## Chapter 6: Turbulent Transport and its Modeling

## Part 4: Vorticity Transport

$$
\frac{\partial \overline{U_{i}}}{\partial t}+\overline{U_{j}} \frac{\partial \overline{U_{i}}}{\partial x_{j}}=-\frac{\partial\left(\frac{\bar{p}}{\rho}+k\right)}{\partial x_{i}}+\nu \nabla^{2} \overline{U_{i}}+\varepsilon_{i j k} \overline{u_{j} \omega_{k}}
$$

Since

$$
\frac{\partial \overline{u_{i} u_{j}}}{\partial x_{j}}=\frac{\partial k}{\partial x_{i}}-\varepsilon_{i j k} \overline{u_{j} \omega_{k}}
$$

$$
\omega_{i}=\Omega_{i}-\overline{\Omega_{i}}
$$

Where $\varepsilon_{i j k}=$ alternating tensor equal to 1 when indices are even permutation of (123), -1 for odd permutation and 0 if any two of the indices are equal.

Note that $\delta_{i j}$ and $\varepsilon_{i j k}$ are the only isotropic $2^{\text {nd }}$ and $3^{\text {rd }}$ order tensors and there is no iotropic $1^{\text {st }}$ order tensor.
$\varepsilon_{i j k}=\varepsilon_{j k i}$ and $\varepsilon_{i j k}=\varepsilon_{k i j}$, i.e., unchanged by moving indices two places right or left. Whereas movement one place changes sign: $\varepsilon_{i j k}=-\varepsilon_{i k j}$.

Also note relation between $\delta_{i j}$ and $\varepsilon_{i j k}$ :

$$
\sum_{k=1}^{3} \varepsilon_{i j k} \varepsilon_{k l m}=\varepsilon_{i j k} \varepsilon_{k l m}=\delta_{i l} \delta_{j m}-\delta_{i m} \delta_{j l}
$$

Returning to rotational form RANS equation in which $\left(\overline{u_{i}} \bar{u}_{j}\right)_{j}$ is replaced by the vorticity flux $\overline{u_{j} \omega_{j}}=$ rate at which $\omega_{j}$ is transported in the $i^{\text {th }}$ direction by $u_{i}$.
$\frac{\bar{p}}{\rho}+k$ can be solved similarly as $\bar{p}$ and since $k=0$ on boundaries, forces and moments readily obtained.

Assume unidirectional channel flow where $\underline{\bar{U}}=(\bar{U}, 0,0)$ and $\underline{\bar{\Omega}}=\left(0,0, \overline{\Omega_{3}}\right)$ such that:

$$
\begin{equation*}
0=-\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x}+v \frac{d^{2} \bar{U}}{d y^{2}}+\overline{v \omega_{3}}-\overline{w \omega_{2}} \tag{1}
\end{equation*}
$$

Taylor derived gradient transport law:

$$
\begin{equation*}
\overline{v \omega_{3}}=-\overline{v L_{2}} \frac{d \overline{\Omega_{3}}}{d y} \tag{2}
\end{equation*}
$$

With $\overline{v L_{2}}=\mathcal{T}_{22} \overline{v^{2}}$, same as momentum gradient transport since $v_{t}$ independent from the quantity being transported.

Substituting Eq. (2) into (1) gives:

$$
0=-\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x}+\left(v+\mathcal{T}_{22} \overline{v^{2}}\right) \frac{d^{2} \bar{U}}{d y^{2}} \quad \overline{\Omega_{3}}=-\frac{d \bar{U}}{d y}
$$

Note that $\overline{w \omega_{2}}=0$ for purely 2D turbulent flow and for gradient transport when $\overline{\Omega_{2}}=0$.

Compare with similar equation using $\left(\overline{u_{i} u_{j}}\right)_{j}$ gradient transport model

$$
0=-\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x}+\frac{d}{d y}\left(\left(v+\mathcal{T}_{22} \overline{v^{2}}\right) \frac{d \bar{U}}{d y}\right)
$$

This shows that in the vorticity transport $v_{t}$ is not differentiated; however, there are difficulties near boundaries since the vorticity flux and $\frac{d \bar{\Omega}}{d y}$ have opposite signs close to the wall, as shown by DNS and Fig. below, which results in unacceptable negative eddy viscosity.


Fig. 6.17 Wall-normal vorticity flux in channel flow. $-\overline{v \omega_{3}} ;--, d \bar{\Omega} / d y$.

