Turbulent flows HW #2

Problem 5.19

Starting from the Reynolds equation show that the mean-kinetic-energy equation is:

$$\frac{\bar{D}\bar{E}}{\bar{D}t} + \nabla \cdot \bar{\boldsymbol{T}} = -\tilde{P} - \bar{\varepsilon}$$

where

$$\tilde{P} \equiv -\overline{u_i u_j}\frac{\partial \bar{U}_i}{\partial x_j},$$

$$\bar{T}_i \equiv \bar{U}_j\overline{u_i u_j} + \frac{\bar{U}_i \bar{p}}{\rho} - 2\upsilon \bar{U}_j \overline{S_{ij}}.$$

Solution:

Reynolds equation:

$$\frac{\bar{D}\bar{U}_j}{\bar{D}t} = \upsilon \nabla^2 \bar{U}_j - \frac{\partial \overline{u_i u_j}}{\partial x_i} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j}$$

By definition, mean-kinetic-energy is:

$$\bar{E} \equiv \frac{1}{2}\bar{\boldsymbol{U}} \cdot \bar{\boldsymbol{U}}$$

To obtain the mean-kinetic-energy, pre-multiply Reynolds equation by $\bar{U}_j$:

$$\bar{U}_j \frac{\bar{D}\bar{U}_j}{\bar{D}t} = \bar{U}_j \left(\upsilon \nabla^2 \bar{U}_j - \frac{\partial \overline{u_i u_j}}{\partial x_i} - \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j} \right)$$

(A (B) (C) (D

Let’s focus on term (A). Decompose the mean substantial derivative:

$$\bar{U}_j \frac{\bar{D}\bar{U}_j}{\bar{D}t} = \bar{U}_j \left(\frac{\partial \bar{U}_j}{\partial t} + \bar{U}_i \frac{\partial \bar{U}_j}{\partial x_i} \right) = \frac{\partial \left(\frac{1}{2}\bar{U}_j \bar{U}_j \right)}{\partial t} + \bar{U}_i \frac{\partial \left(\frac{1}{2}\bar{U}_j \bar{U}_j \right)}{\partial x_i}$$

Which can be rewritten as:

$$\frac{\bar{D}\bar{U}_j}{\bar{D}t}\bar{U}_j = \frac{\partial \bar{E}}{\partial t} + \bar{U}_i \frac{\partial \bar{E}}{\partial x_i} = \frac{\bar{D}\bar{E}}{\bar{D}t}$$

Let’s now focus on term (B):

$$\overline{U_J}\upsilon(\nabla^2\overline{U_J})$$

By definition of Laplacian operator, we get:

$$\upsilon\overline{U_J}\left(\frac{\partial^2\overline{U_J}}{\partial x_i \partial x_i}\right) = \upsilon\overline{U_J}\frac{\partial}{\partial x_i}\left(\frac{\partial\overline{U_J}}{\partial x_i}\right)$$

Now, we can add and subtract the transpose of the mean velocity gradient:

$$\upsilon\overline{U_J}\left(\frac{\partial^2\overline{U_J}}{\partial x_i \partial x_i}\right) = \upsilon\overline{U_J}\frac{\partial}{\partial x_i}\left(\frac{\partial\overline{U_J}}{\partial x_i}+\frac{\partial\overline{U_\iota}}{\partial x_j}-\frac{\partial\overline{U_\iota}}{\partial x_j}\right) = \upsilon\overline{U_J}\frac{\partial}{\partial x_i}\left(\frac{\partial\overline{U_J}}{\partial x_i}+\frac{\partial\overline{U_\iota}}{\partial x_j}\right) - \upsilon\overline{U_J}\frac{\partial}{\partial x_i}\left(\frac{\partial\overline{U_\iota}}{\partial x_j}\right)$$

Using the symmetry of second derivatives, we can rewrite the last term as:

$$-\upsilon\overline{U_J}\frac{\partial}{\partial x_i}\left(\frac{\partial\overline{U_\iota}}{\partial x_j}\right) = -\upsilon\overline{U_J}\frac{\partial}{\partial x_j}\left(\frac{\partial\overline{U_\iota}}{\partial x_i}\right) = 0$$

