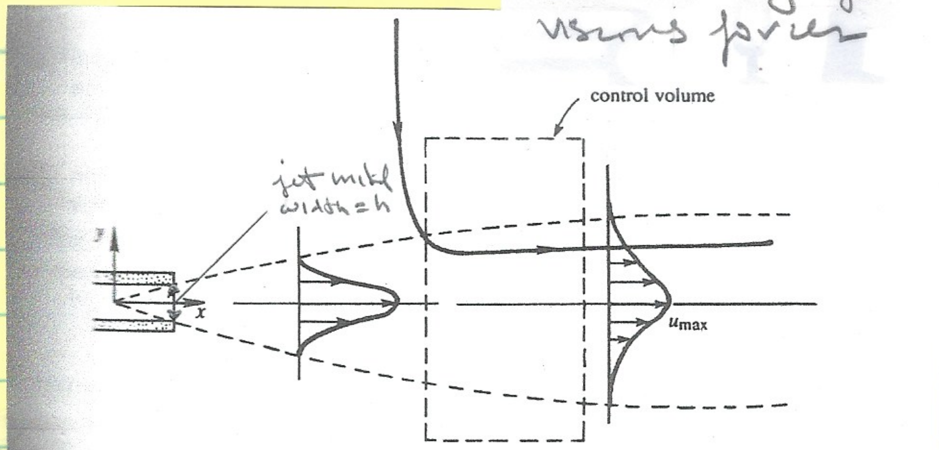
Plane Laminar Jet

($v \neq 0$ due to entrainment)
Entrainment = jet entrains surrounding fluid due to viscous forces



10.28 Laminar two-dimensional jet. A typical streamline showing entrainment of surrounding ... indicated.

near field depends IC, whereas far field achieves similarity due to absence externally imposed length scale in downstream direction

BL assumptions : $Re = \frac{u_0 x}{\nu} \gg 1$ $\quad u_0$ = uniform inlet velocity

$\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x} \quad v \ll u \quad \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$ $\quad$ Note: $Re_{crit} = 4$, $Re_{trans} \sim 30$

$u_x + v_y = 0 \quad (1)$ $\qquad u(y=\pm\infty) = 0$ $\quad$ infinity condition but $v \neq 0$

$u u_x + v u_y = \nu u_{yy} \quad (2)$ $\qquad v(y=0) = 0$ $\quad$ Symmetry condition

inlet condition $\quad u(x=x_0) = u_0 \quad -\frac{h}{2} \le y \le \frac{h}{2}$

$u_y(y=0) = 0$

$$\int_{-\infty}^{\infty} [u(1) + (2)]\, dy =$$

$$\int_{-\infty}^{\infty} 2 u u_x dy + \int_{-\infty}^{\infty} (u v_y + v u_y)\, dy = \nu u_y \Big|_{-\infty}^{\infty}$$

$$\frac{d}{dx} \int_{-\infty}^{\infty} u^2 dy + uv \Big|_{-\infty}^{\infty} = \nu u_y \Big|_{-\infty}^{\infty}$$

$u(y=\pm\infty) = 0$

$u_y(\pm\infty) = 0$

$$\frac{d}{dx} \int_{-\infty}^{\infty} \rho u^2 dy = 0$$

Streamwise momentum flux preserved ie zero drag (zero drag of free stream velocity version BL equation CV flat plate)

$$M = \rho \int_{-\infty}^{\infty} u^2 \, dy = \text{constant} \neq f(x)$$

$$= \rho u_0^2 h \qquad h = \text{width of jet inlet}$$

Similarity Solution:

