Chapter 2 Describing Turbulence: Averages, Correlations and Spectra

Spatial and temporal velocity and pressure fields for incompressible turbulent flow obey the Navier-Stokes equations

$$\nabla \cdot \underline{U} = \frac{\partial U_i}{\partial x_i} = 0$$

$$\rho \frac{D\underline{U}}{Dt} = \rho \left(\frac{\partial \underline{U}}{\partial t} + \underline{U} \cdot \nabla \underline{U} \right) = \rho \underline{g} - \nabla p + \mu \nabla^2 \underline{U}$$

$$\rho \left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right) = \rho g_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 U_i}{\partial x_j^2}$$

$$\underline{U}(\underline{x}, t) = (U_1, U_2, U_3) \qquad \underline{x} = (x_1, x_2, x_3)$$
$$= (U, V, W) \qquad = (x, y, z)$$
$$P = P(\underline{x}, t)$$

inertia = body force due grav − pressure gradient force + viscous diffusion

inertia includes $\underline{U} \cdot \nabla \underline{U}$ nonlinear mechanism by which perturbations may self-reinforce

viscous diffusion derives from molecular momentum diffusion: for nondense gases due to molecular collisions, whereas for liquids due to local intermolecular cohesion arising from their close proximity

Diffusion smooths flow perturbations, whereas nonlinear inertia may strengthen them such that their effects are often in conflict

$\underline{x}^* = \underline{x}/L \quad \underline{U}^* = \underline{U}/U$

$\mu \frac{\partial^2 U_i}{\partial x_j^2} = \mu\, U/L^2 \quad \rho U_j \frac{\partial U_i}{\partial x_j} = \rho U^2/L$

$\mu \frac{\partial^2 U_i}{\partial x_j^2} \Big/ U_j \frac{\partial U_i}{\partial x_j} = \mu \frac{U}{L^2} \Big/ \rho U^2/L$

$= \frac{1}{Re} \qquad Re = UL/\nu$

$\nu = \mu/\rho$

$U = m/s$

$x = m$

$\mu = \frac{Ns}{m^2} = \frac{kg}{ms}$

$\rho = kg/m^3$

$\nu = m^2/s$

Small Re viscous diffusion dominates, whereas for large Re inertia dominates, i.e., Re most important parameter distinguish stability of transition to turbulence

Recall viscous diffusion layers:

Viscous layers:

Sudden acceleration flat plate: $u_t = \nu u_{yy}$

$$\delta = 3.64\sqrt{\nu t}$$

$$u(y,0) = 0$$
$$u(0,t) = U$$
$$u(\infty,t) = 0$$

Layer grows in time due viscous diffusion

Oscillating flat plate: $u_t = \upsilon u_{yy}$

$$\delta = 6.5\sqrt{\nu / \omega}$$

$$u(0,t) = U_0 \cos \omega t$$
$$u(\infty,t) = 0$$

Layer confined constant thickness

Stagnation point flow: $\delta = 2.4\sqrt{\nu / B}$ layer not a function of x since convection balances diffusion

Flat plate boundary layer:

$$u_x + v_y = 0$$
$$uu_x + vu_y = \upsilon u_{yy}$$
$$\delta = 4.9\sqrt{\nu x / U}$$
$$u(x,0) = 0$$
$$u(x,\infty) = U$$

Layer grows with x due convection

Ekman Layer on Free Surface: effects due to wind shear τ(x)

$$\delta = \sqrt{\frac{2\nu}{f}} = \text{ Ekman layer thickness}$$

Coriolis force = $2\underline{\Omega}\times\underline{V} = -fv\hat{i}+fu\hat{j}-2\Omega\cos\theta u\hat{k}$ $f=2\Omega\sin\theta$
= planetary vorticity = 2*vertical component Ω

$-fv=\nu u_{ZZ}$ $fu=\nu v_{ZZ}$

$\mu u_z=\tau$ at z = 0 $\underline{\tau}=\tau\hat{i}=.002\rho_{air}(v_{wind}-u(0))\hat{i}$

$v_z=0$ at z = 0

$(u,v)=0$ at z =-∞

The Ekman layer thickness is constant in time and space since vortex diffusion balances the Coriolis force.

