nu_T \frac{\partial \overline{U}}{\partial y} = -\Phi_{\mathrm{D}}}$$

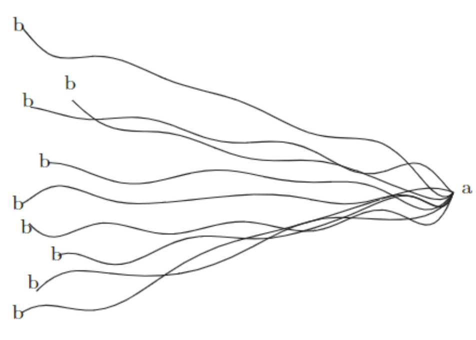
$$\nu_t = \overline{v_a L_2} = \int_{t-\tau}^{t} v\big(\underline{X}(t), t\big) v\big(\underline{X}(s), s\big)\, ds$$

$$\boxed{\begin{gathered} \underline{L} = \int_{t-\tau}^{t} \underline{U}\big(\underline{X}(s), s\big)\, ds \\ \bar{V} = 0 \\ \tau = \tau_D \end{gathered}}$$

Define Lagrangian auto-correlation function:

$$f_{vv}(s) = \frac{\overline{v\big(\underline{X}(t), t\big) v\big(\underline{X}(s), s\big)}}{\overline{v\big(\underline{X}(t), t\big)^2}}$$

$$\nu_t = \overline{v^2}\mathcal{T}_{22} \qquad (6)$$



Where $\mathcal{T}_{22}$ is a Lagrangian integral scale defined by

$$\mathcal{T}_{22} = \int_{-\infty}^{0} f_{vv}(s) ds$$

And $f_{vv}(s) = 0$ for $|s|$ large.

If gradient transport were physically accurate then ν_t in Eq. (6) should approximate the model eddy viscosity

$$\nu_t = -\frac{\overline{uv}}{d\overline{U}/dy} = \text{physical } \nu_t \text{ vs. } \overline{\nu^2}\mathcal{T}_{22}$$

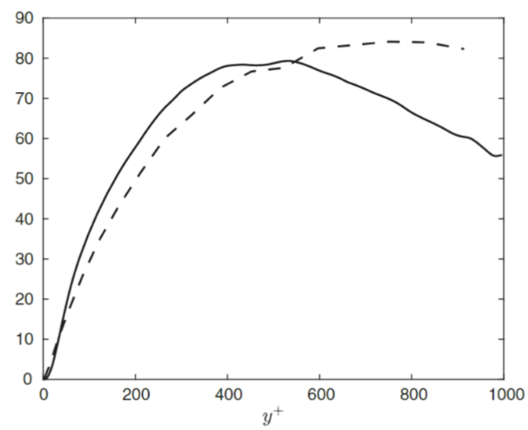


Figure 6.10 Eddy viscosity in channel flow: ––, $\mathcal{T}_{22}^+\overline{v^2}^+$; —, ν_t^+.

Same discrepancies for $y^+ > 500$ where physical $\nu_t =$ constant and modeled decreases; and near wall where physical $<$ modeled. Note $\nu_t > 0$ over whole domain as per $d\overline{U}/dy$, except center channel where both equal 0; and vice versa for upper channel where $d\overline{U}/dy < 0$ and $\overline{uv}$>0.

