The plane laminar jet

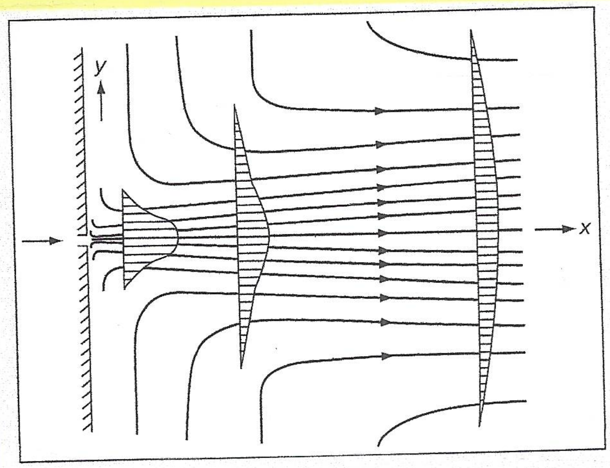

FIGURE 4-23
Definition sketch for a two-dimensional laminar free jet. [*After Schlichting* (*1933a*).]

Since jet spreads at p = constant w/o bounding walls

$$u_x + v_y = 0$$
$$u u_x + v u_y = \nu u_{yy}$$
$$J = \rho \int_{-\infty}^{\infty} u^2 \, dy = \text{constant}$$

$$\psi = \nu^{1/2} x^{1/3} f(\eta) \qquad \eta = y / (3 \nu^{1/2} x^{2/3})$$
$$= a(x) f \qquad \eta = y / \delta(x) \qquad \eta_y = \delta^{-1} \qquad \eta_x = -y \delta^{-2} \delta_x = -\eta \delta_x / \delta$$

$$\frac{\nu^{1/2} x^{1/3}}{3 \nu^{1/2} x^{2/3}} = \frac{x^{-1/3}}{3}$$

$$u = \psi_y = a f' \delta^{-1} = \frac{x^{-1/3}}{3} f'$$
$$v = -\psi_x = -a_x f + a f' \eta \delta_x / \delta$$
$$v = -\frac{\nu^{1/2}}{3} x^{-2/3} f + \frac{x^{-1/3}}{3} (2 \nu^{1/2} x^{-1/3}) \eta f'$$
$$v = -\frac{1}{3} \nu^{1/2} x^{-2/3} \left[f - 2 \eta f' \right]$$

$$a_x = \frac{\nu^{1/2}}{3} x^{-2/3}$$

$$\delta = 3 \nu^{1/2} x^{2/3}$$
$$\delta_x = \frac{2}{3} 3 \nu^{1/2} x^{-1/3} = 2 \nu^{1/2} x^{-1/3}$$

$$(a_x \delta - a \delta_x) f'^2 - a_x \delta f f'' - \nu f''' = 0$$

$$a_x \delta = \frac{\nu^{1/2} x^{-2/3}}{\cancel{3}} \cancel{3} \nu^{1/2} x^{2/3} = \nu$$

