ε equation (scalar equation)

$$\varepsilon = 2\nu \overline{\varepsilon_{ij}\varepsilon_{ij}} = \nu \overline{u_{i,j}^2} + \nu \overline{u_{i,j} u_{j,i}} = \tilde{\varepsilon} + \nu \frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j}$$

$\tilde{\varepsilon} = \nu \overline{u_{i,j} u_{i,j}} \approx \varepsilon$ since $u_{i,j} u_{j,i} <$ few % ε

and $= 0$ for homogeneous isotropic turbulence

$U_i = \overline{U}_i + u_i$

$$\frac{\partial}{\partial x_j}\left[\rho \left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right) = \rho g_i - \frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 U_i}{\partial x_j^2} \right] \times 2\nu \frac{\partial u_i}{\partial x_j} \text{ and average}$$

For example $\frac{\partial U_i}{\partial t}$ term $= 2\nu \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial^2 U_i}{\partial x_j \partial t}} = 2\nu \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j \partial t}} = \frac{\partial \varepsilon}{\partial t}$

assuming $\varepsilon = \tilde{\varepsilon}$

$$\frac{D\epsilon}{Dt} = P_\epsilon^1 + P_\epsilon^2 + P_\epsilon^3 + P_\epsilon^4 + \Pi_\epsilon + T_\epsilon + D_\epsilon - \Upsilon_\epsilon, \quad (3.42)$$

Most terms physical meaning not known

where

$$P_\epsilon^1 = -\epsilon_{ij}^c \frac{\partial \overline{U}_i}{\partial x_j} \quad (3.43)$$

$$P_\epsilon^2 = -\epsilon_{ij} \frac{\partial \overline{U}_i}{\partial x_j} \quad (3.44)$$

$$P_\epsilon^3 = -2\nu \overline{u_k \frac{\partial u_i}{\partial x_j}} \frac{\partial^2 \overline{U}_i}{\partial x_k \partial x_j} \quad (3.45)$$

$$P_\epsilon^4 = -2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_j}} \quad (3.46)$$

$$\Pi_\epsilon = -\frac{2\nu}{\rho} \frac{\partial}{\partial x_i} \left(\overline{\frac{\partial p}{\partial x_j} \frac{\partial u_i}{\partial x_j}} \right) \quad (3.47)$$

$$T_\epsilon = -\nu \frac{\partial}{\partial x_k} \left(\overline{u_k \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}} \right) \quad (3.48)$$

$$D_\epsilon = \nu \nabla^2 \epsilon \quad (3.49)$$

$$\Upsilon_\epsilon = 2\nu^2 \overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_k} \right)^2}. \quad (3.50)$$

Exact equation = processes' dissipation range vs modeled equation used for large scale motions and energy cascade inertial subrange.

$\Pi_\varepsilon, T_\varepsilon, D_\varepsilon$ gradient form so mostly redistribution but $\neq 0$ boundaries so may also be sources/sinks

$\Upsilon_\varepsilon < 0$ so exclusively loss of ε

P_ε terms production $\gtrless 0$

In Eqs (3.43) and (3.44),

$$\epsilon_{ij}^c = 2\nu \overline{\frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}} \quad (3.51)$$

$$\epsilon_{ij} = 2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} \quad (3.52)$$

are referred to as the complimentary dissipation rate tensor and the dissipation rate tensor, respectively. For both of these quantities the trace satisfies $\epsilon_{ii} = 2\epsilon$.

Much simplified homogeneous isotropic and shear flows where many terms = 0 so useful develop model ε equation

10.4.2 The model equation for ε

Quite different approaches are taken in developing the model transport equations for k and ε. The k equation amounts to the exact equation (Eq. (10.35)) with the turbulent flux $\boldsymbol{T}'$ modelled as gradient diffusion (Eq. (10.40)). The three other terms – $\bar{\mathrm{D}}k/\bar{\mathrm{D}}t$, $\mathcal{P}$, and ε – are in closed form (given the turbulent-viscosity hypothesis).

