Chapter 4: Turbulence at Small Scales

Part 3: The smallest scales

Kolmogorov: for sufficiently high Re, universal small-scale equilibrium (dissipation scales) depends on two main parameters, i.e., viscosity ν and the dissipation rate ε. Therefore, using dimensional analysis:

$$\eta = (\nu^3/\varepsilon)^{1/4}$$

Length scale

$$t_d = (\nu/\varepsilon)^{1/2}$$

Time scale = turn over time

$$v_d = \frac{\eta}{t_d} = (\nu\varepsilon)^{1/4}$$

Velocity scale

$\eta \sim 1/k_d$ where k_d represents the peak in the dissipation spectrum. Lower limit since EFD shows $k_d \sim \alpha/\eta$ where $\alpha = 0.1 - 0.15$ and most of the dissipation occurs for $k < 0.5/\eta$. That is, $l_d = 50\eta$ (vs. 60η given in Pope) and $l < 12.5\eta$.

Recall that

$$\varepsilon = 2\upsilon\overline{e_{ij}e_{ij}} = \frac{\upsilon}{2}\overline{\left(u_{i,j} + u_{j,i}\right)^2} \quad (1)$$

$$\varepsilon = \tilde{\varepsilon} + \upsilon\frac{\partial^2\overline{u_i u_j}}{\partial x_i \partial x_j} = \upsilon\left(\overline{u_{i,j}u_{i,j}} + \frac{\partial^2\overline{u_i u_j}}{\partial x_i \partial x_j}\right)$$

and for isotropic turbulence

$$\tilde{\varepsilon} = \upsilon\overline{u_{i,j}u_{i,j}} = \varepsilon \quad (2)$$

since for isotropic turbulence $\frac{\partial^2\overline{u_i u_j}}{\partial x_i \partial x_j} = 0$.

Also, it has been shown that ε is related to $\mathcal{R}_{ii}$

$$\frac{\varepsilon}{\upsilon} = \overline{u_{i,k}u_{i,k}} = -\frac{\partial^2 \mathcal{R}_{ii}}{\partial {r_k}^2}(0) \quad (3)$$

Equation (3) can be evaluated using several approaches to show relationship between ε and $f''(0)$, i.e., λ_f and λ_g; and relationship between $f''(0)$ and $\overline{{u_x}^2}$. Which enable measurements of ε from single point time series or 1D spatial/line statistics.

1. Pope (2000).

Appendix A.4 provides more general derivation considering vector x_i vs. scalar x as per below

$f''(0) = -2/\lambda_f^2$ can be related to $\frac{\partial u}{\partial x}$ and thus ε.

$$\boxed{\frac{\partial f}{\partial r} = f'}$$

$$\overline{u^2} f'(r) = \overline{u(x)\frac{\partial}{\partial r}u(x+r)}$$

$$= \overline{u(x)\frac{\partial u(x')}{\partial x'}\frac{\partial x'}{\partial r}}$$

$$\boxed{\begin{aligned} &x' = x + r \\ &\frac{\partial x'}{\partial r} = 1 \end{aligned}}$$

$$\overline{u^2} f''(r) = \overline{u(x)\frac{\partial}{\partial r}\left(\frac{\partial u(x')}{\partial x'}\right)}$$

$$= \overline{u(x)\frac{\partial}{\partial x'}\left(\frac{\partial u(x')}{\partial x'}\right)\frac{\partial x'}{\partial r}}$$

$$= \overline{u(x)\frac{\partial^2 u(x')}{\partial x'^2}}$$

$$\overline{\frac{\partial}{\partial x}\left(\cancel{uu_{x'}}\right) - {u_{x'}}^2} = \overline{{u_{x'}}^2} + \overline{uu_{x'x'}} - \overline{{u_{x'}}^2}$$

Homogeneous turbulence

Applying $\lim r \to 0, x' \to x$:

$$\overline{u^2} f''(0) = -\overline{u_x{}^2}$$

i.e.,

$$\overline{\left(\frac{\partial u}{\partial x}\right)^2} = \frac{2\overline{u^2}}{\lambda_f^2}$$

Next, show relationship between $\overline{u_x{}^2}$ and ε.