Averages:

For turbulent flow $\underline{V}(\underline{x}, t)$, $p(\underline{x}, t)$ are random functions of time and must be evaluated statistically using averaging techniques: time, ensemble, phase, or conditional.

Time Averaging

For stationary flow, the mean is not a function of time and we can use time averaging.

$$\bar{u} = \frac{1}{T}\int_{t_0}^{t_0+t} u(t)\,dt \qquad T > \text{any significant period of } u' = u - \bar{u}$$

(e.g. 1 sec. for wind tunnel and 20 min. for ocean)

Ensemble Averaging

For non-stationary flow, the mean is a function of time and ensemble averaging is used

$$\bar{u}(t) = \frac{1}{N}\sum_{i=1}^{N} u^i(t) \qquad \text{N is large enough that } \bar{u} \text{ independent}$$

$u^i(t) =$ collection of experiments performed under identical conditions (also can be phase aligned for same t=o).

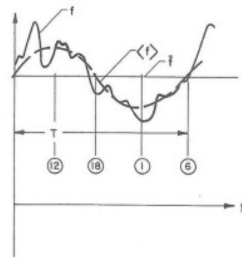


Fig. 6.1 Schematic representation of the triple decomposition: ——, instantaneous signal; - - -, conditionally averaged; ——, mean. The numerals in circles characterize the phase for later reference.

Phase averaging ie $\bar{u}(t)$ is averaged at some t over many realizations

Statistically stationary process = the statistical properties, such as mean, variance and autocorrelation, do not change over time.

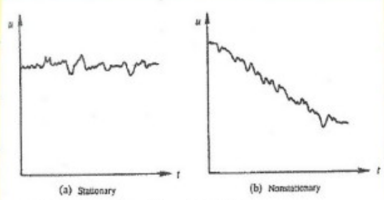


Fig. 12.2 Stationary and nonstationary time series.

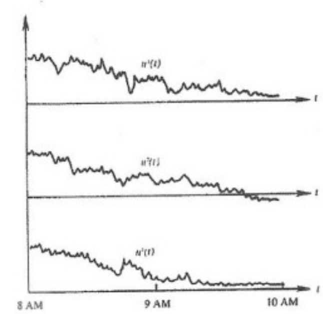


Fig. 12.3 An ensemble of functions $u(t)$.

Phase and Conditional Averaging

Similar to ensemble averaging, but for flows with dominant frequency content or other condition, which is used to align time series for some phase/condition. In this case triple velocity decomposition is used: $u = \bar{u} + u'' + u'$ where u'' is called organized oscillation. Phase/conditional averaging extracts all three components.

Averaging Rules:

$f = \bar{f} + f'$ $\qquad g = \bar{g} + g'$ $\qquad s = x \text{ or } t$

$\overline{f'} = 0$ $\qquad \bar{\bar{f}} = \bar{f}$ $\qquad \overline{\bar{f}g} = \bar{f}\bar{g}$ $\qquad \overline{f'\bar{g}} = 0$

$\overline{f+g} = \bar{f} + \bar{g}$ $\qquad \overline{\frac{\partial f}{\partial s}} = \frac{\partial \bar{f}}{\partial s}$ $\qquad \overline{fg} = \bar{f}\bar{g} + \overline{f'g'}$

$\overline{\int f\,ds} = \int \bar{f}\,ds$

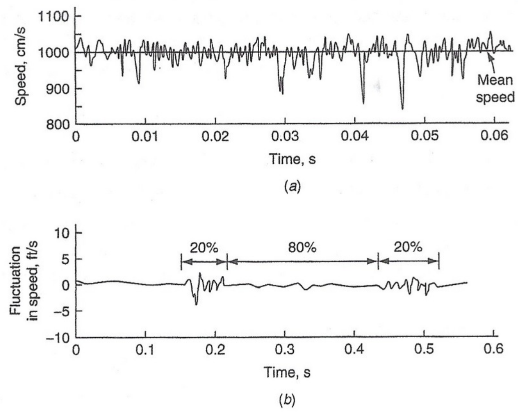


FIGURE 6-3
Hot-wire measurements showing turbulent velocity fluctuations: (*a*) typical trace of a single velocity component in a turbulent flow; (*b*) trace showing intermittent turbulence at the edge of a jet.