The exact equation for ε can also be derived, but it is not a useful starting point for a model equation. This is because (as discussed in Chapter 6) ε is best viewed as the energy-flow rate in the cascade, and it is determined by the large-scale motions, independent of the viscosity (at high Reynolds number). By contrast, the exact equation for ε pertains to processes in the dissipative range. Consequently, rather than being based on the exact equation, the standard model equation for ε is best viewed as being entirely empirical: it is

$$\frac{\bar{\mathrm{D}}\varepsilon}{\bar{\mathrm{D}}t} = \nabla \cdot \left(\frac{\nu_{\mathrm{T}}}{\sigma_\varepsilon} \nabla \varepsilon \right) + C_{\varepsilon 1} \frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k}. \tag{10.53}$$

The standard values of all the model constants due to Launder and Sharma (1974) are

$$C_\mu = 0.09, \; C_{\varepsilon 1} = 1.44, \; C_{\varepsilon 2} = 1.92, \; \sigma_k = 1.0, \; \sigma_\varepsilon = 1.3. \tag{10.54}$$

An understanding of the ε equation can be gained by studying its behaviors in various flows. We first examine homogeneous turbulence, for which the k and ε equations become

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \mathcal{P} - \varepsilon, \tag{10.55}$$

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = C_{\varepsilon 1} \frac{\mathcal{P}\varepsilon}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k}. \tag{10.56}$$

Decaying turbulence

In the absence of mean velocity gradients, the production is zero, and the turbulence decays. The equations then have the solutions

$$k(t) = k_0 \left(\frac{t}{t_0} \right)^{-n}, \quad \varepsilon(t) = \varepsilon_0 \left(\frac{t}{t_0} \right)^{-(n+1)}, \tag{10.57}$$

where k and ε have the values k_0 and ε_0 at the reference time

$$t_0 = n \frac{k_0}{\varepsilon_0}, \tag{10.58}$$

Transport equation for dissipation rate

Exact transport equations for dissipation rates of turbulent kinetic energy, temperature variance and turbulent heat flux are derived using Navier-Stokes and energy equations below. The individual terms in these equations are obtained and Reynolds and Prandtl number dependencies are examined using high spatial resolution DNS (Kozuka *et al.*, 2008).

Nomenclature

k	turbulent kinetic energy, $\overline{u_i'u_i'}/2$
k_θ	temperature variance, $\overline{\theta'\theta'}/2$
p	pressure
t	time
u_i, u, v, w	velocity component
x_1, x	streamwise direction
x_2, y	wall-normal direction
x_3, z	spanwise direction
Greek	
ε	dissipation rate of turbulent kinetic energy, $\nu\left(\partial u_i'/\partial x_j\right)^2$
ε_θ	dissipation rate of temperature variance, $\kappa\left(\partial\theta'/\partial x_j\right)^2$
$\varepsilon_{i\theta}$	dissipation rate of turbulent heat flux, $(\nu+\kappa)\cdot(\partial u_i'/\partial x_j)\cdot(\partial\theta'/\partial x_j)$
κ	thermal diffusivity
θ	temperature
ν	kinematic viscosity
ρ	density
Superscripts	
$(\ )'$	fluctuation component
$\overline{(\ )}$	statistically averaged component

1, Transport equation for ε

Navier-Stokes equation on velocity fluctuation u_i' can be given as follows:

$$\frac{\partial u_i'}{\partial t}+u_k'\frac{\partial \overline{u_i}}{\partial x_k}+\overline{u_k}\frac{\partial u_i'}{\partial x_k}+\frac{\partial}{\partial x_k}\left(u_i'u_k'-\overline{u_i'u_k'}\right)=-\frac{1}{\rho}\frac{\partial p'}{\partial x_i}+\nu\frac{\partial^2 u_i'}{\partial x_k^2}. \tag{1}$$

By differentiating Eq. (1) with x_m, following equation can be obtained:

$$\frac{\partial}{\partial x_m}\left(\frac{\partial u_i'}{\partial t}\right)+\frac{\partial u_k'}{\partial x_m}\frac{\partial \overline{u_i}}{\partial x_k}+u_k'\frac{\partial^2 \overline{u_i}}{\partial x_m\partial x_k}+\frac{\partial \overline{u_k}}{\partial x_m}\frac{\partial u_i'}{\partial x_k}+\overline{u_k}\frac{\partial^2 u_i'}{\partial x_m\partial x_k}$$
$$+\frac{\partial^2}{\partial x_m\partial x_k}\left(u_i'u_k'-\overline{u_i'u_k'}\right)=-\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_m\partial x_i}+\nu\frac{\partial^3 u_i'}{\partial x_m\partial x_k^2}. \tag{2}$$

