e. Flow in the wake of flat plate at zero incidence

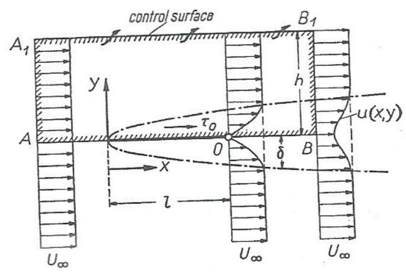


Fig. 9.9. Application of the momentum equation in the calculation of the drag on a flat plate at zero incidence from the velocity profile in the wake

Drag

① x CV

Boundary Layer

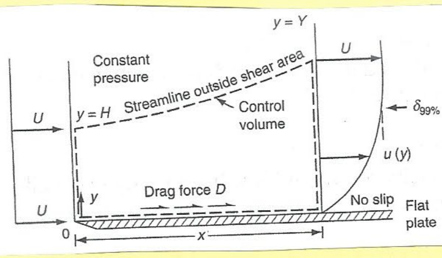


FIGURE 4-1
Definition of control volume for the analysis of flow past a flat plate.

$Y = H + \delta^*$

$$\int_{cs} \rho \underline{V}\cdot\underline{n}\, dA = 0 = -\int_0^H \rho U\, dy + \int_0^Y \rho u\, dy$$

$$UH = \int_0^Y u\, dy = UY + \int_0^Y (u-U)\, dy$$

$$\delta^* = \int_0^Y (1 - u/U)\, dy$$ = displacement thickness

$$\Sigma F_x = -D = \int_{cs} \rho u \underline{V}\cdot\underline{n}\, dA = -\int_0^H \rho U(U\,dy) + \int_0^Y \rho u\,(u\,dy)$$

$$D = \rho U^2 H - \int_0^Y \rho u^2\, dy$$ = fluid force on plate
= − plate force on CV

$$D = \rho U^2 \int_0^Y \frac{u}{U} - \int_0^Y \rho u^2\, dy$$

$$= \int_0^X \tau_w\, dx$$

$$D/\rho U^2 = \theta = \int_0^Y \frac{u}{U}\left(1-\frac{u}{U}\right) dy = \text{momentum thickness}$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 x} = \frac{2\theta}{x} = \frac{1}{x}\int_0^x c_f\, dx \quad \text{per unit span}$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \;\therefore\; c_f = \frac{d}{dx}(x C_D) = 2\frac{d\theta}{dx} \quad \text{ie } \frac{d\theta}{dx} = c_f/2$$

(2) Box CV wake

$$\int_{CS} \rho \underline{V}\cdot\underline{n}\, dA = -\int_A^{A_1} \rho U dy + \int_B^{B_1} \rho u dy + \underbrace{\int_{A_1}^{B_1} \rho v dx}_{\dot{m}_t} + \underbrace{\int_A^B \rho(-v dx)}_{\dot{m}_b}$$

$$\dot{m}_t + \dot{m}_b = \rho U AA_1 - \int_B^{B_1} \rho u dy$$

$$\Sigma F_x = -D = \int_{CS} \rho u \underline{V}\cdot\underline{n}\, dA = -\int_A^{A_1} \rho U(U dy) + \int_B^{B_1} \rho u(u dy)$$

$$+ \underbrace{\int_{A_1}^{B_1} \rho u(v dx) + \int_A^B \rho u(-v dx)}_{\text{assume } u=U}$$

$$= U(\dot{m}_t + \dot{m}_b)$$

$$D = \rho U^2 \cancel{AA_1} - \int_B^{B_1} \rho u^2 dy - U\left(\cancel{\rho U AA_1} - \int_B^{B_1} \rho u dy\right)$$

