2D Far Wake of Nonlifting Bodies

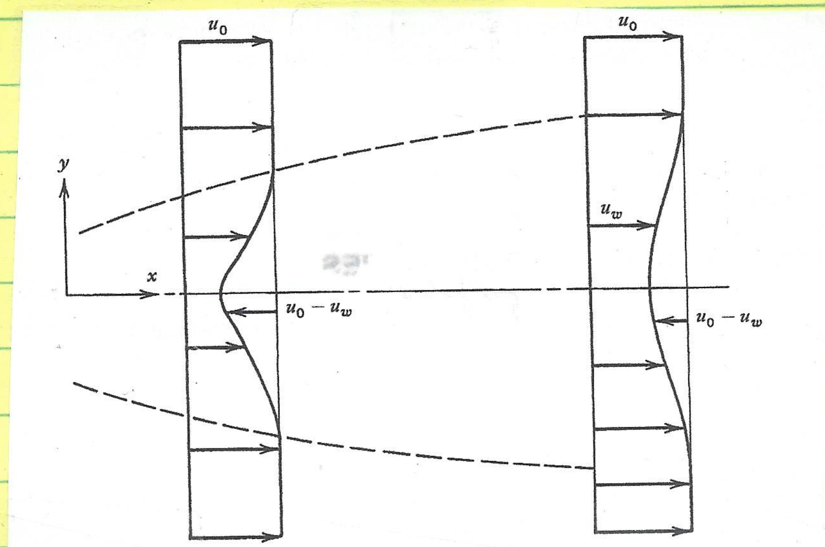


Figure 20.22 Laminar wake of a nonlifting body.

Assume: $Re_{Body} = \frac{u_0 L}{\nu}$ large, similarity for the velocity defect $u = u_0 - u_w$, and BL equations with $p_x = 0$.

$u_x + v_y = 0$ $\quad$ $u u_x + v u_y = \nu u_{yy}$ $\quad$ Subscript w not shown

$u_w = u_0 - u$ $\quad$ $v_w = v$ $\quad$ $u = u_0 - u_w$

= velocity defect

$u_{w_x} = -u_x$

$u_{w_y} = -u_y$

$u_{w_{yy}} = -u_{yy}$

$$(u_0 - u)(-u_x) + v(-u_y) = -\nu(-u_{yy})$$

$$(u_0 - u)u_x + v u_y = \nu u_{yy}$$

Assume $u = \psi_y$ $\quad v = -\psi_x$

$$\psi = A x^p f(\eta) \qquad \eta = y/Bx^q \qquad \eta_y = b^{-1}$$
$$= a(x) f(\eta) \qquad = y/b(x) \qquad \eta_x = -\eta \frac{b_x}{b}$$

A, B, p, q chosen as part of solution.

$$u = \frac{a}{b} f' \qquad v = -a_x f + f'(\eta a b_x/b) \qquad a_x = Ap\, x^{p-1}$$
$$u_x = (ab^{-1})_x f' + (ab^{-1}) f''(-\eta b_x/b) \qquad b_x = Bq\, x^{q-1}$$
$$u_y = (ab^{-2}) f'' \qquad a_x b = ABp\, x^{p+q-1}$$
$$u_{yy} = (ab^{-3}) f''' \qquad a b_x = ABq\, x^{p+q-1}$$

$$u_0 u_x = u_0 \left[(a_x b^{-1} - a b^{-2} b_x) f' - (\eta a b_x / b^2) f'' \right]$$
$$= u_0 \left[(Ap x^{p-1}/Bx^q - ABq x^{p+q-1}/B^2 x^{2q}) f' - (\eta ABq x^{p+q-1}/B^2 x^{2q}) f'' \right]$$
$$= u_0 \left[\left(\frac{Ap}{B} x^{p-q-1} - \frac{Aq}{B} x^{p-q-1} \right) f' - \eta \frac{Aq}{B} x^{p-q-1} f'' \right]$$
$$u_0 u_x = u_0 \frac{A}{B} x^{p-q-1} \left[(p-q) f' - \eta q f'' \right]$$
(1)

$$-u u_x + v u_y = -\left[(Ax^p/Bx^q) f' \right] \left[\frac{A}{B} x^{p-q-1} \left((p-q) f' - \eta q f'' \right) \right]$$
$$\left(-Ap x^{p-1} f + f' (\eta ABq x^{p+q-1}/Bx^q) \right) \underbrace{(Ax^p/B^2 x^{2q})}_{\frac{A}{B^2} x^{p-2q}} f''$$
$$x^{p-1}$$

$$-\frac{A^2}{B^2} x^{2p-2q-1} \left[(p-q) f'^2 - \eta q f' f'' \right]$$

$$-\frac{A^2 p}{B^2} x^{2p-2q-1} f f'' + \eta \frac{A^2}{B^2} q x^{2p-2q-1} f' f''$$

