Noninertial Frame of Reference

NS equations: inertial reference frame ie stationary or moving at constant speed relative stationary reference frame; for our purposes stationary reference frame = stationary with respect to distant stars

$\nabla \cdot \underline{u} = 0 \qquad \rho \frac{Du}{Dt} = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}$ incompressible flow

Note: $\mu \frac{\partial^2 u_i}{\partial x_i^2} = 2\mu \frac{\partial S_{ij}}{\partial x_i} = \mu \frac{\partial}{\partial x_i}(u_{i,j} + u_{j,i}) = -\mu \varepsilon_{jik} \frac{\partial \omega_k}{\partial x_i}$

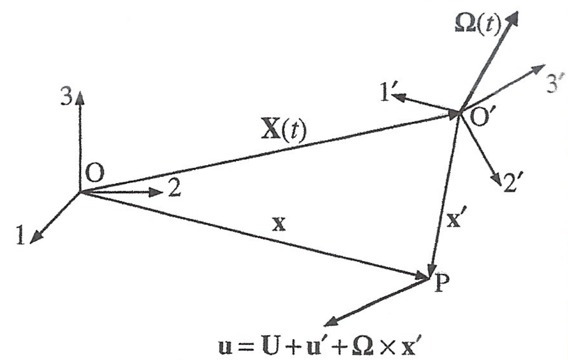
or $\mu \nabla^2 \underline{u} = -\mu \nabla \times \underline{\omega}$ (1) Seeming paradox that net viscous force $f(\underline{\omega})$ even though postulate #2 for $\tau_{ij} = f(e_{ij})$ states that rigid body rotation causes no shear stress, is resolved since involves derivatives of ω_k

(2) $\mu \nabla^2 \underline{u} = 0$ when $\underline{\omega}$ = constant (solid body rotation), which also requires $\frac{\partial S_{ij}}{\partial x_i} = 0$.

However, many applications require noninertial reference frames: rotating machinery; maneuvering vehicles; geophysical flows (atmospheric, oceanic); etc.

The continuity equation is not altered, but the NS is.

FIGURE 4.6 Geometry showing the relationship between a stationary coordinate system O123 and a noninertial coordinate system O′1′2′3′ that is moving, accelerating, and rotating with respect to O123. In particular, the vector connecting O and O′ is $\mathbf{X}(t)$ and the rotational velocity of O′1′2′3′ is $\boldsymbol{\Omega}(t)$. The vector velocity $\mathbf{u}$ at point P in O123 is shown. The vector velocity $\mathbf{u}'$ at point P in O′1′2′3′ differs from $\mathbf{u}$ because of the motion of O′1′2′3′.



noninertial
reference frame O′1′2′3′ : translates at $\frac{d\underline{X}(t)}{dt} = \underline{U}(t)$
& rotates at $\underline{\Omega}(t)$ with respect
to stationary reference frame O123

$\underline{U}$ & $\underline{\Omega}$ can be resolved in either frame
however time is invariant between both
reference frames, i.e., $t' = t$.

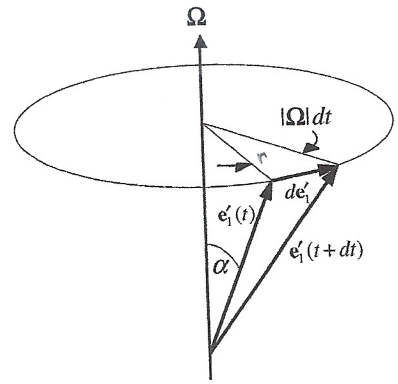
Fluid particle $P = P(\underline{x}')$ or $P(\underline{x})$ where
$x' = (x_1', x_2', x_3')$ & $\underline{X} = (X_1, X_2, X_3)$ &
$\underline{x} = \underline{X} + \underline{x}'$.