Other useful statistical definitions

$$\bar{u} = \frac{1}{T}\int_{t_0}^{t_0+T} u\,dt$$

$u' = u - \bar{u}$ Reynolds decomposition

$\overline{u'} = 0$

$$\overline{u'^2} = \underbrace{\frac{1}{T}\int_{t_0}^{t_0+T} u'^2\,dt}_{\text{mean square}} = \underbrace{\frac{1}{T}\int_{t_0}^{t_0+T} (u-\bar{u})^2\,dt}_{\text{variance}}$$

$\sqrt{\overline{u'^2}}$ = RMS u' = Standard deviation SD u

SD % Mean = coefficient of variation

If integrals independent t_0 u' is statistically stationary

One-Point Statistics

Statistical analysis needed for $\underline{U}$, $\underline{\Omega} = \nabla \times \underline{V}$, P and other variables of interest

Reynolds Decomposition

$\underline{u} = \underline{U} - \overline{\underline{U}}$ $\quad$ $\underline{u}$ = turbulent fluctuation

or $\underline{U} = \overline{\underline{U}} + \underline{u}$ $\quad$ where $\overline{\underline{u}} = 0$ and $\overline{\overline{\underline{U}}} = \overline{\underline{U}}$

The TKE $= K = \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2}) = \overline{u_i u_i}/2$

$= \overline{\underline{u} \cdot \underline{u}}/2$

is of central importance as it characterizes the strength of the turbulence where $\overline{u_i^2}$ are the variances in the x_i coordinate directions, e.g., the inflow $\sqrt{\overline{u^2}}/U$ characterizes the free stream turbulence in a wind tunnel experiment.

If $\overline{u^2} = \overline{v^2} = \overline{w^2}$ the turbulence is isotropic, whereas more generally $\overline{u^2} \neq \overline{v^2} \neq \overline{w^2}$ and the turbulence is nonisotropic, e.g., for near wall turbulence $\overline{v^2} < \overline{u^2}, \overline{w^2}$

Homogeneous turbulence: the time-averaged properties of the flow are uniform and independent of position. For example, whereas $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$ may differ from each other, each must be constant throughout the system.

Isotropic turbulence: Turbulence in which the products and squares of the velocity components and their derivatives are independent of direction, or, more precisely, invariant with respect to rotation and reflection of the coordinate axes in a coordinate system moving with the mean motion of the fluid. Then all the normal stresses are equal, and the tangential stresses are zero.

$\overline{u^2} =$ ¿ $\overline{v^2} =$ ¿ $\overline{w^2}$, $\overline{uv} = \overline{uw} = \overline{vw} = 0$

Isotropy implies homogeneity, but not vice versa.

The normal variances in combination with the off diagonal quadratic products form the Reynolds Stress tensor = symmetric second order tensor

$$R_{ij} = \overline{u_i u_j} = \begin{matrix} \overline{u^2} & \overline{uv} & \overline{uw} \\ \overline{vu} & \overline{v^2} & \overline{vw} \\ \overline{wu} & \overline{wv} & \overline{w^2} \end{matrix}$$

= turbulent velocity correlations / variances and covariances

$$\sigma_t = -\rho R_{ij}$$

$$\overline{\sigma} = -P\delta_{ij} + \mu(\overline{v}_{i,j} + \overline{v}_{j,i}) - \rho\overline{u_i u_j}$$

$R_{12} = \overline{u_1 u_2} = \overline{uv}$ is of central importance for turbulence modeling for fundamental flows such as shear layers and wall flows

ρR_{ij} represents turbulent momentum fluxes

$$M_{ij} = \int \rho v_i v_j n_j dA \qquad v_i = \overline{v}_i + u_i = (\overline{v}+u, v, w)$$

$$\frac{d\overline{M_{xy}}}{dA} = \rho\overline{(\overline{v}+u)v} = \rho\overline{uv} \qquad \overline{\overline{v}v} = 0$$

flux of x momentum in the y direction due to turbulent $u_2 = v$ fluctuation

turbulent transport of momentum and other quantities is one of the primary physical turbulent flow processes!