$$-\frac{A^2}{B^2} x^{2p-2q-1} \left[(p-q) f'^2 - 2\eta q f' f'' + p f f'' \right]$$
(2)

$$u_{yy} = (Ax^p / B^3 x^{3q}) f''' = \frac{A}{B^3} x^{p-3q} f'''$$

$u_0 u_x - u u_x + v u_y = \nu u_{yy}$

$u_0 B^2 x^{2q-1} [(1)] + \frac{B}{A} x^{p+q-1} [-(2)] = \nu f''$

$-(2) = (q-p) f'^2 + 2\eta q f f'' - p f f''$

$$F_D = \rho u_0^2 \theta = \rho u_0^2 \int_{-\infty}^{\infty} \left[\frac{u_w}{u_0} - \left(\frac{u_w}{u_0}\right)^2 \right] dy$$

$u = (a b^{-1}) f'$ $\qquad = \rho u_0^2 \int_{-\infty}^{\infty} u_0^{-2} [u_w u_0 - u_w^2] dy$

$(u_0 - (a b^{-1}) f')(a b^{-1}) f'$

$u_0 (a b^{-1}) f' - (a b^{-1})^2 f'^2$ $\qquad = \rho \int_{-\infty}^{\infty} u_w (u_0 - u_w) dy$ $\qquad \eta = y b^{-1}$

$u_0 \frac{A}{B} x^{p-q} f' - \frac{A^2}{B^2} x^{2p-2q} f'^2$ $\qquad d\eta = dy\, b^{-1}$

$= \rho \int_{-\infty}^{\infty} (u_0 - u)(u)\, dy$ $\qquad B x^q d\eta = dy$

$\times B x^q = u_0 A x^p f' - \frac{A^2}{B} x^{2p-q} f'^2$

$p = 0$ $q = \frac{1}{2}$ indep

$x \sim x \to \infty$

$$F_D = \rho A x^p \int_{-\infty}^{\infty} \left[u_0 - \frac{A}{B} x^{p-q} f' \right] f' d\eta$$

$\frac{u_0 B^2}{2} [-f' - \eta f''] + \frac{B}{A} x^{-1/2} [-(2)] = \nu f'''$

Pink highlight = nonlinear terms drop out Oseen approximation

$\lim x \to \infty$ $\qquad f''' + \frac{u_0 B^2}{2} (f' + \eta f'') = 0$

$F_D = \rho A u_0 \int_{-\infty}^{\infty} f' d\eta$ $\qquad a(x) = A x^p = A$

$b(x) = B x^{1/2}$

$Q = \int \mathbf{u} \cdot \mathbf{n}\, dA$ $\qquad = \rho A u_0 \int_{-\infty}^{\infty} \frac{B x^{1/2}}{A} u \frac{dy}{B x^{1/2}}$ $\qquad d\eta = dy / B x^{1/2}$

$= \int u\, dy$ $\qquad u = \frac{a}{b} f'$

per unit span $\qquad = \rho u_0 \int_{-\infty}^{\infty} u\, dy$ $\qquad f' = \frac{b u}{a} = \frac{B x^{1/2}}{A} u$

$= \rho u_0 Q$

$$B = \left(\frac{4\nu}{u_0}\right)^{1/2} \qquad B^2 = \frac{4\nu}{u_0} \times \frac{u_0}{2\nu} = 2 \Rightarrow f''' + 2(f' + \eta f'') = 0$$

$$f' = e^{-\eta^2} \qquad f'' = -2\eta e^{-\eta^2} \qquad \frac{d}{dx} e^u = e^u \frac{du}{dx}$$

$$f''' = -2e^{-\eta^2} + 4\eta^2 e^{-\eta^2}$$

$$2(f' + \eta f'') = 2(e^{-\eta^2} - 2\eta^2 e^{-\eta^2}) \qquad \text{OK}$$

$$F_D = \rho A u_0 \int_{-\infty}^{\infty} e^{-\eta^2} d\eta = \rho A u_0 \pi^{1/2} \Rightarrow A = \frac{F_D}{\rho u_0 \sqrt{\pi}}$$

$$\eta = y \Big/ \left(\frac{4\nu}{u_0}\right)^{1/2} x^{1/2} = y\sqrt{\frac{u_0}{4\nu x}}$$

$$\chi = \frac{F_D}{\rho u_0 \pi^{1/2}} \underbrace{\int e^{-\eta^2} d\eta}_{\frac{\sqrt{\pi}}{2}\operatorname{erf}(\eta)} \qquad \frac{df}{d\eta} = e^{-\eta^2} \qquad f = \int_0^{\eta} e^{-\eta^2} d\eta$$

$$= \frac{2F_D}{\rho u_0} \operatorname{erf}(\eta) \qquad \operatorname{erf}(\eta) = \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta$$