m, n exponents
n = rate of jet spreading
a, b make χ, η dimensionless

$$\chi = a x^m f(\eta) \qquad \eta = \frac{y}{b x^n}$$

$$u = \chi_y$$

$$v = -\chi_x$$

$$f''(0) = f(0) = 0$$

$$f'(\infty) = 0$$

$$\chi_y \chi_{xy} - \chi_x \chi_{yy} = \nu \chi_{yyy}$$

$$\frac{ab}{\nu} x^{m+n-1} \left[(m-n) f'^2 - m f f'' \right] = f'''$$

Since both LHS & RHS $f(\eta)$: $m+n=1$

$$M = \rho a^2 b^{-1} x^{2m-n} \int_{-\infty}^{\infty} f'^2 \, d\eta = \text{constant} \;:\; 2m-n=0$$

$$m = 1/3 \quad n = 2/3 \quad \text{ie} \quad \eta = \frac{y}{b x^{2/3}} \propto x^{-2/3}$$

jet width increases as $x^{2/3}$

$$f''' + \frac{a b}{3\nu}\left(f'^2 + f f''\right) = 0$$ nonlinear equation has surprisingly simple solution

$$f = \tanh \eta$$

$$u = \frac{a}{b} x^{-1/3} \operatorname{sech}^2 \eta \quad \text{or} \quad \frac{u}{u_{max}} = \operatorname{sech}^2 \eta$$

x, η dimensionless & f, u without constants if

$$a = \left(\frac{9\nu M}{2\rho}\right)^{1/3} \qquad b = \left(\frac{48\nu^2\rho}{M}\right)^{1/3}$$

$$u_{max} = \left(\frac{3M^2}{32\rho^2\nu}\right)^{1/3} x^{-1/3}$$ decays as $x^{-1/3}$

$$Q = \int_{-\infty}^{\infty} u\,dy = \left(\frac{36M\nu}{\rho}\right)^{1/3} x^{1/3}$$ increases as $x^{1/3}$ due to entrainment

$$\frac{dM}{dx} = 0 \qquad \frac{dQ}{dx} > 0 \qquad \underbrace{\frac{d}{dx}\int_{-\infty}^{\infty} u^3\,dy < 0}$$

KE flux decreases due to viscous dissipation

$x = 0$ singularity
practical application use $x - x_0$ where x_0 = virtual origin

M = constant, increase in Q & decay KE are all interrelated

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$$\frac{\dot{m}}{\mu} = \left[\frac{M\rho x}{\mu^2}\right]^{1/3}$$

Theory not valid near slot, only for large $Re = \frac{\dot{m}}{\mu} \propto x^{1/3}$

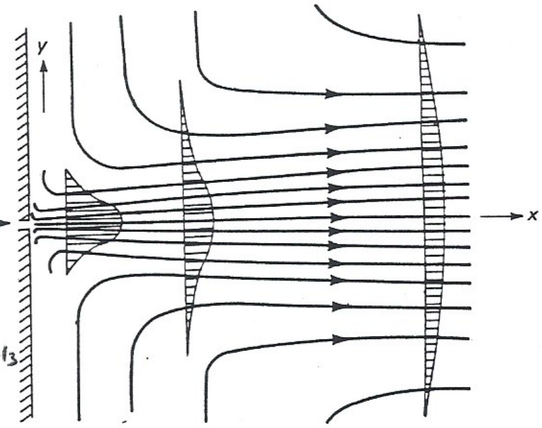


FIGURE 4-18
Definition sketch for the two-dimensional laminar free jet. [*After Schlichting* (*1933a*).]

Summary 2D Laminar Jet (White equations)

$$u(x,y) = u_{max}\,\mathrm{sech}^2\left[.2752\left(\frac{M\rho}{\mu^2 x^2}\right)^{1/3} y\right]$$