Higher order statistics

S = skewness = $\overline{u_1^3} / \overline{u_1^2}^{3/2}$

F = flatness or kurtosis = $\overline{u_1^4} / \overline{u_1^2}^{2}$

S = bias or symmetry between + and − values
F = congregation at large or small values

Probability density functions provide information about range of values u_1 takes and the likelihood to assume specific values in that range

$p(u)$ = probability density function u_1
$p(u)\,du$ = probability u_1 in between u and $u+du$

$$\int_{-\infty}^{\infty} p(u)\,du = 1$$ since u must take on some value

$$\overline{u_1} = \int_{-\infty}^{\infty} u\, p(u)\, du = 0 \qquad \overline{u_1^2} = \int_{-\infty}^{\infty} u^2 p(u)\, du$$

$$\overline{u_1 u_2} = \iint uv\, p(u,v)\, du\, dv$$

= joint probability function for $\overline{uv}$

Two-Point Statistics

In order to reveal the structural characteristics of turbulence such as vortical features (eddy size) two-point correlations are required

velocity correlation

$$R_{ij}(\underline{x},\underline{y},t) = \overline{u_i(\underline{x},t)\,u_j(\underline{y},t)}$$ 2 point 2 velocity

$$S_{ijk}(\underline{x},\underline{y},t) = \overline{u_i(\underline{x},t)\,u_j(\underline{x},t)\,u_k(\underline{y},t)}$$ 2 point 3 velocity

For $\underline{x}=\underline{y}$ $R_{ij}(\underline{x},t) = \overline{u_i u_j}$ = Reynolds stress tensor Eq. (2.19)

Letting $\underline{r} = \underline{y} - \underline{x}$ which is appropriate for homogeneous turbulence where $R(\underline{r},t)$ is the same for all $\underline{x}$ (but also important nonhomogeneous flows where $f(\underline{x})$ dependence implied)

central importance isotropic turbulence

$$R_{ij}(\underline{r},t) = \overline{u_i(\underline{x},t)\,u_j(\underline{x}+\underline{r},t)}$$

$$S_{ijk}(\underline{r},t) = \overline{u_i(\underline{x},t)\,u_j(\underline{x},t)\,u_k(\underline{x}+\underline{r},t)}$$

Note that

$$R_{ij}(0,t) = \overline{u_i u_j}$$

$$R_{ij}(\underline{r} \to \text{large value}, t) = 0$$ velocities uncorrelated for distances greater than largest eddy size

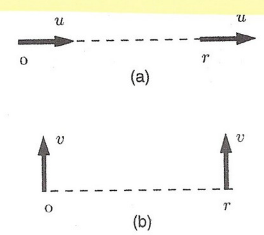
Two-point longitudinal and transverse coefficients $f(r)$ and $g(r)$

$$\overline{u^2}\, f(r) = R_{11}(r\hat{e}_1)$$

$$\overline{v^2}\, g(r) = R_{22}(r\hat{e}_1)$$

where both are $f(t)$ for nonstationary flows

Figure 2.1 (a) Longitudinal and (b) transverse velocities appearing in the definitions of $f(r)$ and $g(r)$, respectively.



$$f(r) = \overline{u(x)\,u(x+r)} \,/\, \overline{u^2(x)} \qquad f(0)=1$$

$$g(r) = \overline{v(x)\,v(x+r)} \,/\, \overline{v^2(x)} \qquad g(0)=1$$

TS $\quad f(r) = 1 + r f_r(0) + \frac{r^2}{2!} f_{rr}(0) + \cdots$

$$\overline{u^2}\, f_r(r) = \overline{u(x)\,u(x+r)_r} = \overline{u(x)\,u_x(x+r)}$$

$$\overline{u^2}\, f_r(0) = \frac{1}{2}\frac{\partial \overline{u^2}}{\partial x}$$

$\therefore$ for homogeneous turbulence where $\overline{u^2}$ = constant

$$f(r) = 1 + \frac{r^2}{2} f_{rr}(0)$$ ie $f(r)$ is parabolic

$$\overline{u^2}\,\frac{df}{dr}(r) = \overline{u(x)\frac{\partial}{\partial r}u(x+r)} \qquad x' = x + r \qquad \frac{\partial x'}{\partial r} = 1$$

$$= \overline{u(x)\,\frac{\partial u}{\partial x'}(x+r)\,\frac{\partial x'}{\partial r}}$$

$$\lim r \to 0 \quad = \overline{u(x)\,\frac{\partial u}{\partial x}(x)}$$

$$\overline{u^2}\,\frac{df}{dr}(0) = \frac{1}{2}\frac{\partial \overline{u^2}}{\partial x} \qquad \text{assuming} \quad \overline{\frac{\partial u^2}{\partial x}} = \frac{\partial \overline{u^2}}{\partial x}$$