By multiplying both sides in Eq. (2) by $2\nu\left(\partial u_i'/\partial x_m\right)$, ensemble-averaged terms in Eq. (2) can be expressed as below:

$$2\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial}{\partial x_m}\left(\frac{\partial u_i'}{\partial t}\right)}=2\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial}{\partial t}\left(\frac{\partial u_i'}{\partial x_m}\right)}=\frac{\partial}{\partial t}\left(\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\right)=\frac{\partial\varepsilon}{\partial t}$$

$$2\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}\frac{\partial \overline{u_i}}{\partial x_k}} = \nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}\left(\frac{\partial \overline{u_i}}{\partial x_k}+\frac{\partial \overline{u_i}}{\partial x_k}\right) = \nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}\left(\frac{\partial \overline{u_i}}{\partial x_k}+\frac{\partial \overline{u_k}}{\partial x_i}\right)$$

$$2\nu\overline{\frac{\partial u_i'}{\partial x_m}u_k'\frac{\partial^2 \overline{u_i}}{\partial x_m \partial x_k}} = \nu\overline{u_k'\frac{\partial u_i'}{\partial x_m}}\frac{\partial^2 \overline{u_i}}{\partial x_m \partial x_k}$$

$$2\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial \overline{u_k}}{\partial x_m}\frac{\partial u_i'}{\partial x_k}} = \nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_k}}\left(\frac{\partial \overline{u_k}}{\partial x_m}+\frac{\partial \overline{u_k}}{\partial x_m}\right) = \nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_k}}\left(\frac{\partial \overline{u_k}}{\partial x_m}+\frac{\partial \overline{u_m}}{\partial x_k}\right)$$

$$2\nu\overline{\frac{\partial u_i'}{\partial x_m}\overline{u_k}\frac{\partial^2 u_i'}{\partial x_m \partial x_k}} = \overline{u_k}\frac{\partial}{\partial x_k}\left(\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\right) = \overline{u_k}\frac{\partial \overline{\varepsilon}}{\partial x_k}$$

$$2\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial^2}{\partial x_m \partial x_k}\left(u_i'u_k' - \overline{u_i'u_k'}\right)} = 2\nu\overline{\frac{\partial u_i'}{\partial x_m}\left\{\frac{\partial^2}{\partial x_m \partial x_k}\left(u_i'u_k'\right) - \frac{\partial^2 \overline{u_i'u_k'}}{\partial x_m \partial x_k}\right\}}$$

$$= 2\nu\overline{\frac{\partial u_i'}{\partial x_m}\left\{\frac{\partial}{\partial x_m}\left(u_k'\frac{\partial u_i'}{\partial x_k} + u_i'\frac{\partial u_k'}{\partial x_k}\right) - \frac{\partial^2 \overline{u_i'u_k'}}{\partial x_m \partial x_k}\right\}}$$

$$= 2\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}\frac{\partial u_i'}{\partial x_k}} + \frac{\partial}{\partial x_k}\left(\nu\overline{u_k'\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\right)$$

$$-2\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{1}{\rho}\frac{\partial^2 p'}{\partial x_m \partial x_i}} = -\frac{2}{\rho}\frac{\partial}{\partial x_k}\left(\overline{\frac{\partial u_k'}{\partial x_m}\frac{\partial p'}{\partial x_m}}\right)$$

$$2\nu\overline{\frac{\partial u_i'}{\partial x_m}\nu\frac{\partial^3 u_i'}{\partial x_m \partial x_k^2}} = 2\nu^2\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial^2 u_i'}{\partial x_k^2}\frac{\partial u_i'}{\partial x_m}} = \nu\frac{\partial^2}{\partial x_k^2}\left(\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\right) - 2\left(\nu\overline{\frac{\partial^2 u_i'}{\partial x_m \partial x_k}}\right)^2$$

$$= \nu\frac{\partial^2}{\partial x_k^2}\left(\overline{\varepsilon}\right) - 2\left(\nu\overline{\frac{\partial^2 u_i'}{\partial x_m \partial x_k}}\right)^2$$