The velocity $\underline{u}$ of P is

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{X}}{dt} + \frac{d\mathbf{x}'}{dt} = \mathbf{U} + \frac{d}{dt}(x_1'\mathbf{e}_1' + x_2'\mathbf{e}_2' + x_3'\mathbf{e}_3')$$

$$= \mathbf{U} + \underbrace{\frac{dx_1'}{dt}\mathbf{e}_1' + \frac{dx_2'}{dt}\mathbf{e}_2' + \frac{dx_3'}{dt}\mathbf{e}_3'}_{u'} + \underbrace{x_1'\frac{d\mathbf{e}_1'}{dt} + x_2'\frac{d\mathbf{e}_2'}{dt} + x_3'\frac{d\mathbf{e}_3'}{dt}}_{\Omega \times x'} = \mathbf{U} + \mathbf{u}' + \boldsymbol{\Omega} \times \mathbf{x}'$$

The cross product $\underline{\Omega} \times \underline{x}' = x_1' e_{1t}' + x_2' e_{2t}' + x_3' e_{3t}'$
derivation is based on geometric
considerations



$\sin\alpha = \frac{r}{|e'_1|} = r$

$r|\Omega|dt = de'_1 =$ arc length

$(r\theta = s)$

FIGURE 4.7 Geometry showing the relationship between Ω, the rotational velocity vector of O'1'2'3', and the first coordinate unit vector e'_1 in O'1'2'3'. Here, the increment de'_1 is perpendicular to Ω and e'_1.

Rotation O'1'2'3' per time increment dt causes, e.g., $\hat{e}'_1$ to trace a small portion of a cone with radius $\sin\alpha$.

$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}| \sin\alpha$

$d|\hat{e}'_1| = \sin\alpha\, |\underline{\Omega}|\, dt \Rightarrow \frac{d|\hat{e}'_1|}{dt} = \sin\alpha\, |\underline{\Omega}|$

$= \underline{\Omega} \times \hat{e}'_1 \qquad |\hat{e}'_1| = 1$

ie $\frac{d\hat{e}'_i}{dt} = \underline{\Omega} \times \hat{e}'_i$

$\therefore\ x'_1 \frac{d\hat{e}'_1}{dt} + x'_2 \frac{d\hat{e}'_2}{dt} + x'_3 \frac{d\hat{e}'_3}{dt} = \underline{\Omega} \times \underline{x}'$

Acceleration:

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \frac{d}{dt}(\mathbf{U} + \mathbf{u}' + \boldsymbol{\Omega} \times \mathbf{x}') = \frac{d\mathbf{U}}{dt} + \mathbf{a}' + 2\boldsymbol{\Omega} \times \mathbf{u}' + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{x}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x}').$$

$\underline{U}_t =$ acceleration O' wrt O

$\underline{a}' =$ acceleration in O' reference frame

$2\underline{\Omega} \times \underline{u}' =$ Coriolis acceleration

$\frac{d\underline{\Omega}}{dt} \times \underline{x}' =$ acceleration in O' due $\dot{\underline{\Omega}}$

$\underline{\Omega} \times (\underline{\Omega} \times \underline{x}') =$ centripetal acceleration

For fluid: $\underline{a} = \frac{du}{dt} = \frac{Du}{Dt}$ and $\underline{a}' = \frac{Du'}{Dt}$ ie derivatives following the motion of the fluid particle

$$\left(\frac{D\mathbf{u}}{Dt}\right)_{O123} = \left(\frac{D'\mathbf{u}'}{Dt}\right)_{O1'2'3'} + \frac{d\mathbf{U}}{dt} + 2\boldsymbol{\Omega}\times\mathbf{u}' + \frac{d\boldsymbol{\Omega}}{dt}\times\mathbf{x}' + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}')$$

Substituting into the NS equation:

$$\rho\left(\frac{D'\mathbf{u}'}{Dt}\right)_{O1'2'3'} = -\nabla' p + \rho\left[\mathbf{g} - \frac{d\mathbf{U}}{dt} - 2\boldsymbol{\Omega}\times\mathbf{u}' - \frac{d\boldsymbol{\Omega}}{dt}\times\mathbf{x}' - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}')\right] + \mu\nabla'^2\mathbf{u}'$$

1 2 3 4

provides the incompressible NS equations in noninertial reference frame where ' denotes differentiation, velocity, and position in the O' reference frame.