For homogeneous isotropic turbulence

$$\overline{u_{i,j} u_{k,l}} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}$$

General form isotropic tensor of 4th order.

For $j = i$, $\overline{u_{i,i} u_{k,l}} = 0$

$$\alpha \delta_{ii} \delta_{kl} + \beta \delta_{ik} \delta_{il} + \gamma \delta_{il} \delta_{ik} = 0$$

$$(3\alpha + \beta + \gamma) \delta_{kl} = 0$$

i.e.,

$$3\alpha + \beta + \gamma = 0 \quad (4)$$

For $k = j$, $\overline{u_{i,j} u_{j,l}}$

$$\underbrace{\frac{\partial}{\partial x_j}\left(\overline{u_i u_{j,l}}\right)}_{\boxed{\substack{=0 \\ \text{homogeneous}}}} = \overline{u_{i,j} u_{j,l}} + \underbrace{\overline{u_i \frac{\partial^2 u_j}{\partial x_j \partial x_l}}}_{\boxed{\substack{=0 \\ \text{continuity}}}} = 0$$

$$\alpha\delta_{ij}\delta_{jl} + \beta\delta_{ij}\delta_{jl} + \gamma\delta_{il}\delta_{ij} = 0$$

$$\alpha + \beta + 3\gamma = 0 \quad (5)$$

Subtracting Eq. (5) from Eq. (4) yields

$$2\alpha - 2\gamma = 0$$

$$\alpha = \gamma$$

$$\beta = -4\gamma$$

$$\alpha = -\beta/4$$

$$\overline{u_{i,j}u_{k,l}} = \beta\left(\delta_{ik}\delta_{jl} - \frac{1}{4}\delta_{ij}\delta_{kl} - \frac{1}{4}\delta_{il}\delta_{jk}\right)$$

$$\overline{u_{1,1}^2} = \beta/2 \qquad \boxed{i = j = k = l = 1}$$

$$\overline{u_{1,2}^2} = \beta\left(\delta_{11}\delta_{22} - \frac{1}{4}\cancel{\delta_{12}\delta_{12}} - \frac{1}{4}\cancel{\delta_{12}\delta_{12}}\right) = \beta \qquad \boxed{\begin{matrix} i = k = 1 \\ j = l = 2 \end{matrix}}$$

$$\overline{u_{1,1}u_{2,2}} = \beta\left(\cancel{\delta_{12}\delta_{12}} - \frac{1}{4}\delta_{11}\delta_{22} - \frac{1}{4}\cancel{\delta_{12}\delta_{12}}\right) = -\frac{\beta}{4} \qquad \boxed{\begin{matrix} i = j = 1 \\ k = l = 2 \end{matrix}}$$

$$\overline{u_{1,2}u_{2,1}} = \beta\left(\cancel{\delta_{12}\delta_{21}} - \frac{1}{4}\cancel{\delta_{12}\delta_{21}} - \frac{1}{4}\delta_{11}\delta_{22}\right) = -\beta/4 \qquad \boxed{\begin{matrix} i = l = 1 \\ j = k = 2 \end{matrix}}$$

$$\overline{u_{1,2}^2} = 2\overline{u_{1,1}^2}$$

$$\overline{u_{1,1}u_{2,2}} = \overline{u_{1,2}u_{2,1}} = -\frac{1}{2}\overline{u_{1,1}^2}$$

$$\tilde{\varepsilon} = \nu\beta\left(\delta_{ii}\delta_{jj} - \frac{1}{4}\delta_{ij}\delta_{ij} - \frac{1}{4}\delta_{ij}\delta_{ij}\right) \qquad \boxed{\begin{matrix} i = k \\ j = l \end{matrix}}$$

$$\tilde{\varepsilon} = \nu\beta\left(9 - \frac{3}{4} - \frac{3}{4}\right) = \frac{30}{4}\nu\beta$$

$$\tilde{\varepsilon} = \frac{60}{4}\nu\overline{u_{1,1}^2} = 15\nu\overline{u_{1,1}^2}$$

