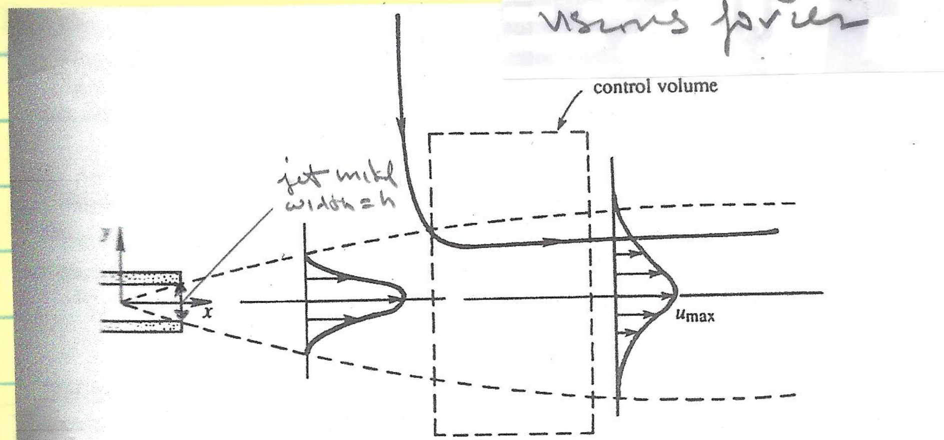
Plane Laminar Jet

($v \neq 0$ due to entrainment)
Entrainment = jet entrains surrounding fluid due to viscous forces



10.28 Laminar two-dimensional jet. A typical streamline showing entrainment of surrounding ... indicated.

near field depends IC, whereas far field achieves similarity due to absence externally imposed length scale in downstream direction

BL assumptions : $Re = \frac{u_0 x}{\nu} >> 1$ $\quad u_0$ = uniform inlet velocity

$\frac{\partial}{\partial y} >> \frac{\partial}{\partial x} \quad v << u \quad \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$ $\quad$ Note: $Re_{cut} = 4$

$Re_{trans} \sim 30$

$\frac{d}{dy}(uv) = u v_y + v u_y$

$u_x + v_y = 0$ $\quad$ (1) $\quad u(y = \pm\infty) = 0$ } infinity condition but $v \neq 0$

$u u_x + v u_y = \nu u_{yy}$ $\quad$ (2) $\quad v(y = 0) = 0$ } $x > 0$ $\quad$ Symmetry condition

inlet condition $\quad u(x = x_0) = u_0 \quad -\frac{h}{2} \le y \le \frac{h}{2}$

$u_y(y = 0) = 0$

$$\int_{-\infty}^{\infty} [u(1) + (2)]\, dy =$$

Similar derivation momentum integral equation #3 except $\int_{-\infty}$

$$\int_{-\infty}^{\infty} 2 u u_x\, dy + \int_{-\infty}^{\infty} (u v_y + v u_y)\, dy = \nu u_y \Big|_{-\infty}^{\infty} = \frac{\tau}{\rho}\Big|_{-\infty}^{\infty} \quad \tau = \mu u_y$$

$$\frac{d}{dx}\int_{-\infty}^{\infty} u^2\, dy + uv\Big|_{-\infty}^{\infty} = \nu u_y\Big|_{-\infty}^{\infty}$$

$u(y = \pm\infty) = 0$

$\therefore u_y(\pm\infty) = 0$

Streamwise momentum flux preserved ie zero drag $\quad \frac{d}{dx}\int_{-\infty}^{\infty} \rho u^2\, dy = 0$ (zero drag of free stream velocity version BL equation cv flat plate)

$$\therefore \int_{-\infty}^{\infty} u^2 dy = \text{Constant} = \int_{-\infty}^{\infty} \tilde{u}^2(y)\, dy = J/\rho$$

J = momentum flux per unit span = constant

where $u = \tilde{u}(y)$ at $x = x_0$ $\quad \tilde{u}(y)$ = known inlet velocity profile