And it is equal to zero since we are considering incompressible flow. Therefore, (B) becomes:

$$\upsilon\overline{U_J}\left(\frac{\partial^2\overline{U_J}}{\partial x_i \partial x_i}\right) = \upsilon\overline{U_J}\frac{\partial}{\partial x_i}\left(\frac{\partial\overline{U_J}}{\partial x_i}+\frac{\partial\overline{U_\iota}}{\partial x_j}\right)$$

and using the definition of mean rate of strain:

$$\overline{S_{\iota J}} = \frac{1}{2}\left(\frac{\partial\overline{U_\iota}}{\partial x_j}+\frac{\partial\overline{U_J}}{\partial x_i}\right)$$

We can rewrite (B) as:

$$\upsilon\overline{U_J}\left(\frac{\partial^2\overline{U_J}}{\partial x_i \partial x_i}\right) = 2\upsilon\overline{U_J}\frac{\partial\overline{S_{\iota J}}}{\partial x_i} = 2\upsilon\left(\frac{\partial\overline{S_{\iota J}}\,\overline{U_J}}{\partial x_i} - \frac{\partial\overline{U_J}}{\partial x_i}\overline{S_{\iota J}}\right)$$

Let's move to (C), we can write:

$$-\overline{U_J}\frac{\partial\overline{u_\iota u_J}}{\partial x_i} = -\frac{\partial\overline{U_J}\,\overline{u_\iota u_J}}{\partial x_i} + \overline{u_\iota u_J}\frac{\partial\overline{U_J}}{\partial x_i}$$

For (D), it is necessary to use the continuity equation to obtain:

$$-\overline{U_\iota}\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_i} = -\frac{1}{\rho}\frac{\partial(\bar{p}\overline{U_\iota})}{\partial x_i}$$

Note that we used i instead of j, because the index is repeated, so any letter can be used without distinction.

Now, we can put all the 4 terms in the same equation:

$$\frac{\overline{D}\overline{E}}{\overline{D}t} = 2\upsilon\left(\frac{\partial \overline{U}_J \overline{S_{\iota J}}}{\partial x_i} - \overline{S_{\iota J}}\frac{\partial \overline{U}_J}{\partial x_i}\right) - \frac{\partial \overline{U}_J \overline{u_\iota u_J}}{\partial x_i} + \frac{\partial \overline{U}_J}{\partial x_i}\overline{u_\iota u_J} - \frac{1}{\rho}\frac{\partial(\overline{p}\overline{U}_\iota)}{\partial x_i}$$

We can rewrite the RHS as:

$$\frac{\overline{D}\overline{E}}{\overline{D}t} = \frac{\partial}{\partial x_i}\left(2\upsilon \overline{U}_J \overline{S_{\iota J}} - \overline{U}_J \overline{u_\iota u_J} - \frac{\overline{p}\overline{U}_\iota}{\rho}\right) - 2\upsilon\left(\frac{\partial \overline{U}_J}{\partial x_i}\overline{S_{\iota J}}\right) + \frac{\partial \overline{U}_J}{\partial x_i}\overline{u_\iota u_J}$$

The first term on the RHS is now equal to minus the divergence of T.

The second term on the RHS can be rewritten as:

$$2\upsilon\left(\frac{\partial \overline{U}_J}{\partial x_i}\overline{S_{\iota J}}\right) = 2\upsilon\left(\left(\frac{\partial \overline{U}_J}{\partial x_i} + \frac{\partial \overline{U}_\iota}{\partial x_j} - \frac{\partial \overline{U}_\iota}{\partial x_j}\right)\overline{S_{\iota J}}\right) = 2\upsilon\left(\overline{S_{\iota J}}\overline{S_{\iota J}}\right) = \bar{\epsilon}$$

The third term on the RHS is equal to $-\tilde{P}$. Therefore, the equation becomes:

$$\frac{\overline{D}\overline{E}}{\overline{D}t} = -\frac{\partial T_{ij}}{\partial x_i} - \tilde{P} - \bar{\epsilon}$$