$$\varepsilon = \tilde{\varepsilon} = 30\nu \frac{\overline{u^2}}{\lambda_f^2}$$

$$= 15\nu \frac{\overline{u^2}}{\lambda_g^2}$$

$$\boxed{\lambda_f = \sqrt{2}\lambda_g}$$

$$f''(0) = -2/\lambda_f^2$$

$$\lambda_f^2 = -2/f''(0)$$

$$\varepsilon = -15\nu\overline{u^2}f''(0)$$

2. Bernard (2019)

$$\mathcal{R}_{ij}(\underline{r}) = \overline{u^2}\left[\left(f + \frac{r}{2}\frac{df}{dr}\right)\delta_{ij} - \frac{r_i r_j}{r^2}\frac{r}{2}\frac{df}{dr}\right] \quad (6)$$

Taking derivative with respect to r_j

$$\frac{\partial \mathcal{R}_{ii}}{\partial r_j} = \overline{u^2}\frac{\partial}{\partial r_j}(3f + rf') = \overline{u^2}\frac{\partial}{\partial r}(3f + rf')\frac{\partial r}{\partial r_j}$$

$$= \overline{u^2}\left(3f' + rf'' + f'\frac{\partial r}{\partial r}\right)\frac{r_j}{r}$$

$$= \overline{u^2}\left(4f'\frac{r_j}{r} + r_j f''\right)$$

Taking another derivative with respect to r_j

$$\frac{\partial^2 \mathcal{R}_{ii}}{\partial {r_j}^2} = \overline{u^2}\frac{\partial}{\partial r_j}\left(4f'\frac{r_j}{r} + r_j f''\right)$$

$$= \overline{u^2}\left(4\frac{r_j}{r}\frac{\partial f'}{\partial r}\frac{\partial r}{\partial r_j} + 4f'\frac{\partial\left(r_j/r\right)}{\partial r_j} + r_j\frac{\partial f''}{\partial r}\frac{\partial r}{\partial r_j} + \frac{\partial r_j}{\partial r_j}f''\right)$$

$$= \overline{u^2}\left(4\frac{r_j}{r}\frac{r_j}{r}f'' + \frac{4f'}{r}\frac{\partial r_j}{\partial r_j} + 4f' r_j\left(-\frac{1}{r^2}\right)\frac{r_j}{r} + \frac{r_j}{r}r_j f''' + 3f''\right)$$

$$= \overline{u^2}\left(4f'' + \frac{12f'}{r} - 4f'\left(-\frac{1}{r}\right) + rf''' + 3f''\right)$$

$$= \overline{u^2}\left(7f'' + \frac{8f'}{r} + rf'''\right) \quad (6)$$

Using Eq. (3)

$$\frac{\varepsilon}{\nu} = -\frac{\partial^2 \mathcal{R}_{ii}}{\partial {r_j}^2}(0)$$

And substituting Eq. (6), we obtain

$$\frac{\varepsilon}{\nu} = -\overline{u^2}\left(7f''(0) + \frac{8f'(0)}{r} + rf'''(0)\right) \quad (7)$$

Using a Taylor expansion around $r = 0$ for f', we obtain

$$f'(r) = \cancel{f'(0)} + rf''(0) + \frac{r^2}{2!}\cancel{f'''}(0) + \frac{r^3}{3!}f^{IV}(0) + \cdots$$

Since f is an even function.

Dividing by r and taking the limit for $r \to 0$

$$\frac{f'(0)}{r} = f''(0)$$

Substituting into Eq. (7)

$$\frac{\varepsilon}{\nu} = -\overline{u^2}(7f''(0) + 8f''(0))$$

$$\frac{\varepsilon}{\nu} = -15\overline{u^2}f''(0) \quad (8)$$

Using $\lambda_f^2 = -\frac{2}{f''(0)}$ And substituting into Eq. (8), we obtain the relationship between ε and λ_f

$$\varepsilon = \frac{30\nu\overline{u^2}}{\lambda_f^2}$$

3. Kundu et al. (2016)

$$\varepsilon = \frac{\nu}{2}\overline{\left(u_{i,j} + u_{j,i}\right)^2} = \nu\overline{\left({u_{i,j}}^2 + u_{i,j}u_{j,i}\right)}$$