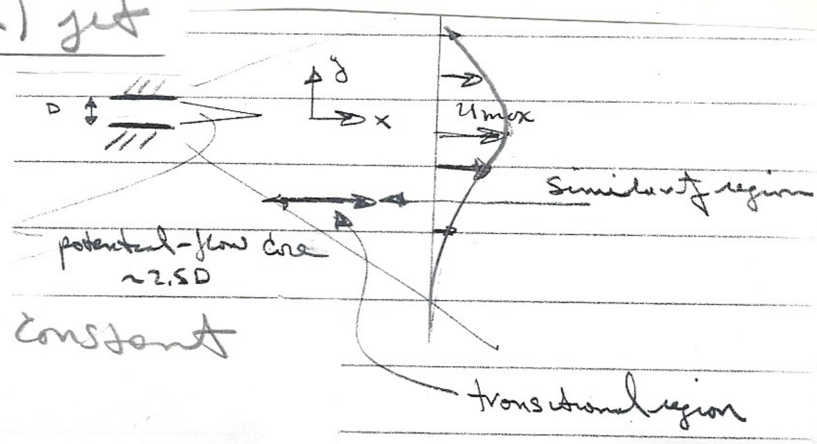
$$u_{max} = .4543\left(\frac{M^2}{\rho\mu x}\right)^{1/3} \qquad \propto x^{-\frac{1}{3}}$$

$$\text{Width} = 2y\Big|_{\frac{u}{u_{max}}=.01} = 21.8\left(\frac{x^2\mu^2}{M\rho}\right)^{1/3} \qquad \propto x^{2/3}$$

$$v_\infty = -.55\left(\frac{M\mu}{\rho x^2}\right)^{1/3}$$

$$\dot{m} = \rho Q = 3.302\,(M\rho\mu x)^{1/3} \qquad \propto x^{1/3}$$

Axisymmetric (round) jet



$$R_e = \frac{u_0 r_0}{\nu} >> 1$$

$$M = 2\pi\rho \int_0^\infty u^2 r\,dr = \text{constant}$$

$$u_x + \frac{1}{r}(r v)_r = 0$$

$$u u_x + v u_r = \frac{\nu}{r}(r u_r)_r \qquad \frac{\partial}{\partial r} >> \frac{\partial}{\partial x}$$

$$v = 0,\ \underbrace{u_y = 0}_{u = u_{max}} \quad r = 0$$

$$u = 0 \quad r = \infty$$

Similarity solution:

$$\psi = x^m F(\eta) \quad \eta = \frac{r}{x^n}$$

m, n determined as with 2D jet st M independent x &

momentum equation LHS & RHS have same order of magnitude

$$u \propto x^{m-2n} \quad u_x \propto x^{m-2n-1} \quad u_r \propto x^{m-3n} \quad \frac{1}{r}(r u_r)_r \propto x^{m-4n}$$

$$2m - 4n + 2n = 0 \qquad 2m - 4n - 1 = m - 4n$$

$$m = n = 1$$

$$\psi = \nu x F(\eta) \quad \eta = r/x \propto x^{-1}$$

jet radius increases linearly

$$u = \frac{\nu}{x}\frac{F'}{\eta} \qquad v = \frac{\nu}{x}\left(F' - \frac{F}{\eta}\right)$$

$$\frac{FF'}{\eta^2} - \frac{F'^2}{\eta} - \frac{FF''}{\eta} = \frac{d}{d\eta}\left(F'' - \frac{F'}{\eta}\right)$$

integration: $FF' = F' - \eta F''$

$F'(0) = F(0)\big|_{y=0}$

① F'/η, F even $f(\eta)$; F' odd

$$F = \frac{\xi^2}{1 + \frac{1}{4}\xi^2} \qquad \xi = c\eta$$

② $F(c\eta) = F(\xi)$ also solution

① & ② used in integration to get particular solution that satisfies boundary conditions

$$c = \left(\frac{3M}{16\pi\rho\nu^2}\right)^{1/2} \quad \text{from } M = \text{Constant} = \frac{16\pi}{3}\rho c^2\nu^2$$

$$u = \frac{3M}{8\pi\mu x}\underbrace{\left(1 + \frac{c^2\eta^2}{4}\right)^{-2}}_{\text{profile shape} \neq \text{sech}^2 \text{ shape plane jet}} \qquad u_{max} \propto x^{-1}$$