[ ] terms are body forces due to the motion of O'

p and net viscous stresses are independent of reference frame Thermodynamic properties and vectors and vector operators $\nabla, \nabla\cdot, \nabla\times, \nabla^2$ (relative) but not their components

For $\underline{U}$ = constant and $\underline{\Omega} = 0$: [ ] = $\underline{g}$ ie inertial reference frame

For $\dot{\underline{U}}$ and/or $\underline{\Omega}$ = constant and $\underline{u}' = 0$: [ ] = $\underline{g} - \dot{\underline{U}} - \underline{\Omega}\times(\underline{\Omega}\times\underline{x}')$

rigid body translation or rotation

Such that $\mu\nabla^2 u' = 0$ and $\nabla p = \rho(\underline{g} - \underline{a})$

(1) $\underline{\dot{v}}$ = acceleration O' relative to O

(2) $-2\,\underline{\Omega} \times \underline{u}'$ = Coriolis acceleration = $f(u') \neq f(x')$

(3) $-\underline{\dot{\Omega}} \times \underline{x}'$ = $f(\underline{\dot{\Omega}})$ ie rate of change $\underline{\Omega}$

(4) $-\underline{\Omega} \times (\underline{\Omega} \times \underline{x}')$ = centrifugal acceleration $f(\Omega, \underline{x}')$

(1) Apparent force pushes person back in seat or tighten grip hand rail when accelerating. Aircraft parabolic trajectory weightless interior when $\underline{\dot{v}} = \underline{g}$

(2) $f(\underline{u}')$ not $\underline{x}$. Important navigation (air/sea) and artillery

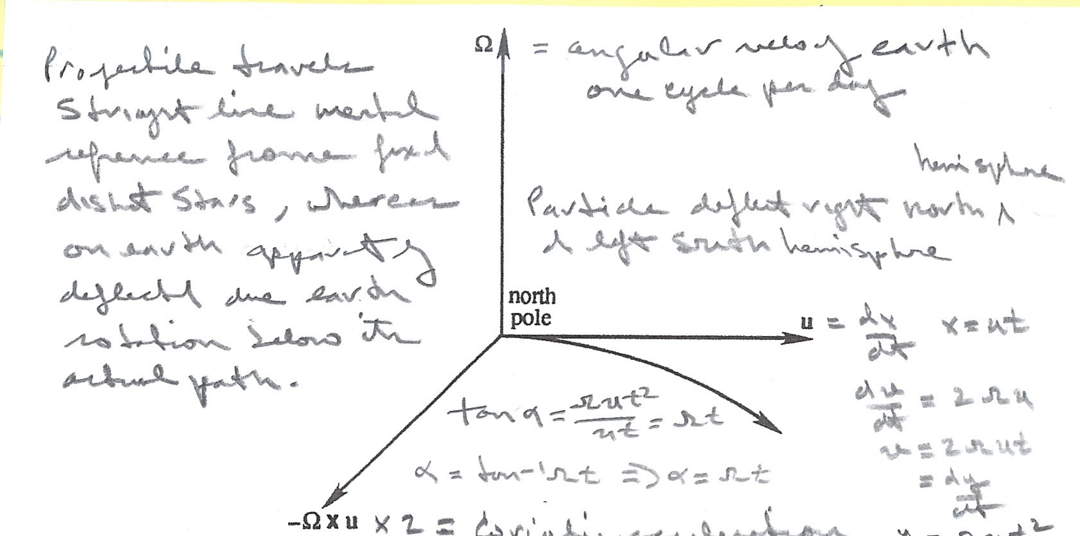


FIGURE 4.8 Particle trajectory deflection caused by the Coriolis acceleration when observed in a rotating frame of reference. If observed from a stationary frame of reference, the particle trajectory would be straight.

Also important geophysical fluid dynamics

③ Important when $\Omega(t)$ or direction rotation change with time

④ Centrifugal acceleration $= f(\underline{\Omega}, \underline{x}')$

distance axis rotation

Consider: $\underline{\Omega} = (0,0,\Omega)$

$x' = (R, \alpha, z)$

$-\underline{\Omega} \times (\underline{\Omega} \times \underline{x}') = \Omega^2 R \hat{e}_r$

$=$ apparent acceleration

$\underline{g} = \underline{g}_n + \Omega^2 R \hat{e}_r =$ effective acceleration

