Part 2: Laminar Boundary Layer: $\boldsymbol{Re_{crit}} =$ $\mathbf{91,000}$; $\boldsymbol{Re_{trans}} = \mathbf{5 \times 10^5 - 3 \times 10^6}$.

Similarity solutions (2D, steady, incompressible): method of reducing PDE to ODE by appropriate similarity transformation; also, because of transformation at least one coordinate lacks origin such that the solution collapses to same form at all length or time scales.

$$u_x + v_y = 0$$
$$uu_x + vu_y = UU_x + \nu u_{yy}$$

BCs: $u(x,0) = v(x,0) = 0$ no slip
$u(x,\infty) = U(x)$ matching outer flow
+ inlet condition

For Similarity $\frac{u(x,y)}{U(x)} = F\left(\frac{y}{g(x)}\right)$ expect $g(x)$ related to $\delta(x)$
Or in terms of stream function Ψ : $u = \psi_y$ $v = -\psi_x$
For similarity $\psi = U(x)g(x)f(\eta)$ $\eta = y/g(x)$

$$u = \psi_y = Uf' \quad v = -\psi_x = -(U_x g f + U g_x f - U g_x \eta f')$$

BC:

$$u(x,0) = 0 \Rightarrow U(x) f'(0) = 0 \Rightarrow f'(0) = 0$$
$$v(x,0) = 0 \Rightarrow U_x(x) g(x) f(0) + U(x) g_x(x) f(0)$$
$$- U(x) g_x(x) \times 0 \times f'(0) = 0$$
$$\Rightarrow \left(U_x(x) g(x) + U(x) g_x(x)\right) f(0) = 0$$
$$\Rightarrow f(0) = 0$$
$$u(x,\infty) = U(x) \Rightarrow U(x) f'(\infty) = U(x) \Rightarrow f'(\infty) = 1$$

Write boundary layer equations in terms of ψ

$$\psi_y\psi_{yx} - \psi_x\psi_{yy} = UU_x + \vartheta\psi_{yyy}$$

Substitute

$$\psi_{yy} = Uf''/g$$

$$\psi_{yyy} = Uf'''/g^2$$

$$\psi_{xy} = U_x f' - Uf''\eta g_x / g$$

Assemble them together:

$$(Uf')\left(U_x f' - \cancel{Uf''\eta\frac{g_x}{g}}\right) - \left(U_x g f + U g_x f - \cancel{U g_x \eta f'}\right)(Uf''/g)$$

$$= UU_x + \nu\left(U\,f'''/g^2\right)$$

$$UU_x f'^2 - UU_x f f'' - (U^2\, g_x/g) f f'' = UU_x + \nu\frac{U}{g^2} f'''$$

$$UU_x f'^2 - \frac{U}{g}(Ug)_x f f'' = UU_x + \nu\frac{U}{g^2} f'''$$

$$f''' + \underbrace{\boxed{\frac{g}{\nu}(Ug)_x}}_{C_1} f f'' + \underbrace{\boxed{\frac{g^2}{\nu}U_x}}_{C_2}\left(1 - f'^2\right) = 0$$

Where for similarity C_1 and C_2 are constant or function η only

- i.e. for a chosen pair of C_1 and C_2 → $U(x)$, $g(x)$ can be found, i.e., potential flow is NOT known a priori.
- Then solution of $f''' + C_1 ff'' + C_2\left(1 - f'^2\right) = 0$ gives $f(\eta)$ → $u(x, y)$, $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_w = \frac{\mu U f''(0)}{g}$, δ, δ^*, θ, H, C_f, C_D

<u>The Blasius Solution for Flat-Plate Flow</u>

U=constant $\rightarrow U_x = 0 \rightarrow C_2 = 0$

Then $C_1 = \frac{U}{\nu} g g_x \neq function(x)$

$$\frac{d}{dx}(g^2) = \frac{2C_1\nu}{U} \Longrightarrow g(x) = [2C_1\nu x/U]^{1/2}$$

Let $C_1 = 1$, then $g(x) = \sqrt{\frac{2\nu x}{U}} \Longrightarrow \eta = y\sqrt{\frac{U}{2\nu x}} = \frac{y}{\sqrt{\frac{2\nu x}{U}}} \propto \frac{y}{\delta}$

Note $\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}}$, i.e., $\delta = \frac{5x}{\sqrt{\frac{Ux}{\nu}}} = 5\sqrt{\frac{\nu x}{U}}$

$$\psi = U[2\nu x/U]^{1/2} f\left(y\sqrt{\frac{U}{2\nu x}}\right) = \sqrt{2\nu U x}\, f(\eta)$$