Similarity solution far enough downstream x_0 where BL equations valid at $\tilde{u}(y)$ w/o influence. Since BL equation same Blasius but different BC seek solution of same form:

$$\psi = u_0(x)\delta(x) f(\eta) \qquad \eta = y/\delta(x) \qquad \delta(x) = [\nu x/u_0]^{1/2}$$

$$u_0(x) = u(x,0) = u_{max}$$

$$u = \psi_y = u_0 \delta f' \eta_y = u_0 f' \qquad u(0) = u_0 \Rightarrow f'(0) = 1$$

$$J/\rho = \int_{-\infty}^{\infty} u^2 dy = u_0^2(x) \int_{-\infty}^{\infty} f'^2(\eta)\, dy \qquad d\eta = dy/\delta$$

$$= u_0^2 \delta \int_{-\infty}^{\infty} f'^2(\eta)\, d\eta = u_0^2 \delta C \qquad C = \text{non dimensional constant}$$

$$= u_0^2 [\nu x / u_0]^{1/2} C$$

$$J/\rho = C u_0^{3/2} (\nu x)^{1/2} \qquad \frac{J^2}{\rho^2} = C^2 u_0^3 \nu x$$

$$u_0^3 = J^2/\rho^2 C^2 \nu x$$

$$u_0(x) = [J^2/\rho^2 C^2 \nu x]^{1/3} \qquad \delta(x) = [C \rho \nu^2 x^2 / J]^{1/3}$$

$$\frac{\nu x\, (\rho^2 C^2 \nu x)^{1/3}}{J^{2/3}}$$

$$(u_0 \delta)^3 = \frac{J^2}{\rho^2 C^2 \nu x} \times \frac{C \rho \nu^2 x^2}{J} = \frac{J \nu x}{\rho C}$$

$$\frac{\nu^{\frac{4}{3}} \rho^{\frac{2}{3}} C^{\frac{2}{3}} x^{4/3}}{J^{2/3}}$$

$$\psi = \underbrace{\left[\frac{J \nu x}{\rho C}\right]^{1/3}}_{a(x)} f(\eta) \qquad \eta = y / \underbrace{[C \rho \nu^2 x^2 / J]^{1/3}}_{\delta(x)}$$

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} = \nu \psi_{yyy}$$

$\psi = a(x) f(\eta)$ $\quad \eta = y/\delta(x)$ $\quad \eta_y = \delta^{-1}$ $\quad \eta_{yy} = 0$

$u = \psi_y = a f' \delta^{-1}$ $\quad = y\delta^{-1}$ $\quad \eta_x = -y\delta^{-2}\delta_x$

$= \frac{a}{\delta} f'$ $\quad = -\eta\, \delta_x/\delta$

$u_x = (a\delta^{-1})_x f' + (a\delta^{-1}) f''(-\eta\delta_x/\delta)$

$u_y = (a\delta^{-1}) f'' \delta^{-1} = (a\delta^{-2}) f''$

$u_{yy} = (a\delta^{-3}) f'''$ $\quad v = -\psi_x = -[a_x f + a f'(-\eta\delta_x/\delta)]$

$= -a_x f + f'(\eta a \delta_x/\delta)$

$(a\delta^{-1}) f' [(a\delta^{-1})_x f' + (a\delta^{-1}) f''(-\eta\delta_x/\delta)]$

$+ [-a_x f + f'(\eta a\delta_x/\delta)](a\delta^{-2}) f'' = \nu (a\delta^{-3}) f'''$

$(a\delta^{-1})(a\delta^{-1})_x f'^2 - (a\delta^{-1})^2(\eta\delta_x/\delta) f'f''$

$- a_x(a\delta^{-2}) f f'' + (a\delta^{-2})(\eta a \delta_x/\delta) f'f'' = \nu(a\delta^{-3}) f'''$

$(a\delta^{-1})(a\delta^{-1})_x f'^2 - a_x(a\delta^{-2}) ff'' - \nu(a\delta^{-3}) f''' = 0$

$\frac{\cancel{a}}{\delta}\left(\frac{a_x}{\delta} - \frac{a}{\delta^2}\delta_x\right) f'^2 - \frac{a_x \cancel{a}}{\delta^2} ff'' - \nu \frac{\cancel{a}}{\delta^3} f''' = 0$