$$= U\int_B^{B_1} \rho u - \int_B^{B_1} \rho u^2 dy$$

$$D = \rho U^2 \int_B^{B_1} \frac{u}{U}\left(1-\frac{u}{U}\right) dy$$

same as x CV

with $B=0$ & $B_1 = Y$

note for $y > Y$ $u/U = 1$

$$uu_x + vu_y = \nu u_{yy}$$

$u_1 = U - u(x,y)$ $\quad u_1 << U$ $\quad$ velocity defect

$u = U - u_1$ $\quad v = v_1 << U$

$u_x = -u_{1x}$ $\quad u_y = -u_{1y}$ $\quad u_{yy} = -u_{1yy}$

$$(U - u_1)(-u_{1x}) + v_1(-u_{1y}) = \nu(-u_{1yy})$$

$U u_{1x} = \nu u_{1yy}$ $\quad$ to first order

$$u_{1y}(0) = 0$$
$$u_1(\infty) = 0$$

Assume similarity for u_1 as per Blasius

$$\eta = y\sqrt{U/\nu x} = y\, U^{1/2} \nu^{-1/2} x^{-1/2}$$

$u_1 = cU\sqrt{\frac{L}{x}}\, g(\eta)$ $\quad$ $x^{-1/2}$ so that D independent x

$$2D = b\rho\int_{-\infty}^{\infty} u(U-u)\,dy = b\rho\int_{-\infty}^{\infty}(U-u_1)(u_1)\,dy$$

$= b\rho U\int_{-\infty}^{\infty} u_1\,dy$ $\quad$ $d\eta = dy\sqrt{U/\nu x}$ $\quad$ $\frac{d\eta}{dx} = y U^{1/2}\nu^{-1/2}\left(\frac{-x^{-3/2}}{2}\right)$

$$= b\rho U\int_{-\infty}^{\infty}\left[cU\sqrt{\frac{L}{x}}\, g(\eta)\right]\sqrt{\frac{\nu x}{U}}\,d\eta$$

$$= b\rho U^2 c\sqrt{\frac{\nu L}{U}}\int_{-\infty}^{\infty} g(\eta)\,d\eta$$

$$u_{1x} = CU\sqrt{L}\left(-\tfrac{1}{2}x^{-3/2}\right) g(\eta) + CU\sqrt{\tfrac{L}{x}}\, g' \left(yU^{1/2}\nu^{-1/2}\left(-\frac{x^{-3/2}}{2}\right)\right)$$

$$u_{1y} = CUL^{1/2}x^{-1/2}g'(\eta)\sqrt{U/\nu x}$$

$$u_{1yy} = CUL^{1/2}x^{-1/2}g''U\nu^{-1}x^{-1}$$

$$= \frac{CU^2}{\nu}L^{1/2}x^{-3/2}g''$$

$$\cancel{CU^2}\cancel{L^{1/2}}\left(-\tfrac{1}{2}\cancel{x^{-3/2}}\right)\left[g + \underbrace{x^{-1/2}yU^{1/2}\nu^{-1/2}}_{y\sqrt{\frac{U}{\nu x}}}g'\right]$$

$$= \cancel{CU^2}\cancel{L^{1/2}}\cancel{x^{-3/2}}g''$$

$$g'' + g/2 + \eta/2\, g' = 0 \qquad \begin{matrix} g' = 0 & \eta = 0 \\ g = 0 & \eta = \infty \end{matrix}$$

$$g' + \tfrac{1}{2}\eta g + A = 0 \qquad g = A\exp(-\eta^2/4) \qquad A = 1$$

$$g'(0) = 0 \Rightarrow A = 0 \qquad g' = -\frac{2\eta}{4}Ae^{-\eta^2/4}$$

or can be absorbed in C

Determine C such that the plate drag

$$2D = b\rho U^2 C\sqrt{\frac{\nu L}{U}}\underbrace{\int_{-\infty}^{\infty} g(\eta)\,d\eta}_{2\sqrt{\pi}} = \text{Blasius solution} = 1.328\, b\rho U^2\sqrt{\frac{\nu L}{U}}$$

$\therefore$ $C = .664/\sqrt{\pi}$

$$\frac{u_1}{U} = \frac{.664}{\sqrt{\pi}}\left(\frac{L}{x}\right)\exp\left(-\frac{1}{4}\frac{y^2U}{x\nu}\right)$$