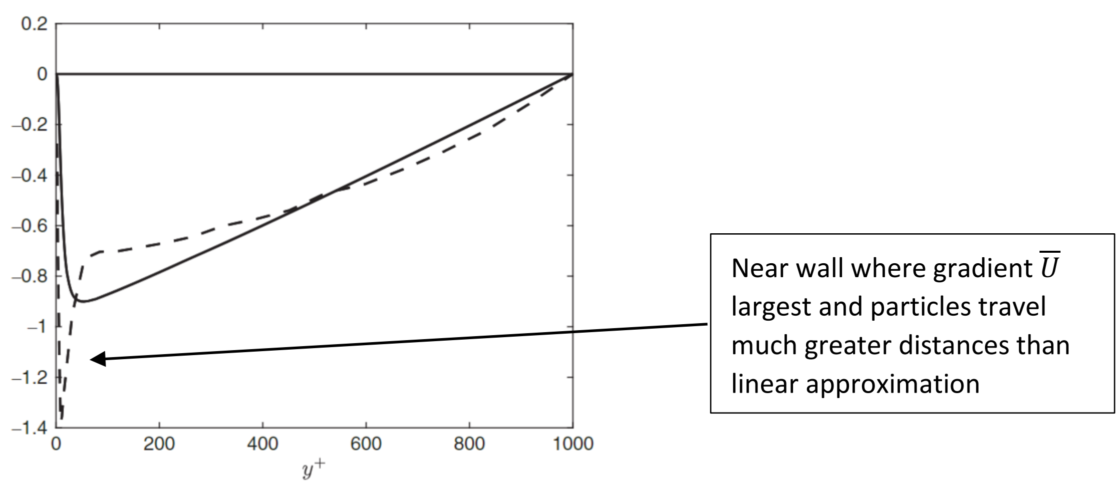
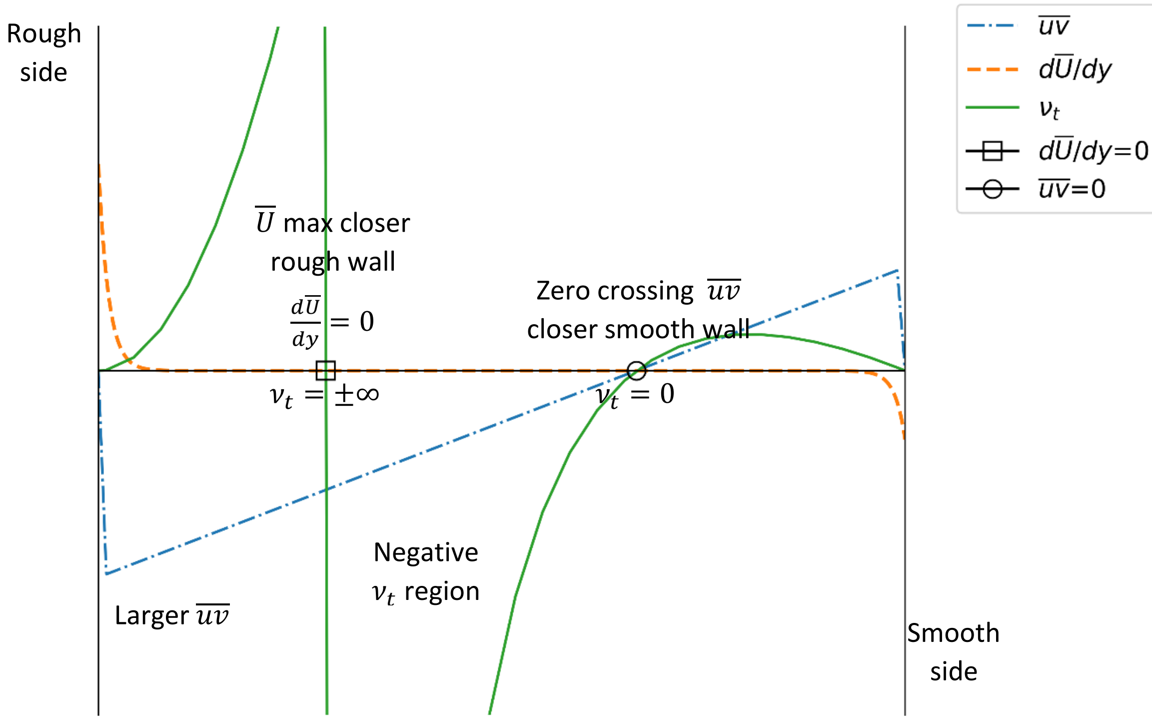


Figure 6.11 Inadequacy of gradient transport physics: —, $\overline{uv}^+$; ––, $-\mathcal{T}_{22}^+\overline{v^2}^+ d\overline{U}^+/dy^+$.

Obvious differences gradient transport vs actual $\overline{uv}$.

Large differences near wall, whereas smaller in outer part ∴ more suitable central part despite ν_t differences shown above.



However, not satisfied for rough wall as ν_t shows unphysical behavior; and numerical methods unstable for $\nu_t < 0$.

$$\nu_t = -\frac{\overline{uv}}{d\overline{U}/dy}$$