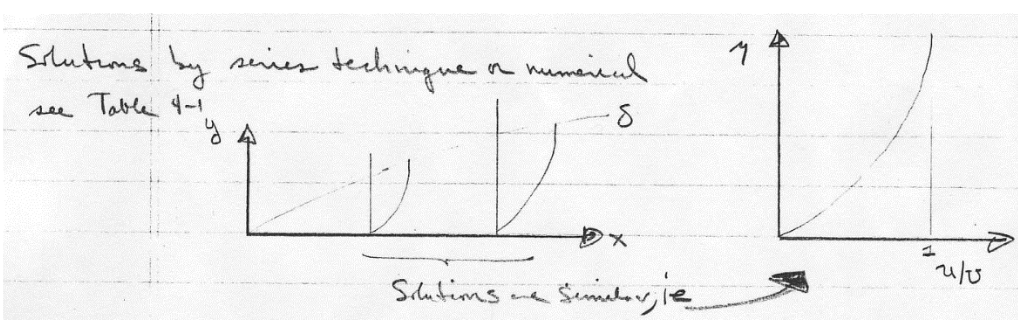
$$u = \psi_y = Uf'$$

$$v = -\psi_x = U g_x(\eta f' - f) = [U\nu/2x]^{1/2}(\eta f' - f) = \frac{U(\eta f' - f)}{\sqrt{2Re_x}}$$

$$\Longrightarrow \boxed{\begin{gathered} f''' + ff'' = 0 \\ f(0) = f'(0) = 0,\ f'(\infty) = 1 \end{gathered}}$$

Blasius equations for Flat Plate Boundary Layer

Solutions by series or numerical methods



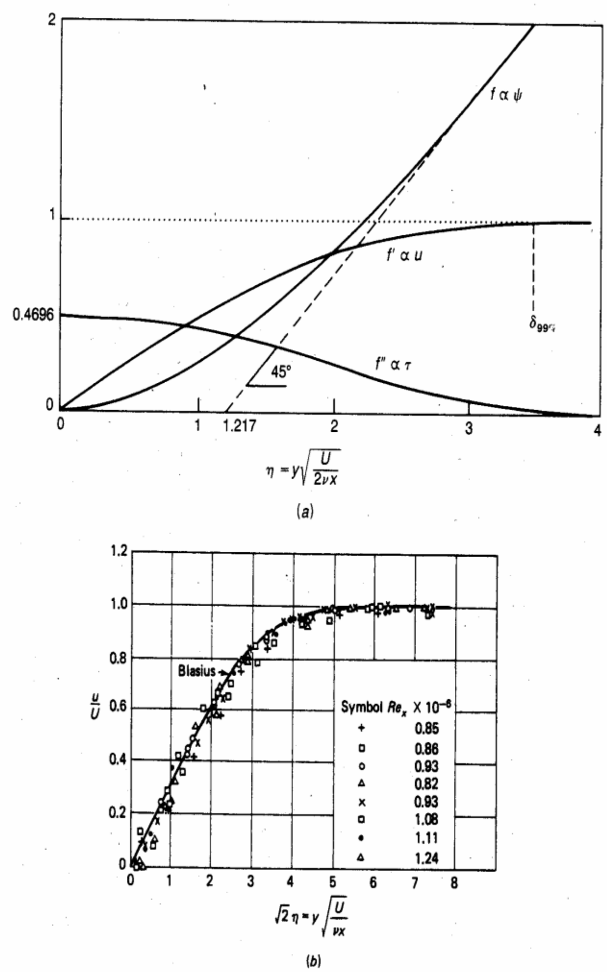


FIGURE 4-6
The Blasius solution for the flat-plate boundary layer: (*a*) numerical solution of Eq. (4-45); (*b*) comparison of $f' = u/U$ with experiments by Liepmann (1943).

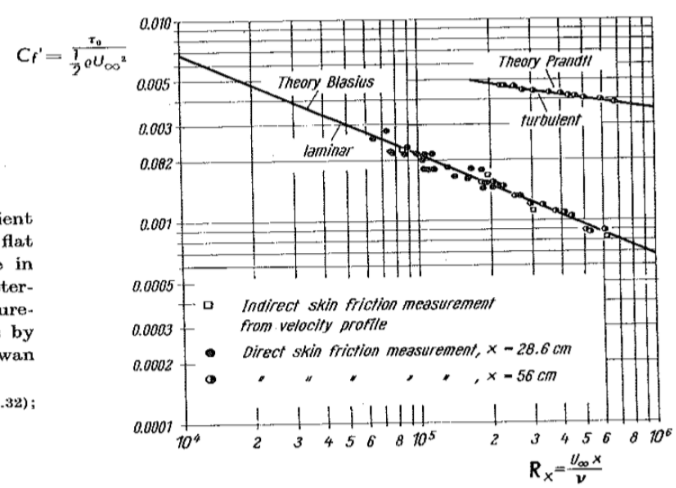


Fig. 7.10. Local coefficient of skin friction on a flat plate at zero incidence in incompressible flow, determined from direct measurement of shearing stress by Liepmann and Dhawan [6, 18]