$\delta^2\left(\frac{a_x}{\delta} - \frac{a}{\delta^2}\delta_x\right) f'^2 - a_x\delta ff'' - \nu f''' = 0$

$(a_x\delta - a\delta_x) f'^2 - a_x\delta\, ff'' - \nu f''' = 0$

$a = \left[\frac{J\nu x}{ec}\right]^{1/3} = \left(\frac{J\nu}{ec}\right)^{1/3} x^{1/3}$ $\quad a_x = \left(\frac{J\nu}{ec}\right)^{1/3} \frac{x^{-2/3}}{3}$

$\delta = \left[\frac{ce\nu^2 x^2}{J}\right]^{1/3} = \left[\frac{ce\nu^2}{J}\right]^{1/3} x^{2/3}$ $\quad \delta_x = \left[\frac{ce\nu^2}{J}\right]^{1/3} \frac{2}{3} x^{-1/3}$

• $a_x\delta = \left(\frac{J\nu}{ec}\right)^{1/3} \frac{x^{-2/3}}{3} \left[\frac{ce\nu^2}{J}\right]^{1/3} x^{2/3}$

$= \left(\frac{J\nu}{ec}\frac{ce\nu^2}{J}\right)^{1/3} \frac{1}{3} = \frac{\nu}{3}$

$$a\,\delta x = \left(\frac{J\nu}{\rho c}\right)^{1/3} x^{1/3} \left(\frac{c\,\rho\,\nu^2}{J}\right)^{1/3} \frac{2}{3} x^{-1/3}$$

$$= \left(\frac{J\nu}{\cancel{\rho c}} \frac{\cancel{c\rho}\nu^2}{\cancel{J}}\right)^{1/3} \frac{2}{3} = \frac{2\nu}{3}$$

$$\left(\frac{\nu}{3} - \frac{2\nu}{3}\right) f'^2 - \frac{\nu}{3} - \nu f''' = 0$$

$$3\frac{d}{d\eta}(f'') + \frac{d}{d\eta}(f'f) = 0$$

ie $3f''' + f''f + f'^2 = 0$ (1)

BC $f' = 0$ $\eta = \pm\infty$ $f' = 1$ $\eta = 0$ $f = 0$ $\eta = 0$

$u(x, \pm\infty) = 0$ $u(x, 0) = u_0(x)$ $v(x, 0) = 0$

integrating (1): $3f'' + f'f = c_1$ (2) at $\eta = \pm\infty$ $f' = 0$ $\therefore f'' = 0$

$\therefore$ $c_1 = 0$

integrating (2): $3f' + f^2/2 = c_2$ (3)

$$3\frac{d}{d\eta}(f') + \frac{d}{d\eta}(f^2/2) = 0$$

at $\eta = 0$ $f' = 1$ $f = 0 \Rightarrow c_2 = 3$

$\frac{df}{d\eta} + f^2/6 = 1$

$df = (1 - f^2/6)\,d\eta$

$$3\frac{df}{d\eta} + f^2/2 = 3 \Rightarrow \int \frac{df}{1 - f^2/6} = \int d\eta$$

$f = \sqrt{6}\tanh\beta$

$df = \sqrt{6}\,\mathrm{sech}^2\beta\, d\beta$

$$\int \frac{\sqrt{6}\,\mathrm{sech}^2\beta\, d\beta}{1 - \tanh^2\beta} = \int d\eta$$

$\beta = \frac{\eta}{\sqrt{6}} + c_3$ $\beta = \tanh^{-1} f/\sqrt{6}$

$$f = \sqrt{6}\tanh\left(\frac{\eta}{\sqrt{6}} + c_3\right)$$

$f(0) = 0 \Rightarrow c_3 = 0$ $f' = \mathrm{sech}^2\left(\frac{\eta}{\sqrt{6}}\right)$

$$u(x,y) = u_0(x)\, f'(\eta) = \left(\frac{J^2}{c^2\rho^2\nu x}\right)^{1/3} \operatorname{sech}^2\left(\frac{y}{\sqrt{6}}\left[\frac{J}{c\rho\nu^2x^2}\right]^{1/3}\right)$$

