$$\frac{D}{Dt}\left(\tfrac{1}{2}v_i v_i\right) = \frac{\partial}{\partial t}\left(\tfrac{1}{2}v_i v_i\right) + v_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}v_i v_i\right)$$

$$= \frac{\partial}{\partial t}\left[\tfrac{1}{2}(v^2+v^2+w^2)\right] + v\frac{\partial}{\partial x}(K) + v\frac{\partial}{\partial y}(K) + w\frac{\partial}{\partial z}(K)$$

$K = \frac{1}{2}v_i v_i$

$$= \frac{\partial K}{\partial t} + v K_x + v K_y + w K_z$$

$$= \frac{DK}{Dt} = \frac{\partial K}{\partial t} + \underline{v}\cdot\nabla K \qquad \text{Scalar}$$

$$\frac{\partial}{\partial x_i}\left(-\frac{\bar{p}v_i}{\rho}\right) = -\rho^{-1}\left[(\bar{p}v)_x + (\bar{p}v)_y + (\bar{p}w)_z\right] \qquad \nabla\cdot\underline{f} = \text{Scalar}$$

$$2\nu\frac{\partial}{\partial x_j}v_i E_{ij} = \nu\frac{\partial}{\partial x_j}\left[v_i(v_{i,j}+v_{j,i})\right] \quad \left[\tfrac{\partial}{\partial x_j}\right]\left[[v_i][\varepsilon_{ij}]\right]$$

column, row, 3x3

$\underbrace{\qquad\qquad}_{\text{vector}}$

$\nabla\cdot\underline{f}$ = Scalar

$$2(v v_x)_x + [v(v_y+v_x)]_y + [v(v_z+w_x)]_z$$
$$+ [v(v_x+v_y)]_x + 2(v v_y)_y + [v(v_z+w_y)]_z$$
$$+ [w(w_x+v_z)]_x + [w(w_y+v_z)]_y + 2(w w_z)_z$$

$$-\frac{\partial}{\partial x_j}\left(\overline{u_i u_j}\, v_i\right) = -\Big[(\overline{u^2}v)_x + (\overline{uv}v)_y + (\overline{uw}v)_z$$
$$+ (\overline{uv}v)_x + (\overline{v^2}v)_y + (\overline{vw}v)_z$$
$$+ (\overline{uw}w)_x + (\overline{vw}w)_y + (\overline{w^2}w)_z\Big]$$

Symmetric

$\therefore v_i\overline{u_i u_j}$

$$-2\nu E_{ij}E_{ij} = -2\nu\tfrac{1}{4}(v_{i,j}+v_{j,i})^2 = -\tfrac{\nu}{2}(v_{i,j}+v_{j,i})^2$$
$$= -\nu(v_{i,j}^2 + v_{i,j}v_{j,i})$$

Inner product two 2nd order tensors = 2nd order tensor & contraction over i,j = Scalar

$$v_x^2 + v_y^2 + v_z^2 \quad + v_x^2 + v_y v_x + v_z w_x$$
$$+ v_x^2 + v_y^2 + v_z^2 \quad + v_x v_y + v_y^2 + v_z w_y$$
$$+ w_x^2 + w_y^2 + w_z^2 \quad + w_x v_z + w_y v_z + w_z^2$$

$$\overline{u_i u_j}\frac{\partial v_i}{\partial x_j} = \overline{u^2}v_x + \overline{uv}v_y + \overline{uw}v_z$$
$$+ \overline{uv}v_x + \overline{v^2}v_y + \overline{vw}v_z$$
$$+ \overline{uw}w_x + \overline{vw}w_y + \overline{w^2}w_z$$

$$\frac{D k}{D t} = \frac{\partial k}{\partial t} + \underline{v} \cdot \nabla k = \frac{\partial k}{\partial t} + \overline{u} k_x + \overline{v} k_y + \overline{w} k_z \quad \text{Scalar}$$

$$k = \tfrac{1}{2} u_i u_i = \tfrac{1}{2} \underline{u} \cdot \underline{u}$$

$$= -\frac{\partial}{\partial x_j} \left[\rho^{-1} \overline{p u_j} + \tfrac{1}{2} \overline{u_i^2 u_j} - 2\nu \overline{u_i \varepsilon_{ij}} \right] - \overline{u_i u_j} \, U_{i,j}$$

$$- 2\nu \overline{\varepsilon_{ij} \varepsilon_{ij}}$$

$$\rho^{-1} \frac{\partial}{\partial x_j} (\overline{p u_j}) = (\overline{p u})_x + (\overline{p v})_y + (\overline{p w})_z \quad \nabla \cdot \underline{f} = \text{Scalar}$$

$$\tfrac{1}{2} \frac{\partial}{\partial x_j} (\overline{u_i^2 u_j}) = (\overline{u^3})_x + (\overline{u^2 v})_y + (\overline{u^2 w})_z$$

u_i^2 ← $u_i u_i$ = Scalar

$$+ (\overline{v^2 u})_x + (\overline{v^3})_y + (\overline{v^2 w})_z$$

$\nabla \cdot \underline{f} =$ Scalar; $\underbrace{u_j}_{\underline{f}}$

$$+ (\overline{w^2 u})_x + (\overline{w^2 v})_y + (\overline{w^3})_z$$

$$-2\nu \frac{\partial}{\partial x_j} (\overline{u_i \varepsilon_{ij}}) = 2(u u_x)_x + [u(u_y + v_x)]_y + [u(u_z + w_x)]_z$$

$\nabla \cdot \underline{f} =$ Scalar

$$+ [v(v_x + u_y)]_x + 2(v v_y)_y + [v(v_z + w_y)]_z$$

$$+ [w(w_x + u_z)]_x + [w(w_y + v_z)]_y + 2(w w_z)_z$$

Contraction i, j $\quad - \overline{u_i u_j} \, U_{i,j} = P =$ Same term MKE equation

$N = 2$

Tensors $\quad 2\nu \overline{\varepsilon_{ij} \varepsilon_{ij}} = \nu \, (\underbrace{u_{i,j}^2 + u_{i,j} u_{j,i}})$

= Scalar

$$u_x^2 + u_y^2 + u_z^2 \quad + u_x^2 + u_y v_x + u_z w_x$$

$$+ v_x^2 + v_y^2 + v_z^2 \quad + v_x u_y + v_y^2 + v_z w_y$$

$$+ w_x^2 + w_y^2 + w_z^2 \quad + w_x u_z + w_y v_z + w_z^2$$