As per turbulent momentum transport, but including possibility of 3D flow:

$$
\begin{gather*}
\omega_{j}^{a}=\Omega_{j}^{a}-\overline{\Omega_{j}^{a}} \\
\omega_{j}^{b}=\Omega_{j}^{b}-\overline{\Omega_{j}^{b}} \\
\omega_{j}^{a}-\omega_{j}^{b}=\left(\overline{\Omega_{j}^{b}}-\overline{\Omega_{j}^{a}}\right)+\left(\Omega_{j}^{a}-\Omega_{j}^{b}\right) \tag{3}
\end{gather*}
$$

Starting from instantaneous vorticity equation

$$
\frac{D \Omega_{i}}{D t}=\frac{\partial \Omega_{i}}{\partial t}+U_{j} \frac{\partial \Omega_{i}}{\partial x_{j}}=\Omega_{j} \frac{\partial U_{i}}{\partial x_{j}}+\nu \nabla^{2} \Omega_{i}
$$

Integrating in time between $t-\tau$ and $t$ :

$$
\Omega_{j}^{a}-\Omega_{j}^{b}=\underbrace{\int_{t-\tau}^{t} \Omega_{k}(s) \frac{\partial U_{j}}{\partial x_{k}}(s) d s}_{\text {Vortex stretching }}+\underbrace{\int_{t-\tau}^{t} v \nabla^{2} \Omega_{j}(s) d s}_{\text {Viscous effects }}
$$

Define flux correlation as $\overline{u_{i}^{a} \omega_{j}^{a}}$ as was done for velocity $\left(\overline{u_{a} v_{b}}\right)$ and using Eq. (3):

$$
\begin{equation*}
\overline{u_{i}^{a} \omega_{j}^{a}}=\overline{u_{i}^{a} \omega_{j}^{b}}+\underbrace{\overline{u_{i}^{a}\left(\overline{\Omega_{j}^{b}}-\overline{\Omega_{j}^{a}}\right)}}_{[1}+\underbrace{\int_{t-\tau}^{t} \overline{u_{i}^{a} \Omega_{k}(s) \frac{\partial U_{j}}{\partial x_{k}}(s)} d s}_{2}+\underbrace{\int_{t-\tau}^{t} v \overline{u_{i}^{a} \nabla^{2} \Omega_{j}(s)} d s}_{2} \tag{4}
\end{equation*}
$$

Where it is assumed that $\tau$ is large enough that mixing condition $\overline{u_{i}^{a} \omega_{j}^{b}}=0$ is satisfied.

Term 1 can be developed similarly as what was done for $\Phi_{D}$ in momentum transport.

Taylor series for $\overline{\Omega_{j}^{b}}$ :

$$
\begin{equation*}
\overline{\Omega_{j}^{b}}=\overline{\Omega_{j}^{a}}-L_{k} \frac{d \overline{\Omega_{j}^{a}}}{d x_{k}} \tag{5}
\end{equation*}
$$

Substituting Eq. (5) in Term 1 gives:

$$
\begin{equation*}
\overline{u_{i}^{a}\left(\overline{\Omega_{j}^{b}}-\overline{\Omega_{j}^{a}}\right)}=\overline{u_{i}^{a}\left(\overline{\Omega_{j}^{a}}-L_{k} \frac{d \overline{\Omega_{j}^{a}}}{d x_{k}}-\overline{\Omega_{j}^{h}}\right)}=-\overline{L_{k} u_{i}^{a}} \frac{d \overline{\Omega_{j}^{a}}}{d x_{k}} \tag{6}
\end{equation*}
$$

Recall definition:

$$
\begin{equation*}
L_{k}=\int_{t-\tau}^{t} U_{k}(\underline{X}(s), s) d s=\int_{t-\tau}^{t}\left(\overline{U_{k}}(\underline{X}(s), s)+u_{k}(\underline{X}(s), s)\right) d s \tag{7}
\end{equation*}
$$

And substitute Eq. (7) into (6) to obtain:

$$
\overline{u_{i}^{a}\left(\overline{\Omega_{j}^{b}}-\overline{\Omega_{j}^{a}}\right)}=-\int_{t-\tau}^{t}\left(\overline{\overline{U_{k}}(\underline{X}(s), s) u_{i}^{a}}+\overline{u_{k}(\underline{X}(s), s) u_{i}^{a}}\right) d s \frac{d \overline{\Omega_{j}^{a}}}{d x_{k}}
$$

Where the first term is 0 under the assumption that $\tau$ is large enough.

