e. Flow in the wake of flat plate at zero incidence

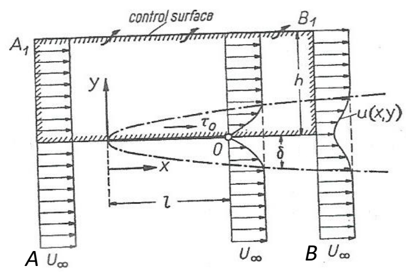


Fig. 9.9. Application of the momentum equation in the calculation of the drag on a flat plate at zero incidence from the velocity profile in the wake

Drag

① x CV Boundary Layer

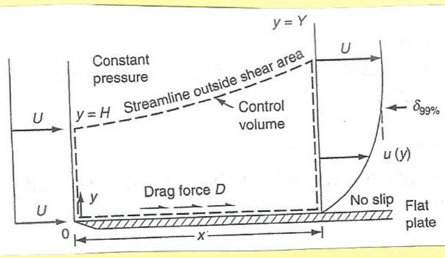


FIGURE 4-1
Definition of control volume for the analysis of flow past a flat plate.

$Y = H + \delta^*$

$$\int_{cs} \rho \underline{V}\cdot\underline{n}\, dA = 0 = -\int_0^H \rho U\, dy + \int_0^Y \rho u\, dy$$

$$UH = \int_0^Y u\, dy = UY + \int_0^Y (u-U)\, dy$$

$$\delta^* = \int_0^Y (1 - u/U)\, dy = \text{displacement thickness}$$

$$\Sigma F_x = -D = \int_{cs} \rho u\, \underline{V}\cdot\underline{n}\, dA = -\int_0^H \rho U (U\, dy) + \int_0^Y \rho u\, (u\, dy)$$

$$D = \rho U^2 H - \int_0^Y \rho u^2\, dy \quad = \text{fluid force on plate}$$
$$= -\text{ plate force on CV}$$

$$D = \rho U^2 \int_0^Y \frac{u}{U} - \int_0^Y \rho u^2\, dy$$

$$= \int_0^X \tau_w\, dx$$

$$D/\rho U^2 = \theta = \int_0^Y \frac{u}{U}\left(1-\frac{u}{U}\right) dy \quad = \text{momentum thickness}$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 x} = \frac{2\theta}{x} = \frac{1}{x}\int_0^x c_f\, dx \quad \text{per unit span}$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \; : \; c_f = \frac{d}{dx}(x C_D) = 2\frac{d\theta}{dx} \quad \text{ie } \frac{d\theta}{dx} = c_f/2$$

(2) Box CV wake

$$\int_{CS} \rho \underline{V}\cdot\underline{n}\, dA = -\int_A^{A_1} \rho U dy + \int_B^{B_1} \rho u dy + \underbrace{\int_{A_1}^{B_1} \rho v dx}_{\dot m_t} + \underbrace{\int_A^B \rho(-v dx)}_{\dot m_b}$$

$$\dot m_t + \dot m_b = \rho U A A_1 - \int_B^{B_1} \rho u dy$$

$$\Sigma F_x = -D = \int_{CS} \rho u \underline{V}\cdot\underline{n}\, dA = -\int_A^{A_1} \rho U(U dy) + \int_B^{B_1} \rho u(u dy)$$

$$+ \underbrace{\int_{A_1}^{B_1} \rho u(v dx) + \int_A^B \rho u(-v dx)}_{\text{assume } u = U}$$

$$= U(\dot m_t + \dot m_b)$$

$$D = \rho U^2 \cancel{AA_1} - \int_B^{B_1} \rho u^2 dy - U\left(\cancel{\rho U AA_1} - \int_B^{B_1} \rho u dy\right)$$

$$= U\int_B^{B_1} \rho u - \int_B^{B_1} \rho u^2 dy$$

$$D = \rho U^2 \int_B^{B_1} \frac{u}{U}\left(1-\frac{u}{U}\right) dy$$

same as x CV
with $B=0$ & $B_1 = Y$
note for $y > Y$ $u/U = 1$

$$u u_x + v u_y = \nu u_{yy}$$

$u_1 = U - u(x,y)$ $\quad u_1 << U$ velocity defect

$u = U - u_1$ $\quad v = v_1 << U$

$u_x = -u_{1x}$ $\quad u_y = -u_{1y}$ $\quad u_{yy} = -u_{1yy}$

$$(U - u_1)(-u_{1x}) + v_1(-u_{1y}) = \nu(-u_{1yy})$$

$U u_{1x} = \nu u_{1yy}$ to first order

$$u_{1y}(0) = 0$$
$$u_1(\infty) = 0$$

Assume similarity for u_1 as per Blasius

$$\eta = y\sqrt{U/\nu x} = y\, U^{1/2} \nu^{-1/2} x^{-1/2}$$

$u_1 = c U \sqrt{\frac{L}{x}}\, g(\eta)$ $\quad x^{-1/2}$ so that D independent x

$$2D = b\rho \int_{-\infty}^{\infty} u(U-u)\,dy = b\rho \int_{-\infty}^{\infty} (U-u_1)(u_1)\,dy$$