Theory: laminar from eqn. (7.32); turbulent from eqn. (21.12)

TABLE 4-3

Highly resolved numerical solution of the Blasius equation for flow over a flat plate, Eq. (4-60)

η	$\eta/\eta_{99\%}$	$f(\eta)$	$f'(\eta)$	$f''(\eta)$	$f'''(\eta)$
0.0	0.00000	0.00000	0.00000	0.46960	0.00000
0.0	0.05761	0.00939	0.09391	0.46931	−0.00441
0.4	0.11521	0.03755	0.18761	0.46725	−0.01755
0.6	0.17282	0.08439	0.28058	0.46173	−0.03896
0.8	0.23042	0.14967	0.37196	0.45119	−0.06753
1.0	0.28803	0.23299	0.46063	0.43438	−0.10121
1.2	0.34563	0.33366	0.54525	0.41057	−0.13699
1.4	0.40324	0.45072	0.62439	0.37969	−0.17114
1.6	0.46084	0.58296	0.69670	0.34249	−0.19966
1.8	0.51845	0.72887	0.76106	0.30045	−0.21899
2.0	0.57606	0.88680	0.81669	0.25567	−0.22673
2.2	0.63366	1.05495	0.86330	0.21058	−0.22215
2.4	0.69127	1.23153	0.90107	0.16756	−0.20636
2.6	0.74887	1.41482	0.93060	0.12861	−0.18196
2.8	0.80648	1.60328	0.95288	0.09511	−0.15249
3.0	0.86408	1.79557	0.96905	0.06771	−0.12158
3.2	0.92169	1.99058	0.98036	0.04637	−0.09230
3.3	0.95049	2.08883	0.98456	0.03781	−0.07899
3.4	0.97929	2.18747	0.98797	0.03054	−0.06679
3.47†	1.00000	2.25856	0.99000	0.02603	−0.05878
3.5	1.00810	2.28641	0.99071	0.02441	−0.05582
3.6	1.03690	2.38559	0.99289	0.01933	−0.04611
3.8	1.09451	2.58450	0.99594	0.01176	−0.03039
4.0	1.15211	2.78389	0.99777	0.00687	−0.01914
4.2	1.20972	2.98356	0.99882	0.00386	−0.01152
4.4	1.26732	3.18338	0.99940	0.00208	−0.00663
4.6	1.32493	3.38330	0.99970	0.00108	−0.00366
4.8	1.38253	3.58325	0.99986	0.00054	−0.00193
5.0	1.44014	3.78323	0.99994	0.00026	−0.00098
5.2	1.49774	3.98323	0.99997	0.00012	−0.00047
5.4	1.55535	4.18322	0.99999	0.00005	−0.00022
5.6	1.61296	4.38322	1.00000	0.00002	−0.00010
5.8	1.67056	4.58322	1.00000	0.00001	−0.00004
6.0	1.72817	4.78322	1.00000	0.00000	−0.00002

†Actual value to 16 significant digits: $\eta_\delta = 3.471886880405967$.

$$\frac{u}{U} = 0.99 \text{ when } \eta = 3.5 \;\rightarrow\; \frac{\delta}{x} = \frac{5}{\sqrt{Re_x}} \qquad Re_x = \frac{Ux}{\nu}$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \int_0^\infty (1 - f')\, d\eta \sqrt{\frac{2\vartheta x}{U}} \;\rightarrow\; \boxed{\frac{\delta^*}{x} = \frac{1.7208}{\sqrt{Re_x}}}$$

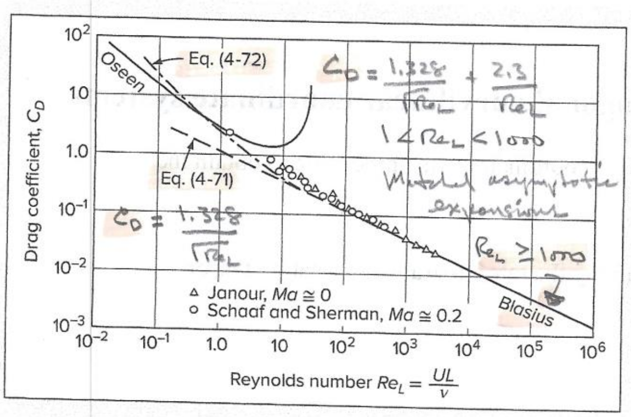
$$\theta = \int_0^\infty \left(1 - \frac{u}{U}\right) \frac{u}{U}\, dy = \int_0^\infty (1 - f')\, f' \sqrt{\frac{2\vartheta x}{U}}\, d\eta \;\rightarrow\; \boxed{\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}}$$

$$\boxed{\frac{\delta^*}{\theta} = H = shape\ parameter\ 2.59}$$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_w = \frac{\mu U f''(0)}{\sqrt{\frac{2\nu x}{U}}} \;\rightarrow\; \boxed{C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}} = \frac{\theta}{x}}$$

$$\boxed{C_D = \frac{D}{\frac{1}{2}\rho U^2 L} = \int_0^L C_f \frac{dx}{L} = \frac{1.328}{\sqrt{Re_L}}} \qquad Re_L = \frac{UL}{\nu}$$

$$\frac{v}{U} = \frac{\eta f' - f}{\sqrt{2 Re_x}} \ll 1 \qquad \text{for} \quad \mathrm{Re}_x >> 1$$