Therefore exact transport equation for dissipation rate ε of turbulent energy can be given as follows:

$$\frac{D\overline{\varepsilon}}{Dt} = P_\varepsilon^{(1)} + P_\varepsilon^{(2)} + P_\varepsilon^{(3)} + P_\varepsilon^{(4)} + T_\varepsilon + V_\varepsilon + \pi_\varepsilon - \gamma_\varepsilon, \tag{3}$$

Mixed Production:

$$P_\varepsilon^{(1)} = \nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}\left(\frac{\partial \overline{u_i}}{\partial x_k}+\frac{\partial \overline{u_k}}{\partial x_i}\right) \tag{4}$$

Production by mean velocity gradient:

$$P_\varepsilon^{(2)} = \nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_k}}\left(\frac{\partial \overline{u_k}}{\partial x_m}+\frac{\partial \overline{u_m}}{\partial x_k}\right) \tag{5}$$

Gradient production:

$$P_\varepsilon^{(3)} = \nu\overline{u_k'\frac{\partial u_i'}{\partial x_m}}\frac{\partial^2 \overline{u_i}}{\partial x_m \partial x_k} \tag{6}$$

Turbulent production:

$$P_\varepsilon^{(4)} = 2\nu\overline{\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}\frac{\partial u_i'}{\partial x_k}} \tag{7}$$

Turbulent diffusion:

$$T_\varepsilon = \frac{\partial}{\partial x_k}\left(\overline{\nu u'_k \frac{\partial u'_i}{\partial x_m}\frac{\partial u'_i}{\partial x_m}}\right) \tag{8}$$

Viscous diffusion:

$$V_\varepsilon = \nu \frac{\partial^2}{\partial x_k^2}\left(\overline{\nu \frac{\partial u'_i}{\partial x_m}\frac{\partial u'_i}{\partial x_m}}\right) \tag{9}$$

Pressure diffusion:

$$\pi_\varepsilon = -\frac{2}{\rho}\frac{\partial}{\partial x_k}\left(\overline{\frac{\partial u'_k}{\partial x_m}\frac{\partial p'}{\partial x_m}}\right) \tag{10}$$

Dissipation:

$$\gamma_\varepsilon = 2\left(\overline{\nu \frac{\partial^2 u'_i}{\partial x_m \partial x_k}}\right)^2 \tag{11}$$

2, Transport equation for ε_θ

By differentiating energy equation on temperature fluctuation θ' expressed by Eq. (12) with x_m and multiplying by $2\kappa\,(\partial\theta'/\partial x_m)$, ensemble-averaged terms in Eq. (12) can be given as below.

$$\frac{\partial \theta'}{\partial t} + u'_k \frac{\partial \overline{\theta}}{\partial x_k} + \overline{u_k}\frac{\partial \theta'}{\partial x_k} + u'_k \frac{\partial \theta'}{\partial x_k} - \frac{\partial \overline{u'_k\theta'}}{\partial x_k} = \kappa \frac{\partial^2 \theta'}{\partial x_k^2} \tag{12}$$

$$\overline{2\kappa \frac{\partial \theta'}{\partial x_m}\frac{\partial}{\partial x_m}\left(\frac{\partial \theta'}{\partial t}\right)} = \frac{\partial}{\partial t}\left\{\overline{\kappa\left(\frac{\partial \theta'}{\partial x_m}\right)^2}\right\} = \frac{\partial \overline{\varepsilon_\theta}}{\partial t}$$

$$\overline{2\kappa \frac{\partial \theta'}{\partial x_m}\frac{\partial}{\partial x_m}\left(u'_k \frac{\partial \overline{\theta}}{\partial x_k}\right)} = 2\kappa \overline{\frac{\partial \theta'}{\partial x_m}\left(\frac{\partial u'_k}{\partial x_m}\frac{\partial \overline{\theta}}{\partial x_k} + u'_k \frac{\partial^2 \overline{\theta}}{\partial x_m \partial x_k}\right)}$$

$$= 2\kappa \overline{\frac{\partial \theta'}{\partial x_m}\frac{\partial u'_k}{\partial x_m}}\frac{\partial \overline{\theta}}{\partial x_k} + 2\kappa \overline{u'_k \frac{\partial \theta'}{\partial x_m}}\frac{\partial^2 \overline{\theta}}{\partial x_m \partial x_k}$$