Therefore, term 1 becomes:

$$
\overline{u_{i}^{a}\left(\overline{\Omega_{j}^{b}}-\overline{\Omega_{j}^{a}}\right)}=-\int_{t-\tau}^{t} \overline{u_{k}(s) u_{i}^{a}} d s \frac{d \overline{\Omega_{j}^{a}}}{d x_{k}}
$$

For term 2, substituting mean and fluctuating quantities for $\Omega_{k}$ and $\frac{\partial U_{j}}{\partial x_{k}}$ gives:

$$
\begin{gathered}
\int_{t-\tau}^{t} \overline{u_{i}^{a}\left(\overline{\Omega_{k}}+\omega_{k}(s)\right)\left(\frac{\partial \overline{U_{j}}}{\partial x_{k}}+\frac{\partial u_{j}}{\partial x_{k}}(s)\right)} d s \\
=\int_{t-\tau}^{t} \overline{u_{i}^{a} \overline{\bar{y}_{k}} \frac{\partial \overparen{U}_{j}}{\partial x_{k}}}+\overline{u_{i}^{a} \frac{\partial u_{j}}{\partial x_{k}}(s)} \overline{\Omega_{k}}+\overline{u_{i}^{a} \omega_{k}(s) \frac{\partial U_{j}}{\partial x_{k}}}+\overline{u_{i}^{a} \omega_{k}(s) \frac{\partial u_{j}}{\partial x_{k}}(s)} d s
\end{gathered}
$$

The last term is neglected since it contains only fluctuating quantities and is nonlinear. Also, the first term is neglected as nonlinear in the mean flow. The third term is neglected for simple shear flow, where $\frac{\partial \overline{U_{j}}}{\partial x_{k}}=0$, according to Bernard, but needs proof.

Therefore, term 2 becomes:

$$
\int_{t-\tau}^{t} \overline{u_{i}^{a} \Omega_{k}(s) \frac{\partial U_{j}}{\partial x_{k}}(s)} d s=\int_{t-\tau}^{t} \overline{u_{i}^{a} \frac{\partial u_{j}}{\partial x_{k}}(s)} d s \overline{\Omega_{k}}
$$

Term 3 is omitted for simplicity.

Equation 4 is equivalent to:

$$
\begin{equation*}
\overline{u_{i}^{a} \omega_{j}^{a}}=-\underbrace{\int_{t-\tau}^{t} \overline{u_{k}(s) u_{i}^{a}} d s \frac{d \overline{\Omega_{j}^{a}}}{d x_{k}}}_{\text {gradient term }}+\underbrace{\int_{t-\tau}^{t} \overline{u_{i}^{a} \frac{\partial u_{j}}{\partial x_{k}}(s)} d s \overline{\Omega_{k}}}_{\text {vortex stretching term }} \tag{8}
\end{equation*}
$$

To obtain a more useful form of Eq. (8), it is necessary to introduce the Lagrangian correlation coefficients $\mathcal{T}_{\alpha \beta}$ and $Q_{\alpha \beta \gamma}(\tau)$ via

$$
\begin{gather*}
\overline{u_{\alpha} u_{\beta}} \mathcal{J}_{\alpha \beta}(\tau)=\int_{t-\tau}^{t} \overline{u_{\alpha}(\underline{x,}, t) u_{\beta}(\underline{X}(s), s)} d s  \tag{9}\\
\overline{u_{\alpha} \frac{\partial u_{\beta}}{\partial x_{\gamma}}} Q_{\alpha \beta \gamma}(\tau)=\int_{t-\tau}^{t} \overline{u_{\alpha} \frac{\partial u_{\beta}}{\partial x_{\gamma}}(s)} d s \tag{10}
\end{gather*}
$$

Such that Eq. (8) becomes:

$$
\begin{equation*}
\overline{u_{\alpha} \omega_{\beta}}=-\mathcal{T}_{\alpha k} \overline{u_{\alpha} u_{k}} \frac{\partial \overline{\Omega_{\beta}}}{\partial x_{k}}+Q_{\alpha \beta k} \overline{u_{\alpha} \frac{\partial u_{\beta}}{\partial x_{k}}} \overline{\Omega_{k}} \tag{11}
\end{equation*}
$$