Problem 5.20

Solution:

Reynolds equation:

$$\frac{\overline{D}\overline{U}_J}{\overline{D}t} = \upsilon\nabla^2\overline{U}_J - \frac{\partial \overline{u_\iota u_J}}{\partial x_i} - \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_j}$$

Navier-Stokes equation:

$$\frac{\partial U_j}{\partial t} + U_i\frac{\partial U_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \upsilon\nabla^2 U_j$$

Using continuity, we can rewrite the second term on LHS as:

$$\frac{\partial U_j}{\partial t} + \frac{\partial U_i U_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + \upsilon\nabla^2 U_j$$

Rewrite Reynolds equation using continuity as:

$$\frac{\partial \overline{U}_J}{\partial t} + \frac{\partial \overline{U_\iota U_J}}{\partial x_i} = \upsilon\left(\frac{\partial^2 \overline{U}_J}{\partial x_i \partial x_i}\right) - \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_j}$$

Subtract Reynolds equation to NS:

$$\frac{\partial U_j}{\partial t}+\frac{\partial U_i U_j}{\partial x_i}-\frac{\partial \bar{U}_j}{\partial t}-\frac{\partial \overline{U_i U_j}}{\partial x_i}=-\frac{1}{\rho}\frac{\partial p}{\partial x_j}+\upsilon\nabla^2 U_j-\upsilon\left(\frac{\partial^2 \bar{U}_j}{\partial x_i \partial x_i}\right)+\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j}$$

Where $U_j=\bar{U}_j+u_j$ and $p=\bar{p}+p'$

$$\frac{\partial(\bar{U}_j+u_j)}{\partial t}+\frac{\partial(\bar{U}_i+u_i)(\bar{U}_j+u_j)}{\partial x_i}-\frac{\partial \bar{U}_j}{\partial t}-\frac{\partial\overline{(\bar{U}_i+u_i)(\bar{U}_j+u_j)}}{\partial x_i}$$
$$=-\frac{1}{\rho}\frac{\partial(\bar{p}+p')}{\partial x_j}+\upsilon\nabla^2(\bar{U}_j+u_j)-\upsilon\left(\frac{\partial^2 \bar{U}_j}{\partial x_i \partial x_i}\right)+\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j}$$

Removing terms with opposite sign:

$$\frac{\partial(u_j)}{\partial t}+\frac{\partial(\bar{U}_i+u_i)(\bar{U}_j+u_j)}{\partial x_i}-\frac{\partial\overline{(\bar{U}_i+u_i)(\bar{U}_j+u_j)}}{\partial x_i}=-\frac{1}{\rho}\frac{\partial(p')}{\partial x_j}+\upsilon\frac{\partial^2(u_j)}{\partial x_i \partial x_i}\qquad(1)$$

Focus on:

$$\frac{\partial(\bar{U}_i+u_i)(\bar{U}_j+u_j)}{\partial x_i}-\frac{\partial\overline{(\bar{U}_i+u_i)(\bar{U}_j+u_j)}}{\partial x_i}$$
$$=\frac{\partial \bar{U}_i\bar{U}_j}{\partial x_i}+\frac{\partial \bar{U}_i u_j}{\partial x_i}+\frac{\partial \bar{U}_j u_i}{\partial x_i}+\frac{\partial u_i u_j}{\partial x_i}-\frac{\partial \bar{U}_i\bar{U}_j}{\partial x_i}-\frac{\partial \overline{u_i u_j}}{\partial x_i}$$

Use continuity:

$$\frac{\partial(\bar{U}_i+u_i)(\bar{U}_j+u_j)}{\partial x_i}-(\bar{U}_i+u_i)\frac{\partial\overline{(\bar{U}_j+u_j)}}{\partial x_i}=\frac{\partial \bar{U}_i u_j}{\partial x_i}+\frac{\partial \bar{U}_j u_i}{\partial x_i}+\frac{\partial u_i u_j}{\partial x_i}-\frac{\partial \overline{u_i u_j}}{\partial x_i}$$
$$=\bar{U}_i\frac{\partial u_j}{\partial x_i}+u_i\frac{\partial \bar{U}_j}{\partial x_i}+u_i\frac{\partial u_j}{\partial x_i}-\frac{\partial \overline{u_i u_j}}{\partial x_i}$$