Stokes flow: Oseen Improvement, i.e., linearized convection

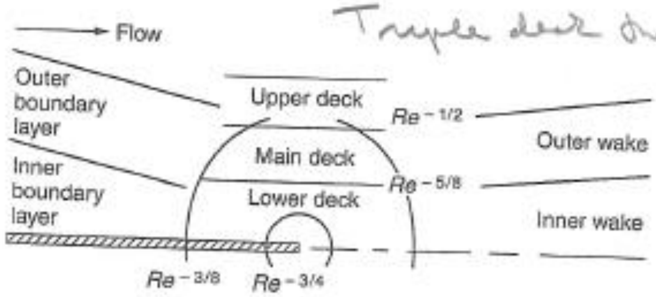
$$C_D = \frac{4\pi}{Re_L[1 - \Gamma + \ln(16/Re_L)]}$$

$\Gamma = .577216 =$ Euler constant

$Re_L << 1$

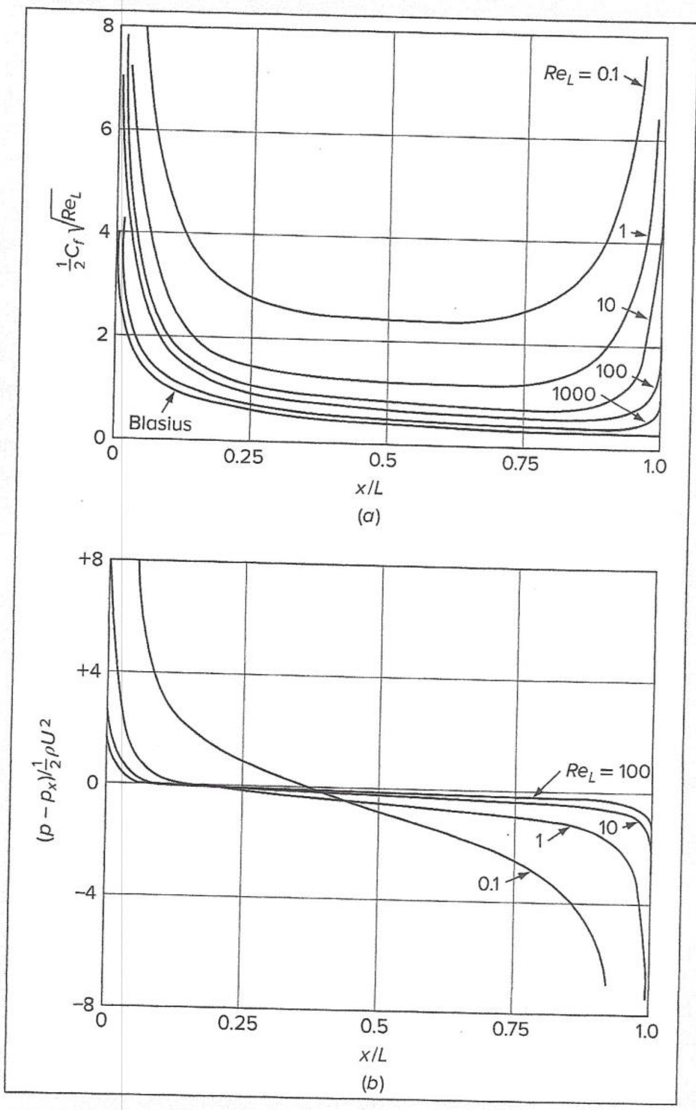
FIGURE 4-11
Theoretical and experimental drag of a flat plate.

Triple deck theory



$$C_D = \frac{1.328}{Re_L^{1/2}} + \frac{2.661}{Re_L^{7/8}} + O(Re_L^{-1})$$

FIGURE 4-40
Sketch of the triple-deck region at the trailing edge of a flat plate, merging into two-layer upstream and downstream regions. [*After Stewartson (1969) and Messiter (1970).*]



NS CFD $.1 < Re_L < 1000$
low Re large LE & TE effects
BL approximation $Re_L \geq 1000$

FIGURE 4-12
Numerical solution of the full Navier–Stokes equations for flat-plate flow at moderate Reynolds numbers: (*a*) local friction coefficient; (*b*) local surface pressure. [*After Dennis and Dunwoody (1966).*]