$$\overline{2\kappa \frac{\partial \theta'}{\partial x_m}\frac{\partial}{\partial x_m}\left(\overline{u_k}\frac{\partial \theta'}{\partial x_k}\right)} = 2\kappa \overline{\frac{\partial \theta'}{\partial x_m}\left(\frac{\partial \overline{u_k}}{\partial x_m}\frac{\partial \theta'}{\partial x_k} + \overline{u_k}\frac{\partial^2 \theta'}{\partial x_m \partial x_k}\right)}$$

$$= 2\kappa \overline{\frac{\partial \theta'}{\partial x_m}\frac{\partial \theta'}{\partial x_k}}\frac{\partial \overline{u_k}}{\partial x_m} + \overline{u_k}\frac{\partial}{\partial x_k}\left(\overline{\kappa \frac{\partial \theta'}{\partial x_k}\frac{\partial \theta'}{\partial x_k}}\right) = 2\kappa \overline{\frac{\partial \theta'}{\partial x_m}\frac{\partial \theta'}{\partial x_k}}\frac{\partial \overline{u_k}}{\partial x_m} + \overline{u_k}\frac{\partial \overline{\varepsilon_\theta}}{\partial x_k}$$

$$\overline{2\kappa \frac{\partial \theta'}{\partial x_m}\frac{\partial}{\partial x_m}\left(u'_k \frac{\partial \theta'}{\partial x_k}\right)} = 2\kappa \overline{\frac{\partial \theta'}{\partial x_m}\frac{\partial u'_k}{\partial x_m}\frac{\partial \theta'}{\partial x_k}} + \kappa \frac{\partial}{\partial x_k}\left(\overline{u'_k \frac{\partial \theta'}{\partial x_m}\frac{\partial \theta'}{\partial x_m}}\right)$$

$$\overline{2\kappa \frac{\partial \theta'}{\partial x_m}\frac{\partial}{\partial x_m}\frac{\partial}{\partial x_m}\left(\overline{u'_k\theta'}\right)} = 0$$

$$\overline{2\kappa \frac{\partial \theta'}{\partial x_m}\frac{\partial}{\partial x_m}\left(\kappa \frac{\partial^2 \theta'}{\partial x_k \partial x_k}\right)} = \frac{\partial}{\partial x_k}\left\{\frac{\partial}{\partial x_k}\kappa^2 \overline{\left(\frac{\partial \theta'}{\partial x_m}\right)^2}\right\} - 2\kappa^2\left(\overline{\frac{\partial^2 \theta'}{\partial x_k \partial x_k}}\right)^2$$

mean vorticity equation

multiply by $\overline{\Omega}_i$ for: mean enstrophy equation (scalar equation)

$$\frac{\partial(\overline{\Omega}_i^2/2)}{\partial t} + U_j\frac{\partial(\overline{\Omega}_i^2/2)}{\partial x_j} = \overline{\Omega}_i\,\overline{\Omega}_j\frac{\partial U_i}{\partial x_j} - \overline{\Omega}_i\frac{\partial}{\partial x_j}\Big(\underbrace{\overline{\omega_i u_j}}_{(a)} - \underbrace{\overline{u_i \omega_j}}_{(b)}\Big) + \nu\frac{\partial^2(\overline{\Omega}_i^2/2)}{\partial x_j \partial x_j} - \nu\frac{\partial\overline{\Omega}_i}{\partial x_j}\frac{\partial\overline{\Omega}_i}{\partial x_j}$$

$$\frac{D(\overline{\Omega}_i^2/2)}{Dt} = \text{material derivative } \overline{\Omega}_i^2/2 = \frac{\overline{\Omega}_i\,\overline{\Omega}_i}{2}$$

$$\frac{D(\overline{\Omega}_i^2/2)}{Dt} = \underbrace{\overline{\Omega}_i\,\overline{\Omega}_j\frac{\partial U_i}{\partial x_j}}_{I} \underbrace{- \frac{\partial}{\partial x_j}\left(\overline{\Omega}_i\,\overline{\omega_i u_j}\right)}_{II} + \underbrace{\overline{\omega_i u_j}\frac{\partial\overline{\Omega}_i}{\partial x_j}}_{III} + \underbrace{\overline{\Omega}_i\frac{\partial}{\partial x_j}\left(\overline{u_i\omega_j}\right)}_{IV} + \underbrace{\nu\frac{\partial^2(\overline{\Omega}_i^2/2)}{\partial x_j\partial x_j}}_{V} - \underbrace{\nu\frac{\partial\overline{\Omega}_i}{\partial x_j}\frac{\partial\overline{\Omega}_i}{\partial x_j}}_{VI}$$