$= 0$ for homogeneous turbulence where $\overline{u^2} = $ constant

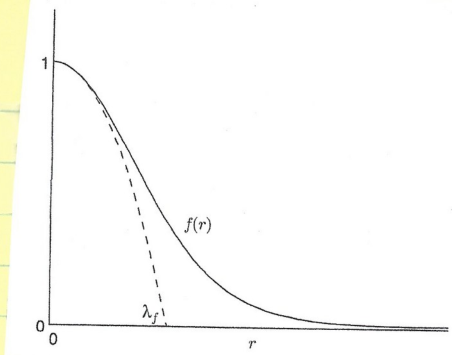


Figure 2.2 Definition of the microscale λ_f.

$f(r) = 0 = 1 + \frac{\lambda_f^2}{2} f_{rr}(0)$ = Taylor microscale
= distance from $r=0$ to r at which $f(r)=0$

$\lambda_f = \left[-2/f_{rr}(0)\right]^{1/2}$

$\lambda_g = \left[-2/g_{rr}(0)\right]^{1/2}$

λ_f & λ_g are measure of size of flow features where viscous effects are important, ie, measure of the smaller scale of turbulence

Whereas a measure of the largest "energetic" scales is given by the integral length scales

$$\Lambda_f = \int_0^\infty f(r)\,dr$$

$$\Lambda_g = \int_0^\infty g(r)\,dr$$

More generally for two random variables
$u_i(\underline{x}_1, t_1)$ & $u_j(\underline{x}_2, t_2)$

$$R_{ij}(\underline{x}_1, t_1, \underline{x}_2, t_2) = \overline{u_i(\underline{x}_1, t_1) u_j(\underline{x}_2, t_2)}$$

R_{ij} specifies how similar $u_i(\underline{x}_1, t_1)$ & $u_j(\underline{x}_2, t_2)$
to each other

$R_{ij} = 0$ $\quad$ $u_i > 0$ & u_i equally likely + or −
ie uncorrelated
or weakly correlated $R_{ij} > 0$ small

$R_{ij} >> 0$ $\quad$ $u_i < 0$ & $u_j < 0$ $\quad$ ie strongly
or $u_i > 0$ & $u_j > 0$ $\quad$ correlated

$R_{ij} << 0$ $\quad$ $u_i > 0$ & $u_j < 0$ $\quad$ ie anti correlated
or $u_i < 0$ & $u_j > 0$

$i \neq j$ $\quad$ cross correlation
$i = j$ $\quad$ auto correlation

$$R_{12}(\underline{x}_1, t_1, \underline{x}_2, t_2) = \overline{u_1(\underline{x}_1, t_1) u_2(\underline{x}_2, t_2)}$$

$$r_{12} = R_{12} / \sqrt{R_{11} R_{22}} = \overline{u_1 u_2} / \sqrt{\overline{u_1^2}\,\overline{u_2^2}}$$

$$R_{11}(\underline{x}_1, t_1, \underline{x}_2, t_2) = \overline{u_1(\underline{x}_1, t_1) u_1(\underline{x}_2, t_2)}$$

$$r_{11} = R_{11} / \sqrt{R_{11} R_{11}} = \overline{u_1 u_1} / \sqrt{\overline{u_1^2}\,\overline{u_1^2}}$$

-1 (anti correlation) $\leq r_{ij} \leq 1$ (correlation) $\quad r_{11}(\underline{x}_1, t_1, \underline{x}_1, t_1) = 1$

$$\overline{u(\underline{x}_1, t_1) u(\underline{x}_2, t_2)} \leq \left[\overline{u^2(\underline{x}_1, t_1)}\right]^{1/2} \left[\overline{u^2(\underline{x}_2, t_2)}\right]^{1/2}$$ Schwartz inequality
analogous $\underline{a} \cdot \underline{b} = ab \cos \alpha$