$$\frac{\partial}{\partial x_i}\overline{u_j^n} = 0, \quad \overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}, \quad \text{and} \quad \overline{\left(\frac{\partial u_1}{\partial x_1}\right)^n} = \overline{\left(\frac{\partial u_2}{\partial x_2}\right)^n} = \overline{\left(\frac{\partial u_3}{\partial x_3}\right)^n}, \qquad (12.36)$$

but relative directions must be respected:

$$\overline{\left(\frac{\partial u_1}{\partial x_2}\right)^n} = \overline{\left(\frac{\partial u_1}{\partial x_3}\right)^n} = \overline{\left(\frac{\partial u_2}{\partial x_1}\right)^n} = \overline{\left(\frac{\partial u_2}{\partial x_3}\right)^n} = \overline{\left(\frac{\partial u_3}{\partial x_1}\right)^n} = \overline{\left(\frac{\partial u_3}{\partial x_2}\right)^n}. \qquad (12.37)$$

Note that the continuity equation requires derivative moments in the third set of equalities of (12.36) to be zero when $n = 1$.

$$\begin{aligned}{u_{i,j}}^2 = \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} &= \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_3} \\ &+ \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_3} \\ &+ \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_1} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_3}\end{aligned}$$

$${u_{i,j}}^2 = 3\left(\frac{\partial u_1}{\partial x_1}\right)^2 + 6\left(\frac{\partial u_1}{\partial x_2}\right)^2$$

$$\begin{aligned}\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} &= \frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1} \\ &+ \frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2} \\ &+ \frac{\partial u_3}{\partial x_1}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_3}\end{aligned}$$

$$\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = 3\left(\frac{\partial u_1}{\partial x_1}\right)^2 + 6\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}$$

$$\varepsilon = 6\nu\left[\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} + \overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2} + \overline{\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}}\right]$$

$$\overline{u_{i,k}u_{j,l}} = -\frac{\partial^2 \mathcal{R}_{ij}}{\partial r_k \partial r_l}(0) \quad \text{Part 1 Eq. (8)}$$

For $i = j = k = l = 1$

$$\overline{{u_{1,1}}^2} = -\frac{\partial^2 \mathcal{R}_{11}}{\partial r_1 \partial r_1}(0)$$

$$= -\overline{u^2}\frac{\partial^2}{\partial {r_1}^2}\left[f + \frac{r}{2}f'\left(1 - \frac{r_1^2}{r^2}\right)\right]$$

Using Taylor expansion for $f(r)$

$$f(r) \approx 1 + \frac{r^2}{2}f''(0)$$

$$\overline{{u_{1,1}}^2} = -\overline{u^2}\frac{\partial^2}{\partial {r_1}^2}\left[1 + \frac{r^2}{2}f''(0) + \frac{r^2}{2}f''(0)\left(1 - \frac{r_1^2}{r^2}\right)\right]$$

$$= -\overline{u^2}\frac{\partial^2}{\partial {r_1}^2}\left[1 + r^2 f''(0) - \frac{r_1^2}{2}f''(0)\right]$$

$$= -\overline{u^2}\frac{\partial}{\partial r_1}\left[2r\frac{r_1}{r}f''(0) - r_1 f''(0)\right] = -\overline{u^2}f''(0)$$

For $i = j = 1,\ k = l = 2$

$$\overline{{u_{1,2}}^2} = -\frac{\partial^2 \mathcal{R}_{11}}{\partial r_2 \partial r_2}(0)$$

$$= -\overline{u^2}\frac{\partial^2}{\partial {r_2}^2}\left[1 + r^2 f''(0) - \frac{r_1^2}{2}f''(0)\right]$$

$$= -\overline{u^2}\frac{\partial}{\partial r_2}\left[2r\frac{r_2}{r}f''(0)\right] = -2\overline{u^2}f''(0)$$