$\overline{\Omega}_i^2/2$ = mean enstrophy

I = mean flow stretching

II = transport by velocity/vorticity interaction

III = gradient production fluctuating vorticity

II + III = turbulence advection

IV = stretching turbulent vorticity by turbulence deformation rate
= turbulence stretching

V = viscous transport (diffusion)

VI = dissipation mean vorticity

$$\underbrace{-\frac{\partial}{\partial x_j}\left(\overline{\Omega}_i\,\overline{\omega_i u_j}\right)}_{II} = \underbrace{-\overline{\omega_i u_j}\frac{\partial\overline{\Omega}_i}{\partial x_j}}_{III} - \overline{\Omega}_i\underbrace{\frac{\partial}{\partial x_j}\left(\overline{\omega_i u_j}\right)}_{(a)}$$

II + III = (a)

IV = (b)

A Reynolds averaged vorticity transport equation can be derived by taking the curl of the Reynolds averaged Navier Stokes equations, which for steady flow is:

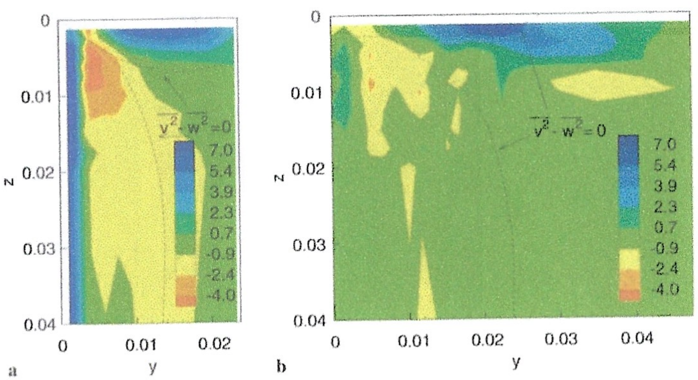
$$\left(\bar{U}\frac{\partial\Omega_x}{\partial x}+\bar{V}\frac{\partial\Omega_x}{\partial y}+\bar{W}\frac{\partial\Omega_x}{\partial z}\right)=\left(\Omega_x\frac{\partial\bar{U}}{\partial x}+\Omega_y\frac{\partial\bar{U}}{\partial y}+\Omega_z\frac{\partial\bar{U}}{\partial z}\right)+\upsilon\left(\frac{\partial^2\Omega_x}{\partial x^2}+\frac{\partial^2\Omega_x}{\partial y^2}+\frac{\partial^2\Omega_x}{\partial z^2}\right)+\frac{\partial}{\partial x}\left(\frac{\partial\overline{u'v'}}{\partial z}-\frac{\partial\overline{u'w'}}{\partial y}\right)+\frac{\partial^2}{\partial y\partial z}(\overline{v'v'}-\overline{w'w'})+\left(\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial y^2}\right)\overline{v'w'}$$

$$\left(\bar{U}\frac{\partial\Omega_y}{\partial x}+\bar{V}\frac{\partial\Omega_y}{\partial y}+\bar{W}\frac{\partial\Omega_y}{\partial z}\right)=\left(\Omega_x\frac{\partial\bar{V}}{\partial x}+\Omega_y\frac{\partial\bar{V}}{\partial y}+\Omega_z\frac{\partial\bar{V}}{\partial z}\right)+\upsilon\left(\frac{\partial^2\Omega_y}{\partial x^2}+\frac{\partial^2\Omega_y}{\partial y^2}+\frac{\partial^2\Omega_y}{\partial z^2}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\overline{v'w'}}{\partial x}-\frac{\partial\overline{v'u'}}{\partial z}\right)+\frac{\partial^2}{\partial z\partial x}(\overline{w'w'}-\overline{u'u'})+\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial z^2}\right)\overline{w'u'}$$

