Axially Symmetric Wake

$u_1(x,r) = U_\infty - u(x,r) << U_\infty$ far downstream

$u = U - u_1$ $\quad v = v_1 << U$ ∴ neglect higher order terms

$u_x = -u_{1x}$

$u_r = -u_{1r}$

$u_{rr} = -u_{1rr}$

$$u u_x + v u_r = \frac{\nu}{r}(r u_r)_r = \nu(u_r/r + u_{rr})$$

$$(U - u_1)(-u_{1x}) + v_1(-u_{1r}) = \nu(-u_{1r}/r - u_{1rr})$$

$$U u_{1x} = \frac{\nu}{r}\frac{\partial}{\partial r}(r u_{1r}) = \nu(u_{1r}/r + u_{1rr})$$

$$D = \rho\int_0^\infty u(U-u)\, r\,dr\,d\theta = 2\pi\rho\int_0^\infty u_1(U-u_1)\,r\,dr$$

$$= 2\pi\rho U\int_0^\infty u_1\,r\,dr = \text{Constant}$$

∴ $u_1 = c\,U f(\eta)/x = a(x) f$

analogous 2D

$$\eta = \frac{1}{2} r\sqrt{\frac{U}{\nu x}} = r\,b(x) \qquad r = \eta/b(x)$$

$r = \eta/b(x)$

$dr = d\eta/b \qquad r\,dr = \dfrac{\eta\,d\eta}{b^2}$

$$D = 2\pi\rho U\int_0^\infty a(x) f(\eta)\,r\,dr$$

$$= 2\pi\rho U\frac{a(x)}{b(x)^2}\int_0^\infty f(\eta)\,\eta\,d\eta \qquad a(x) = \frac{cU}{x}$$

$$= 2\pi\rho U\frac{c\cancel{U}}{\cancel{x}}\frac{4\nu\cancel{x}}{\cancel{U}}\int_0^\infty f(\eta)\,\eta\,d\eta \qquad b = \left[\frac{U}{4\nu x}\right]^{1/2} \quad b^2 = \frac{U}{4\nu x}$$

$$= 8\pi\rho U c\nu\int_0^\infty f(\eta)\,\eta\,d\eta \neq f(x)$$

$u_{1x} = a_x f + a f' \eta_x$

$u_{1r} = a f' \eta_r$

$u_{1rr} = a f'' \eta_r^2$

$\eta_x = r b_x = \eta b_x / b$

$\eta_r = b = \frac{1}{2} (U/\nu)^{1/2} x^{-1/2}$

$b_x = -\frac{1}{4} (U/\nu)^{1/2} x^{-3/2}$

$a = c U x^{-1}$

$a_x = -c U x^{-2}$

$\eta_r / r = b^2/\eta$

$U(a_x f + a f' \eta_x)$

$= \nu (a f' \eta_r / r + a f'' \eta_r^2)$

$\nu a f'' \eta_r^2 + \nu a f' \eta_r / r - U a_x f - U a f' \eta_x = 0$

$f'' (\nu a \eta_r^2) + f' (\nu a \eta_r / r - U a \eta_x) - U a_x f = 0$

$f'' (\nu a b^2) + f' (\nu a b^2/\eta - U a \eta b_x / b) - U a_x f = 0$

$\nu a b^2 = \nu c U x^{-1} \frac{U}{4 \nu x} = c U^2 / 4x^2$

$\nu a b^2/\eta = c U^2 / 4x^2 \eta$

$-U a \eta b_x / b = -U \eta (c U x^{-1})(-\frac{1}{2} x^{-1})$

$= U^2 c \eta / 2x^2$

$-U a_x = -U(-c U x^{-2}) = U^2 c / x^2$

$b_x / b = -\frac{1}{4} \left(\frac{U}{\nu}\right)^{1/2} x^{-3/2}$

$\times 2 \left(\frac{\nu}{U}\right)^{1/2} x^{1/2}$

$= -\frac{1}{2} x^{-1}$

$f'' (c U^2 / 4x^2) + f' (c U^2 / 4x^2 \eta + U^2 c \eta / 2x^2) + U^2 c / x^2 f$