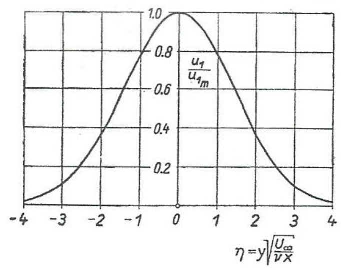
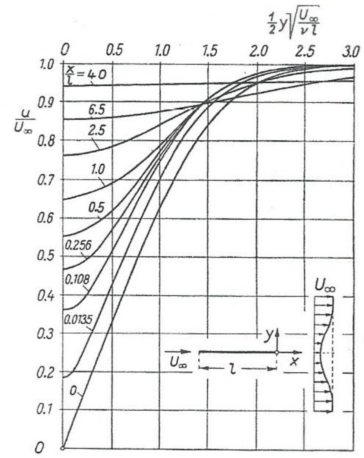


Fig. 9.10. Asymptotic velocity distribution in the laminar wake behind a flat plate, from eqn. (9.35)

Fig. 9.11. Velocity distribution in the laminar wake behind a flat plate at zero incidence ➤



Summary 2D Laminar Wake (White equations)

$$u = u_0 - u_w = C_D\left(\frac{Re_L}{16\pi}\right)^{1/2}\left(\frac{L}{x}\right)^{1/2}\exp\left(-\frac{U_0 y^2}{4x\nu}\right)$$

flat plate: $\dfrac{u(x,0)}{u_0} = \dfrac{.664}{\sqrt{\pi}}\left(\dfrac{L}{x}\right)^{1/2} \quad x > 3L$

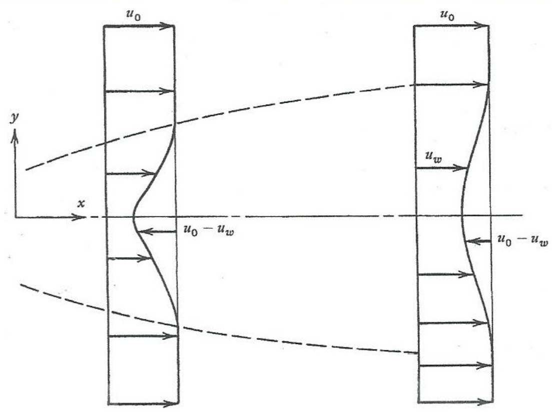


Figure 20.22 Laminar wake of a nonlifting body.

2D Far Wake Nonlifting Body

Recit = 4

Assume : $Re = \frac{u_0 L}{\nu} >> 1$

u_0 = uniform stream
L = length of body

$$F = \text{drag} = \rho u_0^2 \underbrace{\int_{-\infty}^{\infty} \frac{u_w}{u_0}\left(1 - \frac{u_w}{u_0}\right) dy}_{\Theta} = \text{Constant}$$

Θ = wake momentum thickness

$$F = \rho u_0^2 \Theta$$

from x-momentum using continuity.

$u = u_0 - u_w$, u_w = wake velocity
= velocity defect profile

$$C_D = \frac{F}{\frac{1}{2}\rho u_0^2 A}$$

$A = L \times s$, s = span length
or often projected area used for A

$$u_x + v_y = 0$$

$$(u_0 - u) u_x + v u_y = \nu u_{yy}$$

$$u_y = v = 0 \quad y = 0$$

$$u(x, \pm\infty) = 0$$

$u_w = u_0 - u$
$v_w = v$
$u << v$, $\frac{\partial}{\partial x} << \frac{\partial}{\partial y}$
write NS for u_w, v_w
apply BL assumptions for u, v

Similarity Solution:

$$\psi = a x^m f(\eta) \qquad \eta = \frac{y}{b x^n}$$

$$u = \psi_y \qquad v = -\psi_x$$

$$u_0 u_x \propto u_0 b^2 x^{2n-1}\left[(m-n)f' - n f''\right]$$

$$-u u_x + v u_y \propto a b^2 x^{m+n-1}\left[(n-m)f'^2 + 2n\eta f' f'' - m f f''\right]$$

$$\nu u_{yy} \propto \nu f'''$$

$$F = \rho a x^m \int_{-\infty}^{\infty} \left(u_0 - \frac{a}{b} x^{m-n} f'\right) f' \, d\eta$$