$\dot{m} \neq \dot{m}(M)$

slow speed spreads rapidly; high speed spreads slowly.

$$\dot{m} = \rho Q = 8\pi\mu x \qquad \propto x$$

such that former entrains more fluid & $\dot{m}_{fast} = \dot{m}_{slow}$ for some μ, x

Laminar Jet Asymptotic Laws:

		u_{max}	W_{max}	$\dot{m}$	δ
Same solution as for turbulent flow (solutions are indistinguishable except $\nu = \nu_t$)	AXI	x^{-1}		x	x
	AXI with W (rotationally symmetric)	x^{-1}	x^{-2}	x	x
	2D	$x^{-1/3}$		$x^{1/3}$	$x^{2/3}$
	2D turbulent	$x^{-1/2}$			x

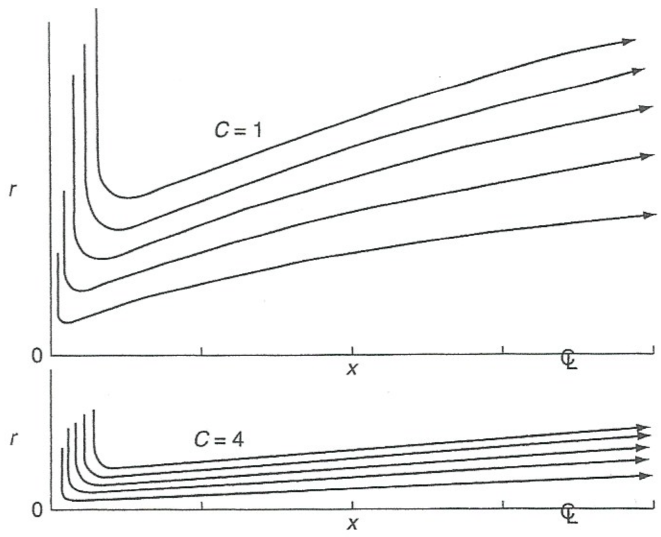


FIGURE 4-37
Streamlines of a round jet for two cases, from the boundary-layer theory of Eq. (4-205).

C = 1 not narrow at BL assumptions marginal
C = 4 narrow $Re = m/\rho\nu^2 > 250$

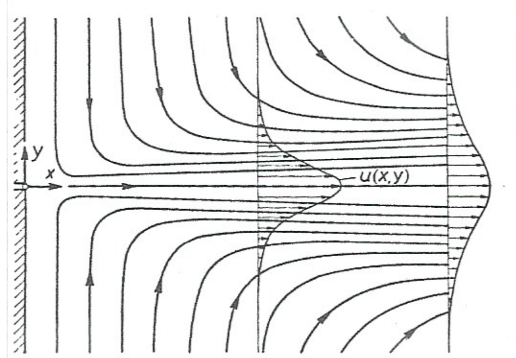


Fig. 11.5. Streamline pattern for a circular laminar jet

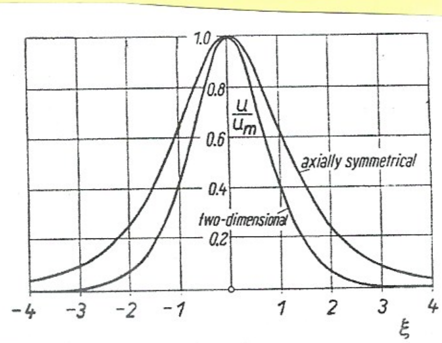


Fig. 9.13. Velocity distribution in a two-dimensional and circular free jet from eqns. (9.44) and (11.15) respectively. For the two-dimensional jet $\xi = 0{\cdot}275\, K^{1/3}\, y/(\nu x)^{2/3}$, and for the circular jet $\xi = 0{\cdot}244\, K'^{1/2}\, y/\nu x$. K and K' denote the kinematic momentum J/ϱ