$$c = \int_{-\infty}^{\infty} f'^2(\eta)\, d\eta = \int_{-\infty}^{\infty} \operatorname{Sech}^4\left(\frac{\eta}{\sqrt{6}}\right) d\eta = \frac{4\sqrt{6}}{3}$$

$$\dot{m} = \int_{-\infty}^{\infty} \rho\, u_0(x)\, f'(\eta)\, dy = \rho\, u_0(x)\, \delta(x) \int_{-\infty}^{\infty} f'(\eta)\, d\eta$$

$$= \rho\, u_0(x)\, \delta(x)\, f \Big|_{-\infty}^{\infty} \qquad f = \sqrt{6}\tanh\left(\frac{\eta}{\sqrt{6}}\right)$$

$$1-(-1) = 2$$

$$u_0(x) = \left[J^2/c^2\rho^2\nu x\right]^{1/3} \propto x^{-1/3}$$

$$= \rho\, u_0(x)\, \delta(x)\, 2\sqrt{6}$$

$$\delta(x) = \left[c\rho\nu^2x^2/J\right]^{1/3} \propto x^{2/3}$$

$$(2\sqrt{6})^3 = 4\times 6\,(2\sqrt{6}) = 4\times 12\times\sqrt{6}$$

$$u_0\delta = \left[\frac{J^2}{c^2\rho^2\nu x}\,\frac{c\rho\nu^2x^2}{J}\right]^{1/3} = \left[\frac{J\nu x}{c\rho}\right]^{1/3} = \left[\frac{3J\nu x}{4\sqrt{6}\,\rho}\right]^{1/3}$$

$$\dot{m} = (36 J\rho^2\nu x)^{1/3} \quad \text{increases downstream due entrainment}$$

$$\propto x^{1/3}$$

$$v = -\psi_x = -a_x f + f'\left(\eta \frac{a\delta_x}{\delta}\right)$$

$$= -\frac{1}{3}\left(\frac{J\nu}{\rho c x^2}\right)^{1/3} f + f'\left(\eta\, \frac{2\nu}{3}\left[\frac{J}{c\rho\nu^2x^2}\right]^{1/3}\right)$$

$$= -\frac{1}{3}\left(\frac{J\nu}{\rho c x^2}\right)^{1/3}\left[f - 2\eta f'\right]$$

or $$v/u_0(x) = \frac{-\left[f - 2\eta f'\right]}{3\sqrt{Re_x}} \qquad Re_x = \frac{x\, u_0(x)}{\nu}$$

$$f(\eta) = \pm\sqrt{6} \quad 2\eta f'(\eta) = 2\eta \operatorname{sech}^2(\eta/\sqrt{6}) = 0 \quad \eta = \pm\infty$$

$$\therefore \quad v/u_0(x) = \mp \frac{\sqrt{6}}{3\sqrt{Re_x}} \qquad \eta = \pm\infty$$

entrainment due v from above/below

The jet spreads as $x^{2/3}$. Using similar when BL δ at u/U 99% define half width jet h_{99} as y location where u falls to 1% $y=0$ (i.e. $.01\,u_0(x)$)

$$\mathrm{sech}^2\left(\frac{h_{99}}{\sqrt{6}}\left[\frac{J}{6\rho\nu^2x^2}\right]^{1/3}\right)=.01 \Rightarrow \frac{h_{99}}{\sqrt{6}}\left[\frac{J}{6\rho\nu^2x^2}\right]^{1/3}=2.2924$$

$$h_{99}=5.6152\left[\frac{6\rho\nu^2x^2}{J}\right]^{1/3}$$

jet width $\propto x^{2/3}$ ρ ν increase width, whereas J decreases width

$$Re_x=\frac{x\,u_0(x)}{\nu}=\left(\frac{3Jx}{4\sqrt{6}\rho\nu^2}\right)^{1/3} \quad Re_{h_{99}}=5.6152\left(\frac{3Jx}{4\sqrt{6}\rho\nu^2}\right)^{1/3}$$

Stability: $Re \gg 1$ due inflection points.