Falkner-Skan Wedge Flows

$$f''' + C_1 f f'' + C_2\left(1 - f'^2\right) = 0 \qquad f = f(\eta)$$

$$\eta = y/g(x)$$

$$u/U = f'(\eta)$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

$$C_1 = \frac{g}{v}(Ug)_x \qquad C_2 = \frac{g^2}{v}U_x$$ (Blasius Solution: C_2=0, C_1=1)

Similarity form of BL equations

Consider $(Ug^2)_x = 2Ugg_x + g^2U_x$

$$= 2Ugg_x + 2g^2U_x - g^2U_x = 2g(Ug_x + gU_x) - g^2U_x$$

$$= 2g(Ug)_x - g^2U_x$$

$$= 2vC_1 - vC_2$$

Hence $(Ug^2)_x = v(2C_1 - C_2)$

Choose C_1=1 and let C_2 = C

Integrate $Ug^2 = v(2 - C)x$ note $g^2 = vC/U_x$

Combine $\frac{UvC}{U_x} = v(2 - C)x$

Rearrange $\frac{\frac{dU}{dx}}{U} = \frac{C}{2-C}\frac{1}{x}$

Integrate $ln\, U = \frac{C}{2-C} ln\, x + \ln k$ where $\ln k$ = constant

$$ln\, U = ln\, x^{\frac{C}{2-C}} + ln\, k = \ln k x^{\frac{C}{2-C}}$$

$$U(x) = kx^{C/(2-C)}$$

$$g(x) = \left[\frac{\upsilon C}{U_x}\right]^{\frac{1}{2}} \text{ note } U_x = k\,\frac{C}{2-C}\,x^{\left(\frac{C}{2-C}-1\right)} = k\,\frac{C}{2-C}\,x^{\left(\frac{-2(1-C)}{2-C}\right)}$$

$$g(x) = \left[\frac{\upsilon C}{k\frac{C}{2-C}x^{\left(\frac{-2(1-C)}{2-C}\right)}}\right]^{\frac{1}{2}} = \left[\frac{\upsilon(2-C)}{k}\,x^{\left(\frac{-2(1-C)}{2-C}\right)}\right]^{\frac{1}{2}} = \sqrt{\frac{\upsilon(2-C)}{k}}\,x^{\frac{1-C}{2-C}}$$

using $a^{1/2}b^{1/2}=(ab)^{1/2}$ and $(a^m)^n=a^{mn}$

Alternatively, $U_x = \frac{C}{2-C}\,\mathrm{k}x^{(C/(2-C))}x^{-1} = \frac{C}{2-C}\,\mathrm{U}x^{-1}$

Such that $g(x) = \left[\frac{\upsilon C}{\frac{C}{2-C}\,\mathrm{U}x^{-1}}\right]^{\frac{1}{2}} = \left[\frac{\upsilon(2-C)x}{\mathrm{U}}\right]^{\frac{1}{2}}$

Change constant: $C = \beta = \frac{2m}{m+1}$ and $m = \frac{\beta}{2-\beta}$

$$U(x) = kx^m$$

$$\eta = \frac{y}{g} = y\sqrt{\frac{m+1}{2}\frac{U}{\nu x}}$$

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0$$

$$f(0) = f'(0) = 0 \text{ and } f'(\infty) = 1$$

Note:

2m/(m+1) = $\frac{2\beta}{2-\beta}/(\frac{\beta}{2-\beta}+1) = \frac{2\beta}{2-\beta}/(\frac{2}{(2-\beta)}) = \beta$

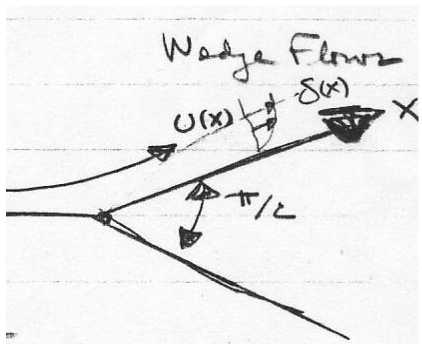
2-C = 2 -2m/(m+1) = 2/(m+1)

$g(x) = \left[\frac{\upsilon(2-C)x}{\mathrm{U}}\right]^{\frac{1}{2}} = \left[\frac{\upsilon 2x}{\mathrm{U(m+1)}}\right]^{\frac{1}{2}}$, i.e., $g(x)^{-1} = \left[\frac{m+1}{2}\frac{U}{\upsilon x}\right]^{\frac{1}{2}}$

Numerical solutions for $-0.19884 \leq \beta \leq 1.0$.

Separation ($\tau_w = 0$)

$$U(x) = kx^m$$



Solutions show many commonly observed characteristics of BL flow:

- The parameter β is a measure of the pressure gradient, dp/dx. For $\beta > 0$, $dp/dx < 0$ and the pressure gradient is favorable. For $\beta < 0$, the $dp/dx > 0$ and the pressure gradient is adverse.

- Negative β solutions drop away from Blasius profiles as separation approached.