## Vorticity transport in Channel Flow

Consider now unidirectional shear flows with mean velocity $\bar{U}(y)$ and $\overline{\Omega_{3}}=$ $-d \bar{U} / d y$ is the only non-zero mean vorticity component. With these assumptions five vorticity flux components are identically zero. As an example, consider $\overline{u \omega_{1}}$ :
The zero vorticity flux components are:

$$
\overline{u \omega_{1}}=\overline{v \omega_{2}}=\overline{w \omega_{3}}=\overline{u \omega_{2}}=\overline{v \omega_{1}}=0
$$

The remaining correlations in Eq. (8) are non-zero and are given by:

$$
\begin{gathered}
\overline{w \omega_{1}}=Q_{313} \overline{\frac{\partial u}{\partial z}} \overline{\Omega_{3}} \\
\overline{w \omega_{2}}=Q_{323} \overline{\frac{\partial v}{\partial z}} \overline{\Omega_{3}} \\
\overline{v \omega_{3}}=-\mathcal{T}_{22} \overline{v^{2}} \frac{d \overline{\Omega_{3}}}{\partial y}+Q_{233} \overline{v \frac{\partial w}{\partial z}} \overline{\Omega_{3}} \\
\overline{u \omega_{3}}=-\mathcal{T}_{12} \overline{u v} \frac{d \overline{\Omega_{3}}}{d y}+Q_{133} \overline{\frac{\partial w}{\partial z}} \overline{\Omega_{3}}
\end{gathered}
$$

Where additional Lagrangian integral scales are defined according to Eqs. (9) and (10):

$$
\begin{aligned}
\overline{u v} \mathcal{T}_{12}(\tau) & =\int_{t-\tau}^{t} \overline{u(\underline{x}, t) v(\underline{X}(s), s)} d s \\
\overline{w \frac{\partial u}{\partial z}} Q_{313}(\tau) & =\int_{t-\tau}^{t} \overline{w(\underline{x}, t) \frac{\partial u}{\partial z}(\underline{X}(s), s)} d s
\end{aligned}
$$

With similar definition for $Q_{323}, Q_{233}$ and $Q_{133}$.
$\overline{w \omega_{1}}$ and $\overline{w \omega_{2}}$ do not originate in gradient physics and would be predicted to be zero if vortex stretching was neglected.

Using DNS best fit data as shown in figures uses $\mathcal{T}_{22}^{+}=4.8, \mathcal{T}_{12}^{+}=12.3, Q_{233}^{+}=$ $5.5, Q_{323}^{+}=9.5, Q_{133}^{+}=16.3$, and $Q_{313}^{+}=0.95$.

Thus, including the first-order vortex stretching model, the essentials of turbulent vorticity flux can be accounted for. Gradient terms capture most of the transport away from the wall, while the stretching terms account for non-gradient transport for $\overline{v \omega_{3}}$ and $\overline{u \omega_{3}}$.


Figure $6.17 \overline{\mathrm{v} \omega_{3}}: \bullet$, DNS results; -, prediction from Eq. (6.62). From [21]. Copyright OSpringer
Figure $6.18 \overline{u \omega_{3}}: \bullet$, DNS results; -, prediction from Eq. (6.63). From [21]. Copyright OSpringer-Verlag.


Figure $6.19 \overline{W \omega_{1}}: \bullet$, DNS results; - , prediction from Eq. (6.60). From [21]. Copyright OSpringer-Verlag.


Figure $6.20 \overline{W \omega_{2}}: \bullet$, DNS results; -, prediction from Eq. (6.61). From [21]. Copyright ©Springer-Verlag.

## Appendix A

## A. 1

$$
\overline{u \omega_{1}}=\overline{v \omega_{2}}=\overline{w \omega_{3}}=\overline{u \omega_{2}}=\overline{v \omega_{1}}=0
$$