For $i = k = 1,\ j = l = 2$

$$\overline{u_{1,2}u_{2,1}} = -\frac{\partial^2 \mathcal{R}_{12}}{\partial r_1 \partial r_2}(0)$$

$$= -\overline{u^2}\frac{\partial^2}{\partial r_1 \partial r_2}\left[-\frac{r_1 r_2}{r^2}\frac{r}{2}\ f'(0)\right]$$

$$\boxed{f'(0) = rf''(0)}$$

$$= \overline{u^2}\frac{\partial^2}{\partial r_1 \partial r_2}\left[\frac{r_1 r_2}{2}\ f''(0)\right]$$

$$= \overline{u^2}\frac{f''(0)}{2}$$

$$\varepsilon = 6\nu\left[\overline{u_{1,1}{}^2} + \overline{u_{1,2}{}^2} + \overline{u_{1,2}u_{2,1}}\right]$$

$$= 6\nu\left[-\overline{u^2}f''(0) - 2\overline{u^2}f''(0) + \overline{u^2}\frac{f''(0)}{2}\right]$$

$$= 6\nu\overline{u^2}f''(0)\left[-1 - 2 + \tfrac{1}{2}\right]$$

$$= -15\nu\overline{u^2}f''(0)$$

Alternatively, ε is determined by the largest scales, i.e., rate at which energy is extracted from the energy containing scales.

t_e = time scale for energy loss from large scales, referred as *eddy turnover time,* represents the life span of eddies so that there is a turnover in their population occurring at this rate.

${u_{rms}}^2$ = energy of the large scales.

l_e = eddy size of the large scales $\approx \Lambda$.

ε =rate of energy loss = ${u_{rms}}^2/t_e$.

$t_e = l_e/u_{rms}$.

l_e = distance over which energy is lost

Thus, $\varepsilon = {u_{rms}}^3/l_e$.

Comparing with small scale $\varepsilon = 15\nu\frac{\overline{u^2}}{\lambda_g^2}$:

$$\varepsilon \sim \frac{\upsilon {u_{rms}}^2}{\lambda^2} \sim \frac{{u_{rms}}^3}{l_e}$$

Therefore

$$\frac{l_e}{\lambda} \sim R_\lambda = \frac{u_{rms}\lambda}{\upsilon} \quad (9)$$

Where $Re \sim {R_\lambda}^2$ is a turbulence Reynolds number with $\lambda = \lambda_f$ or λ_g and R_λ > 10^2 used as criteria for turbulent flow.

$$\sqrt{Re} \sim R_\lambda$$

and

$$Re = \frac{u_{rms} l_e}{\upsilon}$$

Is a turbulence Reynolds number based on the physical size of the flow domain.

Using the definition $\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$ with $\varepsilon \sim \frac{\upsilon {u_{rms}}^2}{\lambda^2}$

$$\frac{\eta}{\lambda} = \frac{1}{\sqrt{R_\lambda}} \sim \frac{1}{Re^{1/4}} \;(10)$$

showing that η is smaller than λ, but not much. Using the ratio of (10) and (9) shows that

$$\frac{\eta}{l_e} \sim {R_\lambda}^{-3/2} \sim Re^{-3/4}$$

Which represents the ratio of the smallest to largest scales in the flow. Also using $u_\eta = v_d = (\varepsilon \nu)^{\frac{1}{4}}$, with $\varepsilon \sim \frac{\upsilon {u_{rms}}^2}{\lambda^2}$ shows that

$$\frac{v_d}{u_{rms}} \sim {R_\lambda}^{-1/2} \sim Re^{-1/4}$$

Represents the ratio of velocities between the smallest and largest eddies.

Alternative reasoning can also be used to determine ratio between the Taylor and Kolmogorov scales [following Pope (2000)]

Noting that

$$\varepsilon = 15\nu \frac{\overline{u^2}}{\lambda_g^2} \qquad (11)$$

ε can also be determined by the largest scales, i.e., rate at which energy is extracted from the energy containing scales.