However, it is not possible to choose m, n such that BL, F independent x, ie, an additional assumption is required to achieve similarity. If $n = 1/2$, $m = 0$, only terms remain with $x^{-1/2}$; therefore must also assume $x \to \infty$, which is why called "far wake" solution.

$$f''' + \frac{u_0^2 \delta}{2\nu}(\eta f'' + f') = 0$$

$$F = \rho a u_0 \int_{-\infty}^{\infty} f' d\eta = \rho u_0 Q$$

$$f' = \exp(-\eta^2) \quad \text{if} \quad \delta = \left(\frac{4\nu}{u_0}\right)^{1/2}$$

$$\Rightarrow a = F/\rho u_0 \pi$$

$\eta = y\sqrt{u_0/4\nu x}$ width increases as $x^{1/2}$

$$\chi = \frac{F}{2\rho u_0} \operatorname{erf}(\eta)$$

$u = \left[\frac{F^2}{4\pi \rho^2 u_0 \nu x}\right]^{1/2} \exp(-\eta^2)$ defect decreases $x^{-1/2}$

In wake $\frac{F}{\rho u_0^2} = \Theta = \text{constant}$

In far wake $Q = u_0 \Theta = u_0 \delta^*$

ie $\Theta = \delta^* = \text{constant}$ in far wake of lifting body

Same solution if make Oseen approximation for convective acceleration term

$u_w \frac{\partial u_w}{\partial x} = u_0 \frac{\partial u}{\partial x}$ $Re < 1$

therefore, solution valid all $Re > 0$.

All bodies produce parabolic wake.

Axisymmetric Wake

$$u = u_0 - u_w$$

$$u_0 u_x = \frac{\nu}{r}(r u_r)_r$$

$u_r = 0 \quad r = 0$

$u = 0 \quad r = \infty$

$$F = 2\pi \rho u_0 \int_0^\infty u r \, dr = \cancel{\text{constant}} = C_D \tfrac{1}{2} \rho u_0^2 L^2$$

$$u = c u_0 \frac{f(\eta)}{x} \qquad \eta = \frac{1}{2} r \sqrt{\frac{u_0}{\nu x}} \qquad \propto x^{1/2}$$

$$(\eta f')' + 2\eta^2 f' + 4\eta f = 0$$

$$f'(0) = 0 \qquad f(\infty) = 0$$

$$f(\eta) = \exp(-\eta^2)$$

Same shape as 2D with r replace y

$$u = \frac{u_0 C_D}{8\pi} Re_L \frac{L}{x} \exp\left(-\frac{u_0 r^2}{4 x \nu}\right) \qquad u_{max} \propto x^{-1}$$

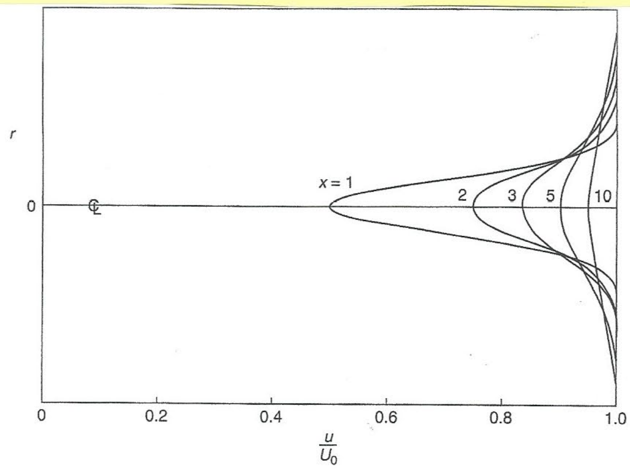


FIGURE 4-38
Velocity profiles illustrating the decay of a round wake from Eq. (4-211). Units on x and r are arbitrary.

Wake Asymptotic Laws:

	$(u_1)_{max}$	δ	
2D	$x^{-1/2}$	$x^{1/2}$	same turbulent
AXI	x^{-1}	$x^{1/2}$	but diff for axi
	$x^{-2/3}$	$x^{1/3}$	