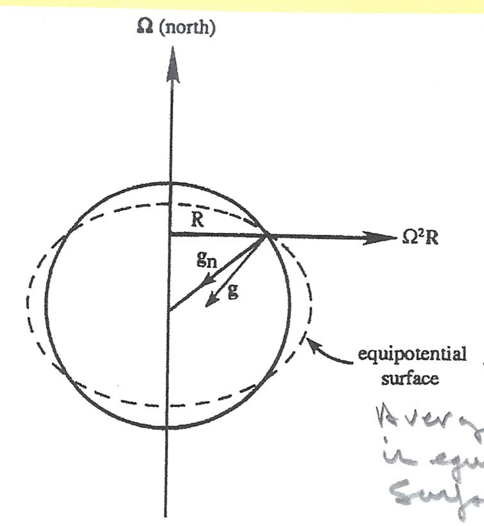
Newtonian $\underline{g}_n$ towards earth center

$\underline{g}_n = -\nabla\Phi$

$\underline{g} = g\hat{z}$

Body force potential can be found for new term, but only important large atmospheric or oceanic scale flows: $-\frac{1}{2}\rho\Omega^2(x'^2 + y'^2)$

Earth surface ellipsoid with equitorial Δ 42 km larger polar diameter



$\perp$ $\underline{g}$

Average sea level is equipotential surface

FIGURE 4.9 The earth's rotation causes it to budge near the equator and this leads to a mild distortion of equipotential surfaces from perfect spherical symmetry. The total gravitational acceleration is a sum of a centrally directed acceleration $\mathbf{g_n}$ (the Newtonian gravitation) and a rotational correction $\Omega^2 R$ that points away from the axis of rotation.

$$\underline{u} = \frac{d\underline{x}}{dt} = \frac{d\underline{X}}{dt} + \frac{d\underline{x}'}{dt} = \underline{U} + \frac{d}{dt}\left(x_1' e_1' + x_2' e_2' + x_3' e_3'\right)$$

$$= \underline{U} + \underline{u}' + \underline{\Omega} \times \underline{x}'$$

$$\underline{u}' = \dot{x}_1' e_1' + \dot{x}_2' e_2' + \dot{x}_3' e_3'$$

$$\underline{\Omega} \times \underline{x}' = x_1' \dot{e}_1' + x_2' \dot{e}_2' + x_3' \dot{e}_3'$$

$$\frac{d\underline{x}'}{dt} = \underline{u}' + \underline{\Omega} \times \underline{x}'$$

$$\frac{d\underline{u}}{dt} = \frac{d\underline{U}}{dt} + \frac{d\underline{u}'}{dt} + \frac{d}{dt}\left(\underline{\Omega} \times \underline{x}'\right)$$

$$= \frac{d\underline{U}}{dt} + \frac{d}{dt}\left(u_1' e_1' + u_2' e_2' + u_3' e_3'\right) + \frac{d\underline{\Omega}}{dt} \times \underline{x}' + \underline{\Omega} \times \frac{d\underline{x}'}{dt}$$

$$= \frac{d\underline{U}}{dt} + \underbrace{\frac{du_1'}{dt} e_1' + \frac{du_2'}{dt} e_2' + \frac{du_3'}{dt} e_3'}_{\underline{a}'} + \underbrace{u_1' \dot{e}_1' + u_2' \dot{e}_2' + u_3' \dot{e}_3'}_{\underline{\Omega} \times \underline{u}'} + \frac{d\underline{\Omega}}{dt} \times \underline{x}'$$

$$+ \underline{\Omega} \times \left(\underline{u}' + \underline{\Omega} \times \underline{x}'\right)$$

$$= \frac{d\underline{U}}{dt} + \underline{a}' + 2\underline{\Omega} \times \underline{u}' + \frac{d\underline{\Omega}}{dt} \times \underline{x}' + \underline{\Omega} \times \left(\underline{\Omega} \times \underline{x}'\right)$$

inertial frame NS: $\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}$

$$\therefore \quad \frac{D\underline{u}}{Dt} = \frac{D\underline{u}'}{Dt} + \dot{\underline{U}} + 2\underline{\Omega} \times \underline{u}' + \frac{d\underline{\Omega}}{dt} \times \underline{x}' + \underline{\Omega} \times \left(\underline{\Omega} \times \underline{x}'\right)$$

$$= -\nabla p/\rho + \underline{g} + \nu \nabla^2 \underline{u}'$$

or $$\rho \frac{D\underline{u}'}{Dt} = -\nabla p + \rho\left[\underline{g} - \dot{\underline{U}} - 2\underline{\Omega} \times \underline{u}' + \frac{d\underline{\Omega}}{dt} \times \underline{x}' + \underline{\Omega} \times \left(\underline{\Omega} \times \underline{x}'\right)\right]$$

$$+ \mu \nabla^2 \underline{u}'$$

Exercise 4.50. Derive (4.43) from (4.42).