Define the following quantities for the large scales:

- ${u_{rms}}^2 = k \rightarrow$ energy of the large scales
- $L \rightarrow$ eddy size of the large scales $\approx \Lambda$
- $t_e \rightarrow$ time scale for energy loss from large scales, referred as *eddy turnover time*

Using dimensional analysis, the rate of energy dissipation can be written as

$$\varepsilon = \frac{{u_{rms}}^2}{t_e} = \frac{{u_{rms}}^3}{l_e}$$

Or equivalently using TKE

$$\varepsilon = \frac{k}{t_e} = \frac{k^{3/2}}{l_e}$$

Where

$$k = \frac{1}{2}\overline{(u^2 + v^2 + w^2)}$$

If isotropic $k = \frac{3}{2}\overline{u^2}$

$$u_{rms} = \left[\frac{2}{3}k\right]^{1/2} \approx k^{1/2}$$

Using the characteristic quantities of the large scales L can be defined as follows

$$L \equiv \frac{k^{3/2}}{\varepsilon} = \frac{{u_{rms}}^3}{\varepsilon}$$

Then, the Reynolds number of the large eddies can be written as

$$Re_L = \frac{k^{1/2}L}{\nu} = \frac{k^2}{\varepsilon\nu} \qquad (12)$$

Combining Eq. (11) and (12),

$$\frac{k^{1/2}L}{\nu} = \frac{k^2}{15\nu\frac{\overline{u^2}}{\lambda_g^2}\nu}$$

And solving for the ratio between the Taylor and large scales yields

$$\frac{\lambda_g}{L} = \sqrt{10}{Re_L}^{-1/2} \qquad (13)$$

Proof in Appendix A.1

Using the Kolmogorov dissipation scale definition, according to dimensional analysis

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \qquad (14)$$

And substituting Eq. (14) into Eq. (11),

$$\lambda_g = \frac{\sqrt{15}\sqrt{\overline{u^2}}}{\nu}\eta^2 \qquad (15)$$

Combining Eq. (13) and (15), it is possible to obtain the ratio between the Kolmogorov and largest scales

$$\frac{\eta}{L} = Re_L^{-3/4} \qquad (16)$$

Proof in Appendix A.2

Finally, combining Eq. (13) and (16), it is possible to obtain the ratio between the Kolmogorov and Taylor scales

$$\frac{\eta}{\lambda_g} = \frac{1}{\sqrt{10}}Re_L^{-1/4}$$

Thus, at high Re, λ_g intermediate in size between η and L.

It is possible to use the Taylor scale to define the Taylor-scale Reynolds number

$$R_\lambda = \frac{u_{rms}\lambda}{\upsilon} \qquad (17)$$

Which is used to characterize grid turbulence.

Combining Eq. (12), (13) and (17), the relationship between Re_L and R_λ is obtained

$$R_\lambda = \left(\frac{20}{3}Re_L\right)^{1/2}$$

Proof in Appendix A.3

These results can be used to estimate the cost of DNS simulations based on grid size and time step required for spatial and temporal resolution of the flow vs. computational power.

Number of mesh points for turbulent flow simulation $\sim O(Re^{\frac{9}{4}})$.

Number of time steps $\sim O(Re^{3/4})$.

Total number of operations $\sim O(Re^{3})$.

Appendix A

A.1

$$\frac{k^{1/2}L}{\nu} = \frac{k^2}{15\nu \frac{\overline{u^2}}{\lambda_g^2} \nu}$$

Rewrite as

$$\frac{\lambda_g^2}{L} = \frac{15\nu^2 k^{1/2}\overline{u^2}}{\nu k^2}$$

Divide by L on both sides and simplify

$$\frac{\lambda_g^2}{L^2} = \frac{15\nu\overline{u^2}}{Lk^{3/2}} = 15 \underbrace{\left(\frac{\nu}{Lk^{1/2}}\right)}_{\boxed{Re_L^{-1}}} \underbrace{\left(\frac{\overline{u^2}}{k}\right)}_{\boxed{2/3}}$$

Apply square root

$$\frac{\lambda_g}{L} = \sqrt{15}\sqrt{2/3}Re_L^{-1/2} = \sqrt{10}Re_L^{-1/2}$$

A.2

$$\frac{\lambda_g}{L} = \sqrt{10} Re_L^{-1/2} \quad (A1)$$

$$\lambda_g = \frac{\sqrt{15}\sqrt{\langle u^2 \rangle}}{\nu}\eta^2 \quad (A2)$$

Rewrite (A1) as

$$\lambda_g = L\sqrt{10} Re_L^{-1/2}$$

And substitute in (A2)

$$L\sqrt{10} Re_L^{-1/2} = \frac{\sqrt{15}\sqrt{\langle u^2 \rangle}}{\nu}\eta^2$$