For temporally stationary flows with u_i sampled at the same point in space $\underline{x} = \underline{x}_1 = \underline{x}_2$, the listing of $\underline{x}$ is not required as the statistics are independent of the time origin such that $\tau = t_2 - t_1$ = time lag and

$$R_{ij}(\tau) = \overline{u_i(t)\,u_j(t+\tau)}$$ correlation

$$R_{11}(\tau) = \overline{u_1(t)\,u_1(t+\tau)}$$ auto correlation

note
$$R_{11}(\tau) = \overline{u_1(t)\,u_1(t+\tau)} = \overline{u_1(t-\tau)\,u_1(t)} = \overline{u_1(t)\,u_1(t-\tau)} = R_{11}(-\tau)$$

$$R_{ij}(\tau) \neq R_{ij}(-\tau) \quad i \neq j$$ cross correlation

$$R_{ij}(0) = u_i u_j$$

Time shift for $\overline{u(t)u(t+\tau)} = 0$ is for equal width crests and troughs $u(t)$

R_{11} is maximum when $\tau = 0$; or when $t = \tau$ and in this case R_{11} is symmetric about t_0

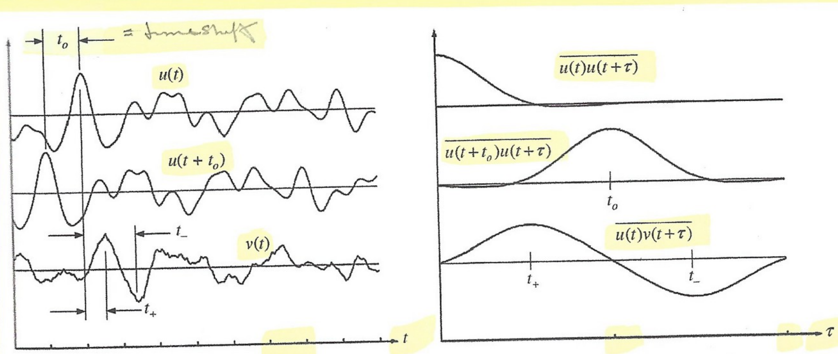


FIGURE 12.4 Sample results for auto- and cross-correlation functions of $u(t)$, $u(t + t_o)$, and $v(t)$. These three time series are shown on the left. The upper curve on the right is the autocorrelation function, $\overline{u(t)u(t+\tau)}$, of the upper time series on the left. The tic marks on the axes represent the same time interval so the width of a peak of $u(t)$ is about equal to the correlation time determined from $\overline{u(t)u(t+\tau)}$. The correlation osf $u(t + t_o)$ and $u(t)$ is shown as the middle curve on the right, and it is just a shifted replica of $\overline{u(t)u(t+\tau)}$. The cross-correlation of $u(t)$ and $v(t)$ is the lower curve on the right. Here the maximum cross-correlation occurs when $\tau = t_+$ and the peaks of u and v coincide. Similarly, u and v are most anti-correlated when peaks in u align with valleys in v at $\tau = t_-$.

Correlation is a mathematical shape-comparison indicator sensitive to space or time alignment

Similar to the longitudinal correlation coefficient $f(r)$ is the temporal correlation coefficient $R_E(\tau) = r_{11}(\tau)$

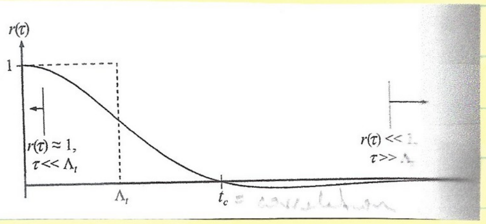
$$r_{11}(\tau) = R_{11}(\tau)/R_{11}(0) = \overline{u(t)u(t+\tau)}\,/\,\overline{u^2} \qquad R_E(\tau)$$

which can be obtained at a fixed point in space. Similarly for r_{22} & r_{33}. Taylor micro and integral macro time "duration" scales can be computed similarly as done previously for the corresponding spatial length scales

$$\Lambda_t = \int_0^\infty r_{11}(\tau)\,d\tau = \frac{1}{\overline{u^2}}\int_0^\infty R_{11}(\tau)\,d\tau$$

