Chapter 6: Turbulent Transport and its Modeling

Part 1: Molecular Momentum Transport

$\rho\overline{u_i u_j}$ = turbulent momentum flux

$\rho\overline{uv} = x$ momentum ρu in y direction due to turbulent v, for $\underline{V} = (U + u, v, w)$

Classical ideas for modeling turbulent transport were based on molecular momentum transport for ideal (non-dense) gas: molecules far apart and inter-molecular forces are weak. Molecules in free flight with brief collisions at which time their direction and speed change.

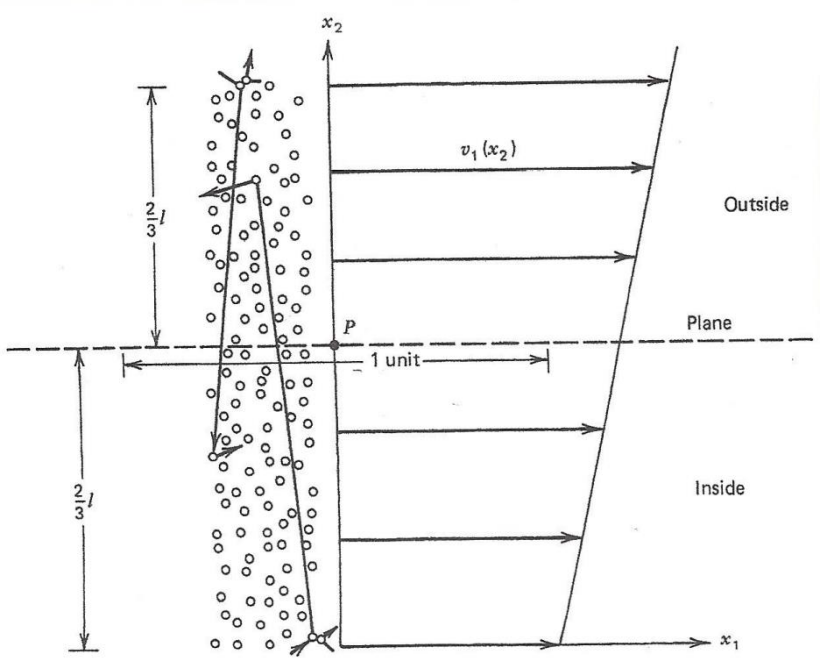

Figure 6.1 Molecular model of the viscosity of a gas.

Across plane separating the gas in two regions, the molecules do not attract or repel each other (contrasting to liquids); therefore, the primary source of shear stress is that due to microscopic transport of momentum due to random molecular motions.

Newtonian fluid stress rate of strain relationship

$$\tau_{21} = \mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)$$

$$= \mu\frac{\partial v_1}{\partial x_2} \qquad \text{For } \underline{V} = v_1(x_2)\hat{e}_1$$

Which can be derived for ideal gas using four facts from kinetic theory:

1) Molecules that cross the plane $x_2 =$ constant begin their free flight on average at distance $\pm\frac{2}{3}l$ ($l =$ mean free path) from the plane.

2) Mean free path $f(d, n)$:

$$l = 1/\sqrt{2}\pi d^2 n$$

$d =$ molecular diameter

$n =$ number density of molecules per unit volume

Since molecules have a distribution of speeds and they are moving relative to each other, $RHS = \frac{3}{4}RHS$ or if assume Maxwellian distribution of velocities RHS = 0.707·RHS.

3) Flux of velocities across x_2-plane per unit area $= \frac{1}{4}n\overline{v}$, where $\overline{v} =$ average molecular speed (without regard direction).

4) Average molecular speed

$$\overline{v} = \left[\frac{8kT}{\pi m}\right]^{1/2} = f(T)$$

$k =$ Boltzmann constant

$m =$ molecular mass

Shear stress x_2-plane

$$\tau_{21} = \frac{x_1 \text{ force}}{\text{unit area}} = \text{net flux momentum across } x_2 \text{ plane}$$

x-momentum of one particle from above:

$$mv_1|_{x_2+\frac{2}{3}l} = m\left[v_1 + \frac{dv_1}{dx_2}\left(\frac{2}{3}l\right) + \cdots\right]_{x_2}$$

x-momentum of one particle from below:

$$mv_1|_{x_2-\frac{2}{3}l} = m\left[v_1 + \frac{dv_1}{dx_2}\left(-\frac{2}{3}l\right) + \cdots\right]_{x_2}$$

Net flux = difference between the momentum of a particle from above minus one particle from below times the rate that the particles cross a unit area $\frac{1}{4}n\overline{v}$:

$$\tau_{21} = \frac{1}{4}n\overline{v} \times \left[mv_1|_{x_2+\frac{2}{3}l} - mv_1|_{x_2-\frac{2}{3}l}\right]$$

$$= \frac{1}{4}n\overline{v} \times m\frac{4}{3}l\frac{dv_1}{dx_2}$$

i.e., comparing with $\tau_{ij} = \mu(v_{i,j} + v_{j,i})$

$$\mu = \frac{1}{3}n\overline{v}ml = \frac{2}{3d^2}\left[\frac{mkT}{\pi^3}\right]^{1/2} = \frac{1}{3}\rho\overline{v}l$$

$$= f(\text{molecular properties and T}) = f(\overline{v}, l)$$

$\mu \uparrow$ m $\uparrow$

$\mu \downarrow$ d $\uparrow$

$\mu \neq f(p)$

$\mu \uparrow$ $\sqrt{T}\uparrow$

More complete theory includes inter molecular forces and better agreement T dependence.