$f'' + f' (\eta^{-1} + 2\eta) + f = 0$

$\underbrace{\eta f'' + f'}_{(\eta f')'} + 2 \eta^2 f' + \eta f = 0$

$(\eta f')' + 2 \eta^2 f' + 4 \eta f = 0$

$u_{1r}(0) = 0 \Rightarrow f'(0) = 0$

$u_1(\infty) = 0 \Rightarrow f(\infty) = 0$

$$f(\eta) = \exp(-\eta^2)$$

$$f' = -2\eta \exp(-\eta^2)$$

$$f'' = 4\eta^2 \exp(-\eta^2) - 2e^{-\eta^2}$$

OK $\quad 4\eta^3 \cancel{e^{-\eta^2}} - 2\eta \cancel{e^{-\eta^2}} - 2\eta \cancel{e^{-\eta^2}} - 4\cancel{\eta^3} e^{-\eta^2} + 4\eta \cancel{e^{-\eta^2}} = 0$

$$u_1 = \frac{cU}{x} e^{-\eta^2} \qquad \eta^2 = \frac{U}{4\nu x} r^2 \qquad \eta = \left[\frac{U}{4\nu x}\right]^{1/2} r$$

$$r = \left[\frac{4\nu x}{U}\right]^{1/2} \eta \propto x^{1/2}$$

$$D = 8\pi\rho U c\nu \int_0^\infty e^{-\eta^2} \eta\, d\eta = 8\pi\rho U c\nu\left[\tfrac{1}{2}\right]$$

$$= 4\pi\rho U c\nu$$

$$\int_0^\infty x^n e^{-ax^2} dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\left(a^{\frac{n+1}{2}}\right)} \qquad n > -1,\ a > 0$$

$a = 1$, $n = 1$ $\quad = \dfrac{\Gamma(1)}{2} = \dfrac{1}{2}$

$$c_D = 4\pi\rho U c\nu / \tfrac{1}{2}\rho U^2 A_p \qquad A_p = \text{projected area}$$

$$= 8\pi c\nu / U A_p$$

$$c = c_D U A_p / 8\pi\nu$$

$\Gamma(n) = (n-1)!$ $\quad (0)! = 1$ $\quad (1)! = 1$

$$u_1 = \frac{c_D U L^2}{8\pi\nu} \frac{U}{x} e^{-\eta^2}$$

$A_p = \dfrac{\pi D^2}{4}$ $\quad c = \dfrac{c_D U \pi D^2/4}{8\pi\nu} = \dfrac{c_D Re_D}{32}$

$Re = UD/\nu$

$$u_1/U = \frac{c_D}{8\pi}(Re)\frac{L}{x} \exp\left(\frac{-Ur^2}{4\nu x}\right) \qquad u_{max} \propto x^{-1}$$

$u_1/U = \dfrac{c_D}{32}(Re)\dfrac{D}{x} e^{-\eta^2}$ Same shape 2D wake with $y = r$ $\quad Re = UL/\nu$

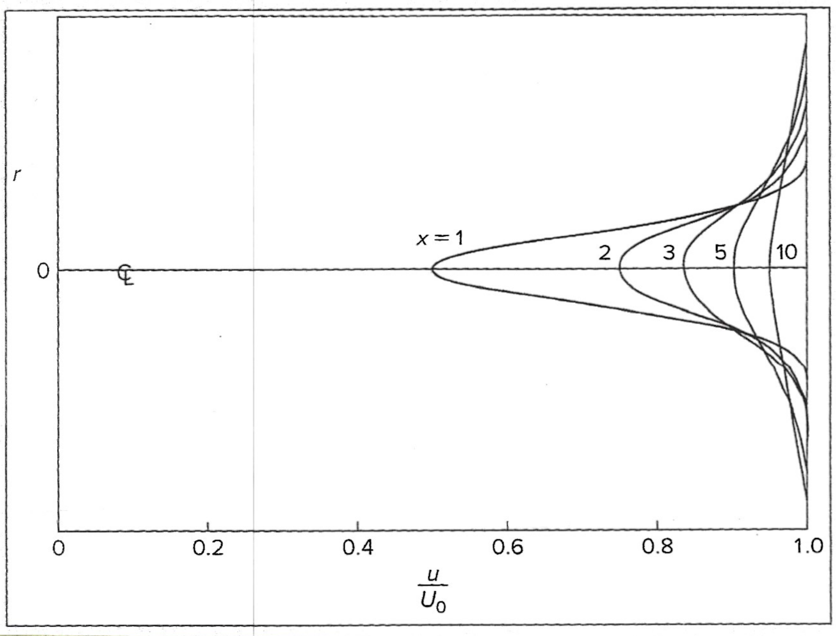

FIGURE 4-43
Velocity profiles illustrating the decay of a round wake from Eq. (4-232). Units on x and r are arbitrary.