- Positive β solutions squeeze closer to wall due to flow acceleration.

- Accelerated flow: $\tau_{\max}$ near wall.

- Decelerated flow: $\tau_{\max}$ moves toward $\delta/2$

Inviscid flow past wedges and corners

$U(x) = Kx^m$ exact solution potential flow past wedge or corner shapes

Plane polar coordinates $\nabla^2 \psi = 0$, $\nabla \times \underline{v} = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\psi_r) + \frac{1}{r^2}\psi_{\theta\theta} = 0 \quad v_r = \frac{1}{r}\psi_\theta \quad v_\theta = -\psi_r$$

$$\psi(r,\theta) = C r^{m+1} \sin[(m+1)\theta] \quad \beta = \frac{2m}{m+1}$$

$$m = \beta/(2-\beta)$$

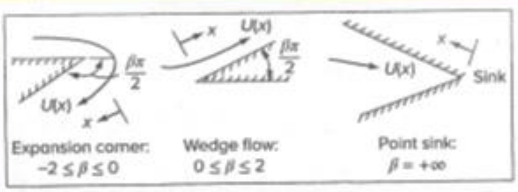
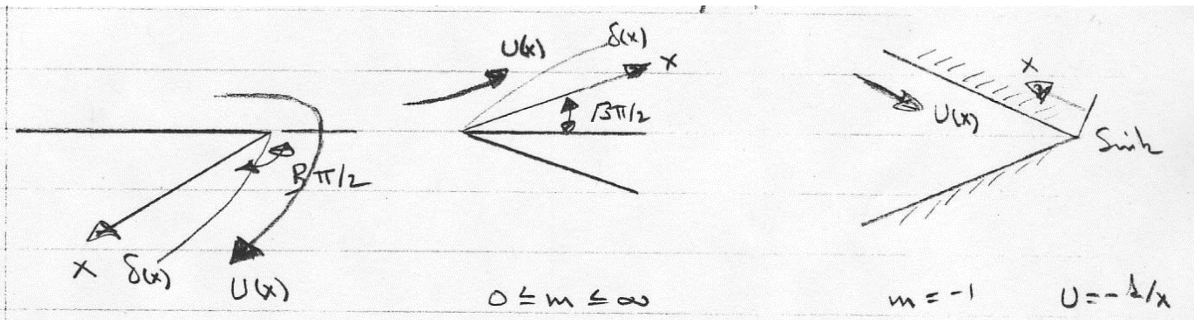


FIGURE 4-15
Some examples of Falkner–Skan potential flows.

This expression yields certain radial streamlines that can be interpreted as the "walls" of a wedge or a corner, as shown in Fig. 4-15, depending on the value of $\beta = 2m/(m+1)$. The velocity along these walls has the form $U = Kx^m$, which represents the freestream driving the boundary layer on the wall, with $x = 0$ at the tip of the wedge. The most prominent cases are:

- $-2 \le \beta \le 0$, $-\frac{1}{2} \le m \le 0$: flow around an expansion corner of turning angle $\beta\pi/2$
- $\beta = 0$, $m = 0$: the flat plate
- $0 \le \beta \le +2$, $0 \le m \le \infty$: flow against a wedge of half-angle $\beta\pi/2$
- $\beta = 1$, $m = 1$: the plane stagnation point (180° wedge)
- $\beta = +4$, $m = -2$: doublet flow near a plane wall
- $\beta = +5$, $m = -\frac{5}{3}$: doublet flow near a 90° corner
- $\beta = +\infty$, $m = -1$: flow toward a point sink [i.e., the boundary-layer version of the Jeffery–Hamel flow in a convergent wedge (Sec. 3-8)].

These are *similar* flows, i.e., for a given β, the velocity profiles all look alike when scaled by $U(x)$ and $\delta(x)$. They may also be used, with modest success, to predict the behavior of nonsimilar flows.



$0 \le m \le \infty$ | $m = -1$ | $U = -A/x$

$-1/2 \le m \le 0$ | $0 \le \beta \le 2$ | $\beta = \infty$ | $\eta = \frac{y}{x}\sqrt{\frac{U}{\nu}}$

$-2 \le \beta \le 0$

Expansion Corner

$\beta = -.19884$ (Separation)