Rewrite (1) as:

$$\frac{\partial(u_j)}{\partial t}+\bar{U}_i\frac{\partial u_j}{\partial x_i}+u_i\frac{\partial u_j}{\partial x_i}+u_i\frac{\partial \bar{U}_j}{\partial x_i}=-\frac{1}{\rho}\frac{\partial(p')}{\partial x_j}+\upsilon\frac{\partial^2(u_j)}{\partial x_i \partial x_i}+\frac{\partial \overline{u_i u_j}}{\partial x_i}$$

By definition:

$$\frac{Du_j}{Dt}=\frac{\partial(u_j)}{\partial t}+(\bar{U}_i+u_i)\frac{\partial u_j}{\partial x_i}$$

Therefore, (1) becomes:

$$\frac{Du_j}{Dt}=-u_i\frac{\partial \bar{U}_j}{\partial x_i}-\frac{1}{\rho}\frac{\partial(p')}{\partial x_j}+\upsilon\frac{\partial^2(u_j)}{\partial x_i \partial x_i}+\frac{\partial \overline{u_i u_j}}{\partial x_i}$$

Turbulent kinetic energy definition:

$$k = \frac{1}{2}\overline{u_i u_i}$$

Multiply the equation for the fluctuating velocity by u_j:

$$u_j \frac{\partial(u_j)}{\partial t} + (\bar{U}_i + u_i) u_j \frac{\partial u_j}{\partial x_i} + u_i u_j \frac{\partial \bar{U}_j}{\partial x_i} = -u_j \frac{1}{\rho}\frac{\partial(p')}{\partial x_j} + \upsilon u_j \frac{\partial^2(u_j)}{\partial x_i \partial x_i} + u_j \frac{\partial \overline{u_i u_j}}{\partial x_i}$$

Using continuity equation:

$$\frac{\partial\left(\frac{1}{2}u_j u_j\right)}{\partial t} + (\bar{U}_i)\frac{\partial\left(\frac{1}{2}u_j u_j\right)}{\partial x_i} + \frac{\partial\left(\frac{1}{2}u_i u_j u_j\right)}{\partial x_i} + u_i u_j \frac{\partial \bar{U}_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial(p' u_j)}{\partial x_j} + \upsilon u_j \frac{\partial^2(u_j)}{\partial x_i \partial x_i} + u_j \frac{\partial \overline{u_i u_j}}{\partial x_i}$$

Taking the mean of this equation:

$$\frac{\partial k}{\partial t} + (\bar{U}_i)\frac{\partial k}{\partial x_i} + \frac{\partial \overline{\left(\frac{1}{2}u_i u_j u_j\right)}}{\partial x_i} + \overline{u_i u_j}\frac{\partial \bar{U}_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial(\overline{p' u_j})}{\partial x_j} + \overline{\upsilon u_j \frac{\partial^2(u_j)}{\partial x_i \partial x_i}}$$

We can rewrite the last term as:

$$\overline{\upsilon u_j \frac{\partial^2(u_j)}{\partial x_i \partial x_i}} = 2\upsilon \frac{\partial \overline{(u_j s_{ij})}}{\partial x_i} - \varepsilon$$

Therefore, we get:

$$\frac{\partial k}{\partial t} + (\bar{U}_i)\frac{\partial k}{\partial x_i} + \frac{\partial \overline{\left(\frac{1}{2}u_i u_j u_j + \frac{p' u_j}{\rho} - 2\upsilon u_j s_{ij}\right)}}{\partial x_i} = -\overline{u_i u_j}\frac{\partial \bar{U}_j}{\partial x_i} - \varepsilon$$

Which can also be written as:

$$\frac{\bar{D}k}{\bar{D}t} + \frac{\partial T_{ij}'}{\partial x_i} = P - \bar{\epsilon}$$