= measure of the memory of the turbulence

FIGURE 12.5 Sample plot of an autocorrelation coefficient showing the integral time scale Λ_t, and the correlation time t_c. The normalization requires $r(0) = 1$. In the limit $\tau \to \infty$, $r(\tau) \to 0$ and thereby indicates that the random process used to construct r becomes uncorrelated with itself when the time shift τ is large enough.



time when $r_{11}(\tau)$ first $= 0$

$$\lambda_t^2 = -2\,/\,r_{11,tt}(0) \qquad \tau_E$$

obtained from curvature of the $r_{11}(\tau)$ peak and indicates where $u_1(t)$ is well correlated with itself.

Spatial Spectra

$$R_{ij}(\underline{x},\underline{r},t) = \overline{u_i(\underline{x},t)\,u_j(\underline{x}+\underline{r},t)} \quad [m^2/s^2]$$

for homogeneous turbulence $R_{ij} \neq f(\underline{x})$ ∴ often dependence not shown, but implied for non-homogeneous flows.

Wave number space vector $\underline{k} = (k_1, k_2, k_3)$. The purpose of the Fourier transform is to map various functions from physical space $\underline{x} = (x_1, x_2, x_3)$ to $\underline{k}$ space, which is useful for structural analysis such as distribution of eddy sizes etc.

energy spectrum tensor $[m^5/s^2]$ $\Sigma_{ij}(\underline{x},\underline{k},t) = (2\pi)^{-3}\int\!\!\int\!\!\int_{-\infty}^{\infty} R_{ij}\, e^{i\underline{k}\cdot\underline{r}}\, d\underline{r}$

Σ_{ij} & R_{ij} decompose turbulence into Fourier modes

$d\underline{k} = dk_1 dk_2 dk_3$ = dV in $\underline{k}$ space

$$R_{ij}(\underline{x},\underline{r},t) = \int\!\!\int\!\!\int_{-\infty}^{\infty} \Sigma_{ij}\, e^{-i\underline{k}\cdot\underline{r}}\, d\underline{k}$$

note: $e^{i\underline{k}\cdot\underline{r}} = \cos \underline{k}\cdot\underline{r} + i \sin \underline{k}\cdot\underline{r}$ spatial Fourier mode, which varies sinusoidally with wave length $\lambda = \frac{2\pi}{|\underline{k}|}$

for $\underline{r} = 0$

$$R_{ij}(\underline{x},0,t) = \int\!\!\int\!\!\int_{-\infty}^{\infty} \Sigma_{ij}(\underline{x},\underline{k},t)\, d\underline{k}$$

= auto correlation tensor with components: $\overline{u_1^2}, \overline{u_2^2}, \overline{u_3^2}$, $\overline{u_1u_2}, \overline{u_1u_3}, \overline{u_2u_3}$ at $\underline{x}$ and t

$$K = \frac{1}{2}\left(\overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}\right) \quad [m^2/s^2]$$

= TKE per unit mass (where integration in $\int_0^\infty$)

R_{ij} and Σ_{ij} are $f(\underline{x})$ such that they provide the ability of decomposing the turbulence correlations into a continuous range of scales as represented by the Fourier components $e^{i\underline{\nu}\cdot\underline{r}}$

$tr(R_{ij})/2$ at $r=0$: spectral decomposition TKE

$$K(t) = \frac{1}{2}\int_{\forall} tr\,\Sigma_{ij}(\underline{k},t)\,d\underline{k} = \frac{1}{2}\int_{\forall} \underbrace{\Sigma_{ii}(\underline{k},t)\,d\underline{k}}_{\text{density energy k space}}$$

$$= \int_0^\infty \underbrace{\left[\frac{1}{2}\int_{|\underline{k}|=k} \Sigma_{ii}(\underline{k},t)\,d\Omega\right]}_{\text{energy spectrum}} dk \qquad d\underline{k} = d\Omega\,dk$$

$d\Omega$ = elemental solid angle

$= E(k,t)$ shows how the kinetic energy is distributed across the different scales of the flow ie