$$u = u_0 - u_w = \frac{A}{B} e^{-\eta^2} = \frac{\dfrac{F_D}{\rho u_0 \sqrt{\pi}}}{\left(\dfrac{4\nu x}{u_0}\right)^{1/2}} e^{-\eta^2}$$

$$u = \left[\frac{F_D^2}{\rho^2 u_0^2 \pi} \times \frac{u_0}{4\nu x}\right]^{1/2} e^{-\eta^2}$$

$$u = \left[\frac{F_D^2}{4\pi \rho^2 u_0 \nu x}\right]^{1/2} e^{-\eta^2} \qquad \text{velocity defect decays } x^{-1/2}$$

$$y = B x^{1/2} \eta = \left(\frac{4\nu}{u_0}\right)^{1/2} x^{1/2} \eta \qquad \text{grows } x^{1/2}$$

momentum thickness wake = constant = $\Theta = F_D / \rho u_0^2$

whereas in the far wake $Q = F_D / \rho u_0 = u_0 \Theta$

$\uparrow$ velocity defect $\dot{V}/s$

It can be shown that $Q = u_0 \delta^*$ thus $\delta^* = \Theta$ in the far wake

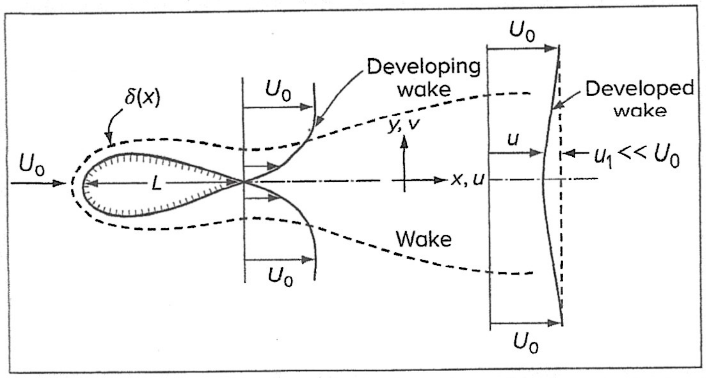
Note we assumed large Re but additionally required $x \to \infty$ ie far wake. It is interesting to note that the same result is obtained using the Oseen equations, which are derived for low Re ie. linearized convection.

$$\int_{\delta^*}^{\infty} u_0 dy = \int_0^{\infty} u_\omega dy = \int_0^{\infty} u_0 dy - \int_0^{\delta^*} u_0 dy$$

ie $$u_0 \delta^* = \int_0^{\infty} (u_0 - u_\omega) dy = \int_0^{\infty} u \, dy = Q \quad \text{velocity defect}$$

ie in the far wake the velocity defect momentum & displacement thickness are equal

Plane laminar wake: far-field approximation



Same as u_1 = wake defect

FIGURE 4-24
Flow in the wake of a body immersed in a stream.

$x > 3L$

wake attains similarity

$u_1(x,y) = U_0 - u(x,y) \ll U_0$

BL assumptions: $uu_x + vu_y = \nu u_{yy}$ with $P_x = 0$

Also Oseen approximation, i.e., linearized

convection: $U_0 u_{1x} = \nu u_{1yy}$ $\quad u_1(x, \pm\infty) = 0$

$u_{1y}(x, 0) = 0$

See prev flat plate for wake solution

$$u_1 = cU_0\left(\frac{L}{x}\right)^{1/2} e\left(-\eta^2/4\right) \qquad \eta = y\left(U_0/\nu x\right)^{1/2}$$

$$\eta^2/4 = y^2\left(U/4\nu x\right)$$

per unit depth

$$F = \int_{-\infty}^{\infty} \rho U_0 u_1\, dy = \rho U_0\, cU_0\left(\frac{L}{x}\right)^{1/2}\int_{-\infty}^{\infty} e^{-\eta^2/4}\, d\eta \left(\frac{\nu x}{U_0}\right)^{1/2}$$

$$= \rho U_0^2 c\left(\frac{L\nu x}{xU_0}\right)^{1/2}\underbrace{\int_{-\infty}^{\infty} e^{-\eta^2/4}\, d\eta}_{2\pi^{1/2}}$$

$$\frac{1}{2} C_D \cancel{\rho} \cancel{U_0^2} L = 2\cancel{\rho}\cancel{U_0^2} c\left(\frac{L\nu\pi}{U_0}\right)^{1/2}$$

$$\frac{1}{4}\left(\frac{U_0}{L\nu\pi}\right)^{1/2} C_D L = c = \left(\frac{U_0 L}{16\nu\pi}\right)^{1/2} C_D = C_D\left(\frac{Re_L}{16\pi}\right)$$

$$u_1/U_0 = C_D\left(\frac{Re_L}{16\pi}\right)^{1/2}\left(\frac{L}{x}\right)^{1/2} e\left(-\eta^2/4\right) \propto C_D$$

valid $x > 3L$