$$a \delta_x = \nu^{1/2} x^{1/3} \, 2 \nu^{1/2} x^{-1/3} = 2\nu$$

$\therefore \quad \underbrace{f''' + ff'' + f'^2}_{\frac{d}{d\eta}(f'f)} = 0$

BC: $\quad u_y = 0 \quad v = 0 \quad y = 0$

$u = 0 \qquad y = \pm\infty$

$\therefore \quad f''(0) = 0 \quad f(0) = 0 \quad y = 0$

$f'(\pm\infty) = 0$

$f'' + f'f = c_1 = 0 \qquad f' = f'' = 0 \quad \eta = \pm\infty$

let $\beta = \alpha\eta \quad f = 2\alpha F(\beta)$

$\beta' = \alpha \quad f' = 2\alpha F'\alpha = 2\alpha^2 F'$

$f'' = 2\alpha^3 F''$

$2\alpha^3 F'' + 2\alpha^2 F' 2\alpha F = 0$

$F'' + 2FF' = 0$

$\beta = 0 \quad F = 0 \quad \beta = \pm\infty \quad F' = 0$

Also $f'(0) = 1$

$u = \frac{x^{-1/3}}{3} f'$

$u(0) = u_{max} f'(0)$

$\therefore f'(0) = 1$

$\alpha\ F'(0) = 1$

Riccati type $\quad F' + F^2 = c_2 = 1$

$$\beta = \int_0^F \frac{dF}{1-F^2} = \frac{1}{2}\ln\frac{1+F}{1-F}$$

$$= \tanh^{-1} F$$

or $\quad F = \tanh\beta = \dfrac{1-\exp(-2\beta)}{1+\exp(-2\beta)}$

$$\frac{dF}{d\beta} = 1 - \tanh^2\beta$$

$$u = \frac{2}{3}\alpha^2 x^{-1/3}\left(1 - \tanh^2\beta\right)$$

$$J = \rho \int_{-\infty}^{\infty} u^2\, dy = \text{constant}$$

$$u = \frac{1}{3x^{1/3}} f' \qquad f' = 2\alpha^2 F' = 2\alpha^2 \operatorname{sech}^2 \beta$$

$$= \frac{2}{3x^{1/3}} \alpha^2 (1 - \tanh^2 \beta) \qquad = 2\alpha^2 (1 - \tanh^2 \beta)$$

$$J = \rho \int_{-\infty}^{\infty} \left(\frac{2\alpha^2}{3x^{1/3}} \operatorname{sech}^2 \alpha\eta \right)^2 3\nu^{1/2} x^{2/3}\, d\eta \qquad dy = \delta(x)\, d\eta = 3\nu^{1/2} x^{2/3}\, d\eta$$

$$= \frac{16}{9} \rho \nu^{1/2} \alpha^3 \quad \text{ie} \quad \alpha = \left(\frac{9J}{16\sqrt{\rho\mu}} \right)^{1/3}$$

$$\approx .8255 \frac{J^{1/3}}{(\rho\mu)^{1/6}}$$

$$u = \frac{2}{3} x^{-1/3} \alpha^2 \operatorname{sech}^2 \alpha\eta$$

$$u_{max} = u(0) = \frac{2\alpha^2}{3} x^{-1/3} = \frac{2}{3} \left(\frac{9J}{16\sqrt{\rho\mu}} \right)^{2/3} x^{-1/3}$$

$$= \frac{2}{3} \left(\frac{9}{16} \right)^{2/3} \frac{J^{2/3}}{(\rho\mu x)^{1/3}} \approx .4543 \left(\frac{J^2}{\rho\mu x} \right)^{1/3}$$

$$\propto x^{-1/3}$$

$$u = u_{max} \operatorname{sech}^2 \alpha\eta = u_{max} \operatorname{sech}^2 \Big[\underbrace{.2752 \left(\frac{\rho J}{\mu^2 x^2} \right)^{1/3} y}_{\alpha\eta} \Big]$$

note $\operatorname{sech}^2 3 \approx .01$

δ = width jet = 2 y distance where $u = .01\, u_{max}$

$$= 2y\big|_{1\%} \approx 21.8 \left(\frac{\mu^2}{\rho J} \right)^{1/3} x^{2/3} \quad \propto x^{2/3}$$

$$\dot{m} = \rho \int_{-\infty}^{\infty} u\, dy = (36 \mu \rho J x)^{1/3} \approx 3.302 (\mu \rho J x)^{1/3} \quad \propto x^{1/3}$$

jet drags ambient fluid along its path as it develops. Solution not valid near $x = 0$, as BL approximation not valid since $Re = \frac{\dot{m}}{\mu} \sim \left(\frac{\rho J x}{\mu^2} \right)^{1/3}$ per unit depth not large. Profile S shaped with inflection point ∴ unstable transition turbulence $Re_{crit} \sim 30$.