$$\underbrace{\left(\bar{U}\frac{\partial\Omega_z}{\partial x}+\bar{V}\frac{\partial\Omega_z}{\partial y}+\bar{W}\frac{\partial\Omega_z}{\partial z}\right)}_{(A)}=\underbrace{\left(\Omega_x\frac{\partial\bar{W}}{\partial x}+\Omega_y\frac{\partial\bar{W}}{\partial y}+\Omega_z\frac{\partial\bar{W}}{\partial z}\right)}_{(B)}+\underbrace{\upsilon\left(\frac{\partial^2\Omega_z}{\partial x^2}+\frac{\partial^2\Omega_z}{\partial y^2}+\frac{\partial^2\Omega_z}{\partial z^2}\right)}_{(C)}+\underbrace{\frac{\partial}{\partial z}\left(\frac{\partial\overline{w'u'}}{\partial y}-\frac{\partial\overline{w'v'}}{\partial x}\right)}_{(D)}+\underbrace{\frac{\partial^2}{\partial x\partial y}(\overline{u'u'}-\overline{v'v'})}_{(E)}+\underbrace{\left(\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial x^2}\right)\overline{u'v'}}_{(F)}$$

Term (A) represents the material derivative of the mean vorticity. The first term of term (B) is the vorticity amplification by the stretching and the other terms provide vortex-line bending effects. Term (C) presents the vorticity damping by the viscous diffusion, and the other terms (D), (E), and (F) are the vorticity production by inhomogeneity in the Reynolds stress.

Longo, J., Huang, H.P., and Stern, F., "Solid-Fluid Juncture Boundary Layer and Wake," Experiments in Fluids, Vol. 25, No. 4, September 1998, pp. 283 –297



Frederick Stern, "Integrated High-Fidelity Validation Experiments and LES for a Surface-Piercing Truncated Cylinder for Sub-and Critical Reynolds and Froude Numbers," AVT-246: Progress and Challenges in Validation Testing for CFD, Avila, Spain, 26-28 September 2016.

Anisotropy

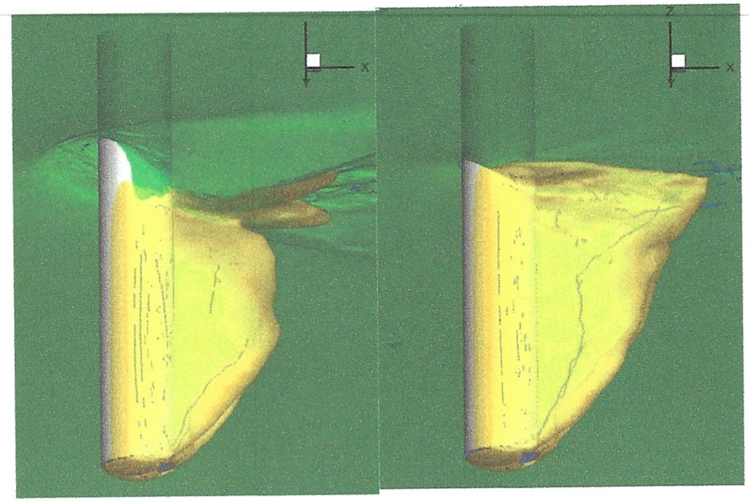


Figure 11-19: Iso-surface of instantaneous Q-criterion (Q=2): (a) sub-critical Re, (b) critical Re; Mean flow separation pattern with vortex core line: visualized approximately using the iso-surfaces of the stagnation C_p=-0.3, (c) sub-critical Re, (d) critical Re.

$$\frac{D\underline{\Omega}}{Dt}=\underline{\Omega}\cdot\nabla\underline{U}-\frac{\partial}{\partial x_j}(\overline{\omega_i u_j}-\overline{u_i\omega_j})+\nu\nabla^2\underline{\Omega}$$

$$\frac{D\underline{\Omega}}{Dt}=\underline{\Omega}\cdot\nabla\underline{U}-\overline{\underline{u}\cdot\nabla\underline{\omega}}+\overline{\underline{\omega}\cdot\nabla\underline{u}}+\nu\nabla^2\underline{\Omega}$$