Multiply both sides by L and rewrite as

$$\frac{\eta^2}{L^2} = \frac{\nu\sqrt{10} Re_L^{-1/2}}{L\sqrt{15}\sqrt{\langle u^2 \rangle}} = \sqrt{\frac{2}{3}}\frac{\nu Re_L^{-1/2}}{L\sqrt{\langle u^2 \rangle}}$$

Using the relation

$$k = \frac{3}{2}\langle u^2 \rangle \Rightarrow k^{1/2} = \sqrt{\frac{3}{2}\langle u^2 \rangle}$$

Yields

$$\frac{\eta^2}{L^2} = \frac{\nu Re_L^{-1/2}}{L k^{1/2}} = Re_L^{-1} Re_L^{-1/2} = Re_L^{-3/2}$$

Apply square root

$$\frac{\eta}{L} = Re_L^{-3/4}$$

A.3

$$Re_L = \frac{k^{1/2}L}{\nu} = \frac{k^2}{\varepsilon\nu}$$

$$R_\lambda = \frac{\sqrt{\langle u^2\rangle}\lambda}{\upsilon}$$

$$\frac{\lambda_g}{L} = \sqrt{10}Re_L^{-1/2} \quad (A3)$$

Multiply Eq. (A3) by $\sqrt{\langle u^2\rangle}/\upsilon$

$$\sqrt{\langle u^2\rangle}\frac{\lambda_g}{\upsilon L} = \frac{\sqrt{\langle u^2\rangle}}{\upsilon}\sqrt{10}Re_L^{-1/2}$$

$$R_\lambda = \frac{L\sqrt{\langle u^2\rangle}}{\upsilon}\sqrt{10}Re_L^{-1/2}$$

Multiply and divide by $\sqrt{\frac{3}{2}}$

$$R_\lambda = \sqrt{\frac{2}{3}}\underbrace{\sqrt{\frac{3}{2}}\frac{L\sqrt{\langle u^2\rangle}}{\upsilon}}_{\boxed{Re_L}}\sqrt{10}Re_L^{-1/2}$$

$$R_\lambda = \sqrt{\frac{20}{3}}Re_L^{1/2}$$

A.4

Pope (2019) Turbulent Flow.

Derivation of Eq. (6.56).

$$\begin{cases}\vec{y}=\vec{x}+r\vec{e}\\ y_1=x_1+r_1\end{cases}$$

From Eq. 6.45. $f(r)=R_{11}/u'^2 \Rightarrow u'^2 f(r)=\overline{u_1(\vec{x})u_1(\vec{y})}$

$$\frac{\partial}{\partial r}\left[u'^2 f(r)\right]=\frac{\partial}{\partial r}\overline{\left[u_1(\vec{x})u_1(\vec{y})\right]}=\overline{u_1(\vec{x})\frac{\partial u_1(\vec{y})}{\partial r}}+\overline{u_1(\vec{y})\frac{\partial u_1(\vec{x})}{\partial r}}$$

$$=\overline{u_1(\vec{x})\frac{\partial u_1(\vec{y})}{\partial y_1}}\frac{\partial y_1}{\partial r}+\cancel{\overline{u_1(\vec{y})\frac{\partial u_1(\vec{x})}{\partial x_1}}\overset{=0}{\frac{\partial x_1}{\partial r}}}\rightarrow 0=\overline{u_1(\vec{x})\frac{\partial u_1(\vec{y})}{\partial y_1}}\overset{=\frac{r}{r_1}}{\frac{\partial r_1}{\partial r}}\Rightarrow$$

$$\boxed{u'^2 f'(r)=\overline{u_1(\vec{x})\frac{\partial u_1(\vec{y})}{\partial y_1}}\frac{r}{r_1}} \qquad (1)$$