Problem 5.25

Obtain the following relationship between the dissipation ε and the pseudo-dissipation $\tilde{\varepsilon}$:

$$\varepsilon \equiv 2\upsilon\overline{s_{ij}s_{ij}} = \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}} = \tilde{\varepsilon} + \upsilon\frac{\partial^2\overline{u_i u_j}}{\partial x_i \partial x_j}$$

Solution:

The pseudo-dissipation $\tilde{\varepsilon}$ is defined by:

$$\tilde{\varepsilon} = \upsilon\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}$$

The true dissipation is defined by:

$$\varepsilon = \upsilon\overline{\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right)} = \upsilon\left(\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}} + \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}}\right)$$

Where the first term on the RHS is exactly equal to $\tilde{\varepsilon}$. To obtain the desired relationship, it is sufficient to proof that

$$\upsilon\frac{\partial^2\overline{u_i u_j}}{\partial x_i \partial x_j} = \upsilon\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}}$$

Let's rewrite the LHS:

$$\upsilon\frac{\partial^2\overline{u_i u_j}}{\partial x_i \partial x_j} = \upsilon\frac{\partial}{\partial x_i}\left(\frac{\partial\overline{u_i u_j}}{\partial x_j}\right)$$

Using the product rule:

$$\upsilon\frac{\partial}{\partial x_i}\left(\frac{\partial\overline{u_i u_j}}{\partial x_j}\right) = \upsilon\frac{\partial}{\partial x_i}\left(\overline{\frac{\partial u_i}{\partial x_j}u_j + u_i\frac{\partial u_j}{\partial x_j}}\right)$$

where the second term on the RHS is zero, according to the continuity equation for the velocity fluctuation field. Using the product rule again:

$$\upsilon\frac{\partial}{\partial x_i}\left(\frac{\partial\overline{u_i u_j}}{\partial x_j}\right) = \upsilon\frac{\partial}{\partial x_i}\overline{\left(\frac{\partial u_i}{\partial x_j}u_j\right)} = \upsilon\overline{\left(\frac{\partial^2 u_i}{\partial x_i \partial x_j}u_j + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right)}$$

Now, the first term on the RHS is zero using the symmetry of second derivatives and the continuity equation for the velocity fluctuation field.

$$\upsilon \frac{\partial^2 \overline{u_\iota u_J}}{\partial x_i \partial x_j} = \upsilon \frac{\partial}{\partial x_i}\left(\frac{\partial \overline{u_\iota u_J}}{\partial x_j}\right) = \upsilon \left(\overline{\frac{\partial u_\iota}{\partial x_J}\frac{\partial u_J}{\partial x_\iota}}\right)$$

Problem 5.28

I will use the index P instead of L since it could be confusing to have both i and l indices because they look similar.

Solution:

$$\left(\overline{\frac{\partial u_\iota}{\partial x_J}\frac{\partial u_k}{\partial x_p}}\right) = \alpha\delta_{ij}\delta_{kp} + \beta\delta_{ik}\delta_{jp} + \gamma\delta_{ip}\delta_{jk} \qquad (2)$$

Considering the continuity equation:

$$\frac{\partial u_i}{\partial x_i} = 0$$

We can rewrite:

$$\left(\overline{\frac{\partial u_\iota}{\partial x_\iota}\frac{\partial u_k}{\partial x_p}}\right) = \alpha\delta_{ii}\delta_{kp} + \beta\delta_{ik}\delta_{ip} + \gamma\delta_{ip}\delta_{ik}$$

δ_{ii} is the trace of the unit 2nd order tensor and it is equal to 3.

The product of $\delta_{ik}\delta_{ip}$ and $\delta_{ip}\delta_{ik}$ is equal to δ_{kp} because i is a repeating index.

$$\left(\overline{\frac{\partial u_\iota}{\partial x_\iota}\frac{\partial u_k}{\partial x_p}}\right) = 3\alpha\delta_{kp} + \beta\delta_{kp} + \gamma\delta_{kp} = (3\alpha + \beta + \gamma)\delta_{kp} = 0$$

Which leads to the condition:

$$(3\alpha + \beta + \gamma) = 0$$

For homogeneous turbulence:

$$\frac{\partial}{\partial x_j}\left(\overline{u_\iota \frac{\partial u_J}{\partial x_p}}\right) = 0 \qquad (3)$$