Consider $\overline{u \omega_{1}}$ :

$$
\begin{gathered}
\overline{u \omega_{1}}=-\mathcal{J}_{11} \overline{u_{1} u_{1}} \frac{\partial \overline{\Omega_{1}}}{\partial x_{1}}+Q_{111} \overline{u_{1} \frac{\partial u_{1}}{\partial x_{1}}} \overline{\Omega_{1}}-\mathcal{T}_{12} \overline{u_{1} u_{2}} \frac{\partial \overline{\Omega_{1}}}{\partial x_{2}}+Q_{112} \overline{u_{1} \frac{\partial u_{1}}{\partial x_{2}}} \overline{\Omega_{2}} \\
-\mathcal{J}_{13} \overline{u_{1} u_{3}} \frac{\partial \overline{\Omega_{1}}}{\partial x_{3}}+Q_{113} \overline{\partial u_{1}} \frac{\partial u_{1}}{\partial x_{3}} \overline{\Omega_{3}}
\end{gathered}
$$

It can be immediately recognized that all the terms, except for the last one, are identically zero, since $\overline{\Omega_{1}}=\overline{\Omega_{2}}=0$ in a channel flow.

The last term

$$
Q_{113} \overline{u_{1} \frac{\partial u_{1}}{\partial x_{3}}} \overline{\Omega_{3}}
$$

Can be rewritten as

$$
\begin{equation*}
\overline{u \omega_{1}}=Q_{113} \overline{\frac{1}{2} \frac{\partial u_{1}^{2}}{\partial x_{3}}} \overline{\Omega_{3}}=0 \tag{1A}
\end{equation*}
$$

This expression shows that also the last term needs to be zero in order to satisfy the requirement of symmetry with respect to reflections in the $x-y$ plane.

Similarly, for $\overline{v \omega_{2}}$ and $\overline{w \omega_{3}}$, the only term containing $\overline{\Omega_{3}}$ is:

$$
\begin{align*}
& \overline{v \omega_{2}}=Q_{223} \overline{u_{2} \frac{\partial u_{2}}{\partial x_{3}}} \overline{\Omega_{3}}=Q_{223} \overline{\frac{\partial u_{2}^{2}}{\partial x_{3}} \overline{\Omega_{3}}=0}  \tag{2A}\\
& \overline{w \omega_{3}}=Q_{333} \overline{u_{3} \frac{\partial u_{3}}{\partial x_{3}}} \overline{\Omega_{3}}=Q_{333} \overline{\frac{\partial u_{3}^{2}}{\partial x_{3}}} \overline{\Omega_{3}}=0 \tag{3A}
\end{align*}
$$

The remaining two vorticity fluxes are $\overline{u \omega_{2}}$ and $\overline{v \omega_{1}}$.

Recall vorticity components definition:

$$
\begin{aligned}
& \omega_{1}=w_{y}-v_{z} \\
& \omega_{2}=u_{z}-w_{x} \\
& \omega_{3}=v_{x}-u_{y}
\end{aligned}
$$

Substituting them into Eqs. (1A), (2A), (3A):

$$
\begin{aligned}
& \overline{u \omega_{1}}=\overline{u w_{y}}-\overline{u v_{z}}=0 \rightarrow \overline{u w_{y}}=\overline{u v_{z}} \\
& \overline{v \omega_{2}}=\overline{v u_{z}}-\overline{v w_{x}}=0 \rightarrow \overline{v u_{z}}=\overline{v w_{x}} \\
& \overline{w \omega_{3}}=\overline{w v_{x}}-\overline{w u_{y}}=0 \rightarrow \overline{w v_{x}}=\overline{w u_{y}}
\end{aligned}
$$

And using several identities appropriate to channel flow gives:

$$
\begin{equation*}
\overline{u w_{y}}=\overline{u v_{z}}=-\overline{v u_{z}}=-\overline{v w_{x}}=\overline{w v_{x}}=\overline{w u_{y}}=-\overline{u w_{y}} \tag{4A}
\end{equation*}
$$

And the equality of the first and last terms in this relation implies that each of the correlations are zero.

For $\overline{u \omega_{2}}$ and $\overline{v \omega_{1}}$ the only term containing $\overline{\Omega_{3}}$ is:

$$
\begin{aligned}
\overline{u \omega_{2}} & =Q_{123} \overline{u_{1} \frac{\partial u_{2}}{\partial x_{3}}} \overline{\Omega_{3}}=Q_{x y z} \overline{u v_{z}} \overline{\Omega_{3}}=0 \\
\overline{v \omega_{1}} & =Q_{213} \overline{u_{2} \frac{\partial u_{1}}{\partial x_{3}}} \overline{\Omega_{3}}=Q_{213} \overline{v u_{z}} \overline{\Omega_{3}}=0
\end{aligned}
$$

And they are both zero according to the equalities shows in Eq. (4A).