$= b\rho U \int_{-\infty}^{\infty} u_1\,dy$ $\quad d\eta = dy\sqrt{U/\nu x}$ $\quad \frac{d\eta}{dx} = y U^{1/2} \nu^{-1/2}\left(\frac{-x^{-3/2}}{2}\right)$

$$= b\rho U \int_{-\infty}^{\infty} \left[c U \sqrt{\frac{L}{x}}\, g(\eta)\right] \sqrt{\frac{\nu x}{U}}\, d\eta$$

$$= b\rho U^2 c \sqrt{\frac{\nu L}{U}} \int_{-\infty}^{\infty} g(\eta)\,d\eta$$

$$u_{1,x} = cU\sqrt{L}\left(-\frac{1}{2}x^{-3/2}\right) g(\eta) + cU\sqrt{\frac{L}{x}}\, g'\left(yU^{1/2}\nu^{-1/2}\left(-\frac{x^{-3/2}}{2}\right)\right)$$

$$u_{1,y} = cUL^{1/2}x^{-1/2}g'(\eta)\sqrt{U/\nu x}$$

$$u_{1,yy} = cUL^{1/2}x^{-1/2}g''U\nu^{-1}x^{-1}$$

$$= \frac{cU^2L^{1/2}}{\nu}x^{-3/2}g''$$

$$\cancel{cU^2L^{1/2}}\left(-\frac{1}{2}\cancel{x^{-3/2}}\right)\Big[g + \underbrace{x^{-1/2}yU^{1/2}\nu^{-1/2}}_{y\sqrt{\frac{U}{\nu x}}}\,g'\Big]$$

$$= \cancel{cU^2L^{1/2}x^{-3/2}}\,g''$$

$$g'' + g/2 + \eta/2\, g' = 0 \qquad \begin{matrix} g'=0 & \eta=0 \\ g=0 & \eta=\infty \end{matrix}$$

$$g' + \frac{1}{2}\eta g + A = 0 \qquad g = A\exp(-\eta^2/4) \qquad A=1$$

$$g'(0)=0 \Rightarrow A=0 \qquad g' = -\frac{2\eta}{4}Ae^{-\eta^2/4}$$

or can be absorbed in C

Determine C such that the plate drag

$$2D = b\rho U^2 c\sqrt{\frac{\nu L}{U}}\underbrace{\int_{-\infty}^{\infty} g(\eta)\,d\eta}_{2\sqrt{\pi}} = \text{Blasius solution} = 1.328\, b\rho U^2\sqrt{\frac{\nu L}{U}}$$

$\therefore$ $C = .664/\sqrt{\pi}$

$$\frac{u_1}{U} = \frac{.664}{\sqrt{\pi}}\left(\frac{L}{x}\right)^{\frac{1}{2}}\exp\left(-\frac{1}{4}\frac{y^2U}{x\nu}\right) \propto x^{-1/2}$$

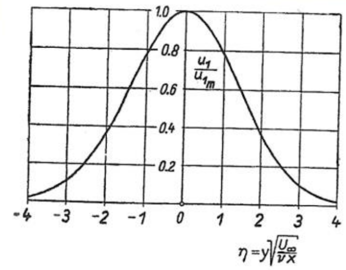
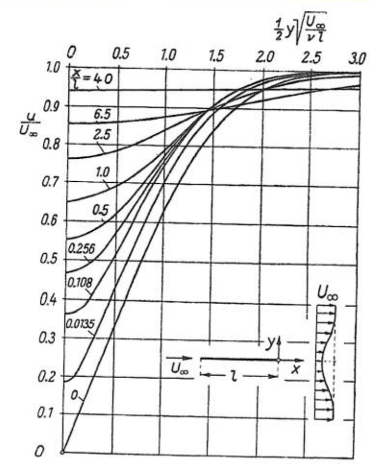


Fig. 9.10. Asymptotic velocity distribution in the laminar wake behind a flat plate, from eqn. (9.35)

Fig. 9.11. Velocity distribution in the laminar wake behind a flat plate at zero incidence ➤



Note that solution only valid in far wake as per assumption $u_1 << U$, neglecting higher order terms, and Blasius type similarity. Therefore, $x > 3L$.

Remarkable velocity distribution Gaussian

$Re_{crit} \sim 4$.