Therefore, we can show that:

$$\overline{\frac{\partial u_\iota}{\partial x_J}\frac{\partial u_J}{\partial x_p}} = 0$$

by using the product rule on (3):

$$\frac{\partial}{\partial x_j}\left(\overline{u_i \frac{\partial u_j}{\partial x_p}}\right) = \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_p}} + u_i \frac{\partial^2 u_j}{\partial x_j \partial x_p} = 0$$

The second term on the RHS is zero using the symmetry of second derivatives and the continuity equation for the velocity fluctuation field.

$$\frac{\partial}{\partial x_j}\left(\overline{u_i \frac{\partial u_j}{\partial x_p}}\right) = \overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_p}} = 0$$

Therefore:

$$\left(\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_p}}\right) = \alpha\delta_{ij}\delta_{jp} + \beta\delta_{ij}\delta_{jp} + \gamma\delta_{ip}\delta_{jj}$$

δ_{jj} is the trace of the unit 2nd order tensor and it is equal to 3.

The product of $\delta_{ij}\delta_{jp}$ is equal to δ_{ip} because j is a repeating index.

$$\left(\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_p}}\right) = \alpha\delta_{ip} + \beta\delta_{ip} + 3\gamma\delta_{ip} = (\alpha + \beta + 3\gamma)\delta_{ip} = 0$$

Which leads to the condition:

$$(\alpha + \beta + 3\gamma) = 0$$

Now, consider Eq. (2):

$$\left(\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_p}}\right) = \alpha\delta_{ij}\delta_{kp} + \beta\delta_{ik}\delta_{jp} + \gamma\delta_{ip}\delta_{jk}$$

We can use the two conditions we obtained before:

$$(3\alpha + \beta + \gamma) = 0$$

$$(\alpha + \beta + 3\gamma) = 0$$

To generate a system of 2 equations in 3 unknowns. Leaving β as the free parameter of the system, we get: $\alpha = \gamma = -\frac{1}{4}\beta$. Substituting into Eq. (2):

$$\left(\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_p}}\right) = \beta\left(\delta_{ik}\delta_{jp} - \frac{1}{4}\delta_{ij}\delta_{kp} - \frac{1}{4}\delta_{ip}\delta_{jk}\right)$$

Eq. 5.169

$$\overline{\frac{\partial u_1}{\partial x_1}^2} = \left(\overline{\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_1}}\right) = \beta\left(\delta_{11}\delta_{11} - \frac{1}{4}\delta_{11}\delta_{11} - \frac{1}{4}\delta_{11}\delta_{11}\right) = \frac{1}{2}\beta$$

$$\overline{\frac{\partial u_1}{\partial x_2}^2} = \left(\overline{\frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_2}}\right) = \beta\left(\delta_{11}\delta_{22} - \frac{1}{4}\delta_{12}\delta_{12} - \frac{1}{4}\delta_{12}\delta_{12}\right) = \beta = 2\overline{\frac{\partial u_1}{\partial x_1}^2}$$

Eq. 5.170

$$\left(\overline{\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}}\right) = \beta\left(\delta_{12}\delta_{12} - \frac{1}{4}\delta_{11}\delta_{11} - \frac{1}{4}\delta_{12}\delta_{12}\right) = -\frac{1}{4}\beta = -\frac{1}{2}\overline{\frac{\partial u_1}{\partial x_1}^2}$$

Eq. 5.171

$$\varepsilon = \upsilon\left(\overline{\frac{\partial u_\iota}{\partial x_j}\frac{\partial u_\iota}{\partial x_j}}\right) = \upsilon\beta\left(\delta_{ii}\delta_{jj} - \frac{1}{4}\delta_{ij}\delta_{ij} - \frac{1}{4}\delta_{ij}\delta_{ij}\right) = \upsilon\beta\left(9 - \frac{3}{4} - \frac{3}{4}\right) = \frac{15}{2}\upsilon\beta = 15\upsilon\overline{\frac{\partial u_1}{\partial x_1}^2}$$