Viscous liquids need more advanced models considering intermolecular forces, but results in same $\tau_{ij} = \mu(v_{i,j} + v_{j,i})$ relationship.

Modelling Turbulent Transport by Analogy to Molecular Transport

Newtonian fluids (incompressible flow)

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij}$$
$$= -p\delta_{ij} + \tau_{ij}$$

$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

μ = isotropic viscosity
= property of the fluid

In analogy the turbulent Reynolds stresses are modeled using the eddy viscosity concept

$$a_{ij} = -\rho\overline{u_i u_j} + \frac{2}{3}\rho k\delta_{ij} = \rho\nu_t \underbrace{\left(\overline{U_{i,j}} + \overline{U_{j,i}}\right)}_{\text{Mean flow rate of strain}} = \rho\nu_t S_{ij}$$

Anisotropic RS is modeled using isotropic eddy viscosity ν_t or $\mu_t = \rho\nu_t$, which may be contrasted with μ definition for ideal gas; however, no reason to believe turbulent motions are without directional biases that are not aligned with S_{ij}.

Nonetheless eddy viscosity concept forms the basis of traditional RANS modeling, which focuses on modeling of ν_t.

For example, consider 1D shear flow:

$$\frac{\tau_{12}}{\rho} = -\overline{uv} = \nu_t \frac{d\overline{U}}{dy}$$

Large scale turbulent eddies are most important in transporting momentum across the flow, which are mostly driven by inertia and pressure forces vs. viscosity. Assume $-\rho\overline{uv}$ due to turbulent eddies with transverse size l and intensity characterized by velocity scale u_0.

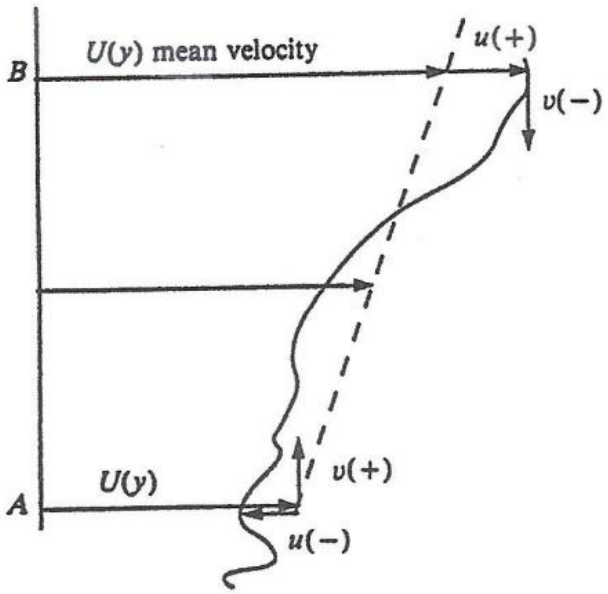

Figure 26.4 Mean velocity profile and the fluctuations that contribute to the Reynolds stresses.

Therefore, $\overline{uv} < 0$, as per previous discussion Chapter 3 RANS equations.

Assume $A_p \rightarrow B$ and viceversa $B_p \rightarrow A$, where $p =$ fluid particles, which interact and merge with the flow (transporting momentum)

$$\left.\begin{aligned} -\rho\overline{uv} &= f\left(\rho, l, u_0, \frac{du}{dy}\right) \\ -\frac{\overline{uv}}{u_0^2} &= f\left(\frac{l}{u_0}\frac{du}{dy}\right) \end{aligned}\right\}$$ Dimensional analysis

Assume linear relationship and eddy viscosity:

$$-\overline{uv} = Cu_0^2 \frac{l}{u_0}\frac{du}{dy}$$

$$= Clu_0 \frac{du}{dy}$$

i.e.,

$$\nu_t = Clu_0$$

$u_0 =$ turbulent velocity scale

$l =$ turbulent length scale

Which is also consistent with ideal gas theory:

$$\mu = \frac{1}{3}\rho\bar{u}l$$

The time scale for eddy turnover times is:

$$l/u_0$$

And time scale for mean flow

$$\left|\frac{du}{dy}\right|^{-1}$$

Therefore assume:

$$\frac{l}{u_0} = \left|\frac{du}{dy}\right|^{-1}$$

i.e.,

$$u_0 = l\left|\frac{du}{dy}\right|$$

$$\nu_t = Clu_0 = Cl^2\left|\frac{du}{dy}\right|$$

Or

$$\mu_t = C_\mu\rho\, l^2\left|\frac{du}{dy}\right|$$

1) This approach is the Prandtl mixing length theory. l depends on the type of flow.
   $l \propto$ larger scale eddies
   Free shear-flow: $l = c\delta$ where $c = f$(mixing layer, jet, wake),
   BL: $l = ky$, i.e., eddy size $\propto y$ near wall

   $= c\delta$ away from the wall

   δ = appropriate width of viscous flow

2) $k-\varepsilon$ model

$u=\sqrt{k}$ and $l=k^{3/2}/\varepsilon=$ length associated eddy turnover time l/u

$\mu_t=C_\mu\rho\frac{k^2}{\varepsilon}\rightarrow$ additional equations needed to model k and ε.

Eddy viscosity concept is based on ideal gas molecular transport; thus, assumes:

1) Mixing occurs over well-defined mixing time.
2) Momentum preserved between collisions.
3) Linear velocity variation over the mixing length.