Wedge Flow

$m=0$, $\beta=0$ } Flat Plate

$m=1$, $\beta=1$ } Stagnation Point

Sink: Simple Sol involving tanh²

Sec. 4.37

BL Sol of Jeffery-Hamel flow

Also, $\beta = 4$, $m = -2$: doublet flow near a plane wall

$\beta = 5$, $m = -5/3$: doublet near a 90° corner

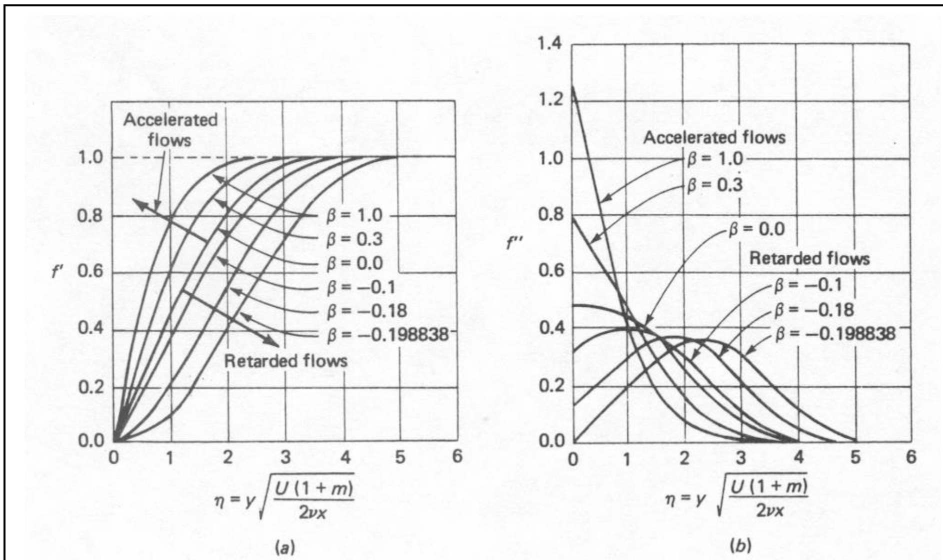


FIGURE 4-11
(*a*) Velocity profiles and (*b*) shear-stress profiles for the Falkner-Skan equation.

$u = U(x) f'$ where $U(x) = k x^m$

& velocity at edge of BL

$\beta = \frac{2m}{m+1}$ $\quad \eta = y/g(x) = y\left[\frac{m+1}{2}\,\frac{U(x)}{\nu x}\right]^{1/2}$

$\delta(x) = \left[\frac{2\nu}{(m+1)k}\right] x^{(1-m)/2}$ $\quad = y/\underbrace{\left[\frac{2}{m+1}\,\frac{\nu x}{k x^m}\right]^{1/2}}_{\delta(x)}$

$\delta^*(x) = \left(\frac{2}{m+1}\right)^{1/2}\left(\frac{\nu x}{U}\right)^{1/2}\int_0^\infty (1-f')\,d\eta$ $\quad \left(\frac{2\nu}{(m+1)k}\right) x^{(1-m)/2}$

$\tau_w(x) = \mu\left(\frac{m+1}{2}\right)^{1/2}\left(\frac{U^3}{\nu x}\right)^{1/2} f''(0)$ $\quad \mu\left(\frac{(m+1)k^3}{2\nu}\right) x^{(3m-1)/2}$

$\delta \propto x^{(1-m)/2}$ $\quad \delta^* \propto x^{(1-m)/2}$ $\quad \tau_w \propto x^{(3m-1)/2}$

Special Cases:

$U(x) = \text{constant}$ 1. Blasius: $m = 0$ $\quad \delta \propto x^{1/2}$ $\quad \delta^* \propto x^{1/2}$ $\quad \tau_w \propto x^{-1/2}$

$U(x) = kx$ 2. Stagnation Point: $m = 1$ $\quad \delta, \delta^* = \text{constant}$ $\quad \tau \propto x$

linearly increasing U thinning of BL exactly balanced by viscous diffusion such that $\delta, \delta^* = \text{constant}$

$U = kx^3$ 3. $m = 3$: $\delta \propto x^{-1}$ $\quad \delta^* \propto x^{-1}$ $\quad \tau_w = \text{constant}$

acceleration makes flow tendency increase τ_w exactly balanced by viscous diffusion spread momentum away from surface to reduce τ_w

Alternative derivation Falkner-Skan equation #1

$u_x + v_y = 0$ ie $v_y = -u_x$

Leibniz rule

$$\int_{a(x)}^{b(x)} \left[\frac{\partial}{\partial x} u(x,y)\right] dy = \frac{d}{dx}\int_a^b u\,dy + u(x,a)\frac{da}{dx} - u(x,b)\frac{db}{dx}$$

$$\int_0^v dv = -\int_0^y \frac{\partial u}{\partial x}\,dy = -\frac{\partial}{\partial x}\int_0^y u\,dy - u(x,y)\frac{dy}{dx} + u(\cancel{x},0)\frac{d0}{dx}$$

$$v = -\frac{\partial}{\partial x}\int_0^y u\,dy$$

$= 0$ since integration for $x =$ constant

x BL equation: $uu_x + vu_y = UU_x + \nu u_{yy}$

$$uu_x - \left(\frac{\partial}{\partial x}\int_0^y u\,dy\right)u_y = RHS$$

integro differential equation u only

Combine x and y single variable $\eta(x,y)$ to transform pde to ode in $f(\eta)$ only to achieve similarity.