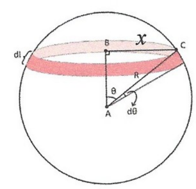
$$K(t) = \int_0^\infty E(k,t)\,dk$$

$E(k,t)$ $[m^3/s^2]$

collect energy shells of fixed distance $k=|\underline{k}|$ from the origin

Calculate the surface area of a sphere with a radius of R:

$dS = Cdl$
where $dl = Rd\theta$; C is the perimeter (circumference) of the red circle, and
$C = 2\pi x = 2\pi R\sin(\theta)$
$dS(R) = 2\pi R\sin(\theta)Rd\theta$



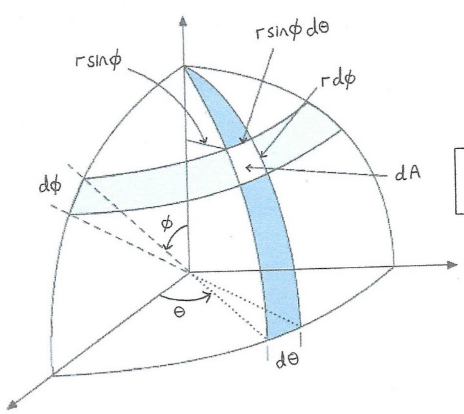
$$\oint dS(k) = \int_0^{\pi} 2\pi k\sin(\theta)kd\theta$$

$$= -2\pi k^2\cos(\theta)|_0^{\pi}$$

$$= 2\pi k^2 - (-2\pi k^2) = 4\pi k^2$$

$$\oint dS(k) = 4\pi k^2 \qquad (6.194)$$

Alternately, the area element can be written as $dS(k) = k^2\sin\varphi d\theta d\varphi$ according to the following figure.



$$dA = r\sin\phi d\theta\, rd\phi = r^2\sin\phi d\phi d\theta$$

$$\oint d\,S(k) = \int_0^{\pi}\int_0^{2\pi} k^2\sin\varphi d\varphi d\theta = \int_0^{\pi} 2\pi k^2\sin\varphi d\varphi = -2\pi k^2\cos\varphi|_0^{\pi} = 4\pi k^2$$

Time Spectra

Fourier Transform pair

$$\hat{R}_E(\omega') = \int_{-\infty}^{\infty} e^{-i\tau\omega'} R_E(\tau)\, d\tau \quad [s] \qquad R_E(\tau) = \frac{\overline{u(t)u(t+\tau)}}{\overline{u^2}}$$

$$R_E(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau\omega'} \hat{R}_E(\omega')\, d\omega' \qquad \omega' = 2\pi\omega \ \frac{rad}{s}$$

ω hz

$$R_E(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{R}_E(\omega')\, d\omega' = 1 = \int_{-\infty}^{\infty} \underbrace{\hat{R}_E(2\pi\omega)}_{= \hat{E}_{11}(\omega)/2\overline{u^2}}\, d\omega$$

$$\int_{-\infty}^{\infty} \frac{\hat{E}_{11}(\omega)}{2\overline{u^2}}\, d\omega = 1$$

define 1D energy spectrum

$$\overline{u^2} = \frac{1}{2}\int_{-\infty}^{\infty} \hat{E}_{11}(\omega)\, d\omega$$

$$\widehat{E_{11}}(\omega) = 2\overline{u^2}\widehat{R_E}(\omega')$$

$\widehat{E_{11}}$ $[m^2/s]$

For stationary flow: $\overline{u(t)u(t+\tau)} = \overline{u(t-\tau)u(t)}$

$$R_E(\tau) = R_E(-\tau)$$

$$\Rightarrow \hat{R}_E(\omega') = 2\int_0^{\infty} \cos\tau\omega'\, R_E(\tau)\, d\tau$$

$$\hat{R}_E(-\omega') = \hat{R}_E(\omega')$$

$$R_E(\tau) = \frac{1}{\pi}\int_0^{\infty} \cos\tau\omega'\, \hat{R}_E(\omega')\, d\omega'$$

$$\overline{u^2} = \int_0^{\infty} \hat{E}_{11}(\omega)\, d\omega$$

Note: $R_E(\tau)$ then $\hat{E}_{11}(\omega)$ can be obtained single point measurement vs $E(r,t)$ requires volume or line measurements and if Taylor frozen turbulence hypothesis used can be transformed from time to space as approximation for spatial spectra.