Take derivative with respect to r again:

$$u'^2\frac{\partial f'(r)}{\partial r}=u'^2 f''(r)=\frac{\partial}{\partial r}\left[\overline{u_1(\vec{x})\frac{\partial u_1(\vec{y})}{\partial y_1}}\frac{r}{r_1}\right]\Rightarrow$$

$$u'^2 f''(r)=\frac{r}{r_1}\overline{\frac{\partial u_1(\vec{y})}{\partial y_1}\frac{\partial}{\partial r}[u_1(\vec{x})]}+\frac{r}{r_1}\overline{u_1(\vec{x})\frac{\partial}{\partial r}\left[\frac{\partial u_1(\vec{y})}{\partial y_1}\right]}+\overline{u_1(\vec{x})\frac{\partial u_1(\vec{y})}{\partial y_1}}\frac{\partial}{\partial r}\left[\frac{r}{r_1}\right]$$

$$u'^2 f''(r)=\frac{r}{r_1}\cancel{\overline{\frac{\partial u_1(\vec{y})}{\partial y_1}\frac{\partial u_1(\vec{x})}{\partial x_1}}\overset{=0}{\frac{\partial x_1}{\partial r}}}+\frac{r}{r_1}\overline{u_1(\vec{x})\frac{\partial^2 u_1(\vec{y})}{\partial y_1^2}}\frac{\partial y_1}{\partial r}+\overline{u_1(\vec{x})\frac{\partial u_1(\vec{y})}{\partial y_1}}\left[\frac{1}{r_1}+r\frac{\partial}{\partial r}\left(\frac{1}{r_1}\right)\right]\Rightarrow$$

$$u'^2 f''(r)=\frac{r}{r_1}\overline{u_1(\vec{x})\frac{\partial^2 u_1(\vec{y})}{\partial y_1^2}}\frac{\partial r_1}{\partial r}+\overline{u_1(\vec{x})\frac{\partial u_1(\vec{y})}{\partial y_1}}\left[\frac{1}{r_1}-r\frac{1}{r_1^2}\overset{=\frac{r}{r_1}}{\frac{\partial r_1}{\partial r}}\right]$$

$$u'^2 f''(r)=\frac{r}{r_1}\overline{u_1(\vec{x})\frac{\partial^2 u_1(\vec{y})}{\partial y_1^2}}\frac{r}{r_1}+\overline{u_1(\vec{x})\frac{\partial u_1(\vec{y})}{\partial y_1}}\left[\frac{1}{r_1}-\overset{=1}{\frac{r^2}{r_1^2}}\frac{1}{r_1}\right]$$

$$u'^2 f''(r)=\overset{=1}{\frac{r^2}{r_1^2}}\overline{u_1(\vec{x})\frac{\partial^2 u_1(\vec{y})}{\partial y_1^2}}=\overline{u_1(\vec{x})\frac{\partial^2 u_1(\vec{y})}{\partial y_1^2}}\Rightarrow\boxed{u'^2 f''(r)=\overline{u_1(\vec{x})\frac{\partial^2 u_1(\vec{y})}{\partial y_1^2}}}$$

$r=0, \quad u'^2 f''(0) = \overline{u_l(\vec{x})\,\frac{\partial^2 u_l(\vec{x})}{\partial x_l^2}} \Rightarrow$

$$u'^2 f''(0) = \cancelto{0}{\frac{\partial}{\partial x_l}\overline{\left[u_l(\vec{x})\frac{\partial u_l(\vec{x})}{\partial x_l}\right]}} - \overline{\left(\frac{\partial u_l(\vec{x})}{\partial x_l}\right)^2} \Rightarrow$$

homogeneous

$u'^2 f''(0) = -\overline{\left[\frac{\partial u_l(\vec{x})}{\partial x_l}\right]^2}, \qquad f''(0) = -2/\lambda_f^2 \Rightarrow$

$u'^2 f''(0) = -\overline{\left[\frac{\partial u_l(\vec{x})}{\partial x_l}\right]^2} = -\frac{2u'^2}{\lambda_f^2} \Rightarrow$

scalar $\rightarrow \boxed{\overline{\left[\frac{\partial u_l(\vec{x})}{\partial x_l}\right]^2} = \frac{2u'^2}{\lambda_f^2}}$ $\quad \boxed{l=1, \ \overline{\left[\frac{\partial u_1(\vec{x})}{\partial x_1}\right]^2} = \frac{2u'^2}{\lambda_f^2} \quad 6.56}$