Solution 4.50. The entire statement of equation (4.42) is:

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} = \frac{d\mathbf{X}}{dt} + \frac{d\mathbf{x}'}{dt} = \mathbf{U} + \frac{d}{dt}\left(x_1'\mathbf{e}_1' + x_2'\mathbf{e}_2' + x_3'\mathbf{e}_3'\right)$$

$$= \mathbf{U} + \frac{dx_1'}{dt}\mathbf{e}_1' + \frac{dx_2'}{dt}\mathbf{e}_2' + \frac{dx_3'}{dt}\mathbf{e}_3' + x_1'\frac{d\mathbf{e}_1'}{dt} + x_2'\frac{d\mathbf{e}_2'}{dt} + x_3'\frac{d\mathbf{e}_3'}{dt} = \mathbf{U} + \mathbf{u}' + \mathbf{\Omega}\times\mathbf{x}',$$

Thus, we see that $d\mathbf{x}'/dt = \mathbf{u}' + \mathbf{\Omega}\times\mathbf{x}'$. Now time differentiate $\mathbf{u}$, to find:

$$\frac{d\mathbf{u}}{dt} \equiv \mathbf{a} = \frac{d\mathbf{U}}{dt} + \frac{d\mathbf{u}'}{dt} + \frac{d}{dt}(\mathbf{\Omega}\times\mathbf{x}')$$

$$= \frac{d\mathbf{U}}{dt} + \frac{d}{dt}\left(u_1'\mathbf{e}_1' + u_2'\mathbf{e}_2' + u_3'\mathbf{e}_3'\right) + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{x}' + \mathbf{\Omega}\times\frac{d\mathbf{x}'}{dt}$$

$$= \frac{d\mathbf{U}}{dt} + \frac{du_1'}{dt}\mathbf{e}_1' + \frac{du_2'}{dt}\mathbf{e}_2' + \frac{du_3'}{dt}\mathbf{e}_3' + u_1'\frac{d\mathbf{e}_1'}{dt} + u_2'\frac{d\mathbf{e}_2'}{dt} + u_3'\frac{d\mathbf{e}_3'}{dt} + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{x}' + \mathbf{\Omega}\times\left(\mathbf{u}' + \mathbf{\Omega}\times\mathbf{x}'\right)$$

Here the various terms written in component form may be identified. The second through fourth terms are the fluid particle acceleration, $\mathbf{a}'$, observed in the non-inertial frame of reference. The fifth through seventh terms, which involve the time derivatives of the unit vectors, can be written in terms of a cross product:

$$u_1'\frac{d\mathbf{e}_1'}{dt} + u_2'\frac{d\mathbf{e}_2'}{dt} + u_3'\frac{d\mathbf{e}_3'}{dt} = \mathbf{\Omega}\times\mathbf{u}',$$

as depicted in Figure 4.7 and described in paragraph below (4.42). With these replacements, the last equality for the fluid particle acceleration becomes:

$$\mathbf{a} = \frac{d\mathbf{U}}{dt} + \mathbf{a}' + \mathbf{\Omega}\times\mathbf{u}' + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{x}' + \mathbf{\Omega}\times\mathbf{u}' + \mathbf{\Omega}\times\left(\mathbf{\Omega}\times\mathbf{x}'\right)$$

$$= \frac{d\mathbf{U}}{dt} + \mathbf{a}' + 2\mathbf{\Omega}\times\mathbf{u}' + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{x}' + \mathbf{\Omega}\times\left(\mathbf{\Omega}\times\mathbf{x}'\right),$$

which matches (4.43).

Right Circular Cone

Let R = radius of base, s = slant height.

$$s = \sqrt{R^2 + h^2}$$
$$S = \pi R s = \pi R\sqrt{R^2 + h^2}$$
$$T = \pi R(R + s) = \pi R(R + \sqrt{R^2 + h^2})$$
$$V = \tfrac{1}{3}\pi R^2 h$$