Problem 11.1

Solution:

Poisson equation:

$$\nabla^2 p = -\rho\frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i}$$

Divide by ρ:

$$\frac{1}{\rho}\nabla^2 p = -\frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i}$$

Using the product rule for differentiation:

$$-\frac{\partial^2 U_i U_j}{\partial x_i \partial x_j} = -\frac{\partial}{\partial x_i}\left(\frac{\partial U_i U_j}{\partial x_j}\right) = -\left(\frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i} + U_i\frac{\partial^2 U_j}{\partial x_i \partial x_j}\right)$$

Where the second term is zero using the symmetry of second derivatives and the continuity equation.

Now, use Reynolds decomposition for p and U:

$$p = \bar{p} + p'$$

$$U_i = \bar{U_\iota} + U_i{}'$$

Rewrite the Poisson equation as:

$$\frac{1}{\rho}\nabla^2(\bar{p}+p') = -\frac{\partial^2(\bar{U_i}+U_i')(\bar{U_j}+U_j')}{\partial x_i \partial x_j}$$

$$\frac{1}{\rho}\nabla^2(\bar{p}+p') = -\left(\frac{\partial^2(\bar{U_i})(\bar{U_j})}{\partial x_i \partial x_j} + \frac{\partial^2(\bar{U_i})(U_j')}{\partial x_i \partial x_j} + \frac{\partial^2(U_i')(\bar{U_j})}{\partial x_i \partial x_j} + \frac{\partial^2(U_i')(U_j')}{\partial x_i \partial x_j}\right)$$

Apply time average to all terms to obtain:

$$\frac{1}{\rho}\nabla^2(\bar{p}) = -\left(\frac{\partial^2(\bar{U_i})(\bar{U_j})}{\partial x_i \partial x_j} + \overline{\frac{\partial^2(U_i')(U_j')}{\partial x_i \partial x_j}}\right)$$

And using continuity equation for the first term on RHS, we get:

$$\frac{1}{\rho}\nabla^2(\bar{p}) = -\left(\frac{\partial \bar{U_i}}{\partial x_j}\frac{\partial \bar{U_j}}{\partial x_i} + \frac{\partial^2\overline{(U_i')(U_j')}}{\partial x_i \partial x_j}\right)$$

Which is equal to 11.17.

For the fluctuation pressure, subtract the Poisson equation for $\bar{p}$ to the total Poisson equation:

$$\frac{1}{\rho}\nabla^2(p') = -\frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i} + \frac{\partial \bar{U_i}}{\partial x_j}\frac{\partial \bar{U_j}}{\partial x_i} + \frac{\partial^2\overline{(U_i')(U_j')}}{\partial x_i \partial x_j}$$

Where

$$\frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i} = \frac{\partial \bar{U_i}}{\partial x_j}\frac{\partial \bar{U_j}}{\partial x_i} + 2\frac{\partial \bar{U_i}}{\partial x_j}\frac{\partial U_j'}{\partial x_i} + \frac{\partial^2(U_i')(U_j')}{\partial x_i \partial x_j}$$

Therefore:

$$\frac{1}{\rho}\nabla^2(p') = -\frac{\partial \bar{U_i}}{\partial x_j}\frac{\partial \bar{U_j}}{\partial x_i} - 2\frac{\partial \bar{U_i}}{\partial x_j}\frac{\partial U_j'}{\partial x_i} - \frac{\partial^2(U_i')(U_j')}{\partial x_i \partial x_j} + \frac{\partial \bar{U_i}}{\partial x_j}\frac{\partial \bar{U_j}}{\partial x_i} + \frac{\partial^2\overline{(U_i')(U_j')}}{\partial x_i \partial x_j}$$

$$\frac{1}{\rho}\nabla^2(p') = -2\frac{\partial \bar{U_i}}{\partial x_j}\frac{\partial U_j'}{\partial x_i} - \frac{\partial^2(U_i')(U_j')}{\partial x_i \partial x_j} + \frac{\partial^2\overline{(U_i')(U_j')}}{\partial x_i \partial x_j}$$