assume: $u(x,y) = U(x)f'(\eta)$ where η dimensionless and $U(x)$ not arbitrary but part of solution

$u_x = U_x f' + U f''\eta_x$

$u_y = U f''\eta_y = U f'' h$

$u_{yy} = U f'''\eta_y^2 + U f''\eta_{yy}$ let $\eta = y\,h(x)$

$\eta_x = y h_x = \frac{\eta h_x}{h(x)}$

$\eta_y = h(x)$ $\eta_{yy} = 0$

$$Uf'\left(U_x f' + U f''\frac{\eta h_x}{h}\right) - \left(\frac{\partial}{\partial x}\int_0^y U f'\,dy\right)U f'' h = UU_x + \nu U f''' h^2$$

$$\int_0^y f'\,dy = h^{-1}\int_0^{\eta/h}\frac{df}{d\eta}\,d\eta = f/h \quad \frac{\partial}{\partial x}(Uf/h) = U_x f/h - \frac{Uf h_x}{h^2} + \frac{U}{h}f'\frac{\eta h_x}{h}$$

Since $U(x)$ & $h(x) \neq f(y)$ or $f(\eta)$ for fixed x

$$v f'\left(v_x f' + v f'' \frac{\eta h_x}{h}\right) + \left(-v_x f/h + \frac{v f h_x}{h^2} - \frac{v}{h^2} f' \eta h_x\right) v f'' h$$
$$= v v_x + \nu v f''' h^2$$

$$v v_x f'^2 + \cancel{\frac{v^2 \eta h_x}{h}} f' f'' - v v_x f f'' + \frac{v^2 h_x}{h} f f'' - \cancel{\frac{v^2 \eta h_x}{h}} f' f''$$

$$= v v_x + \nu v h^2 f'''$$

$$v v_x \left(f'^2 - f f'' - 1\right) + \frac{v^2 h_x}{h} f f'' = \nu v h^2 f'''$$

$$f''' = \underbrace{\frac{v_x}{\nu h^2}}_{\neq f(x)} \left(f'^2 - f f'' - 1\right) + \underbrace{\frac{v h_x}{\nu h^3}}_{\neq f(x)} f f''$$

for similarity only $f(\eta)$

assume $\eta = c_y x^a$ $\quad v = K x^m$ $\quad m = 2a + 1$

$h = c x^a$ $\quad v_x = K m x^{m-1}$ $\quad a = \frac{m-1}{2}$ $\quad a - 1 = \frac{m-1}{2} - 1 = \frac{m-3}{2}$

$h_x = a c x^{a-1}$

$$h = \left[K \frac{m+1}{2\nu}\right]^{1/2} x^{\frac{m-1}{2}} = \left[K \frac{m+1}{2\nu} x^m x^{-1}\right]^{1/2} = \left[\frac{m+1}{2} \frac{v}{x\nu}\right]^{1/2} = g(x)^{-1}$$

$$\frac{K m x^{m-1}}{\nu c^2 x^{m-1}} = \frac{K m}{\nu c^2}$$

$$\frac{K x^m \left(\frac{m-1}{2}\right) c x^{\frac{m-3}{2}}}{\nu c^3 x^{\frac{3(m-1)}{2}}} = \frac{K \left(\frac{m-1}{2}\right) c x^{\frac{3(m-1)}{2}}}{\nu c^3 x^{\frac{3(m-1)}{2}}} = \frac{K \frac{m-1}{2}}{\nu c^2}$$

$2m + m - 3$

$3m - 3 = 3(m-1)$

$$f''' + \frac{v_x}{\nu h^2}\left(1 - f'^2 + f f''\right) - \frac{v h_x}{\nu h^3} f f'' = 0$$

$K\left(m - \frac{m-1}{2}\right) / \nu c^2$

$K\left(\frac{m+1}{2}\right) / \nu c^2$

$$f''' + \frac{v_x}{\nu h^2}\left(1 - f'^2\right) + \underbrace{\left(\frac{v_x}{\nu h^2} - \frac{v h_x}{\nu h^3}\right)}_{1} f f'' = 0$$

if $= 1$ $\quad c^2 = K(m+1)/2\nu$

$$\beta = \frac{K m}{\nu c^2} = \frac{K m}{\nu \frac{K(m+1)}{2\nu}} = 2m/m+1$$

$$\therefore \quad f''' + \beta\left(1 - f'^2\right) + f f'' = 0$$

Alternative derivation $\underbrace{\text{Falkner-Skan}}$ equation #2

What types of external flows are similarity solutions BL eq.?

Assume self similar velocity profile: $u(x,y) = u_e(x) f'(\eta)$

$\eta = y/\delta(x)$

u_e, δ determined as part of solution.

$d\psi = \psi_x dx + \psi_y dy$

for x = constant $\quad \psi(x,y) - \psi(x,0) = \int_0^y \psi_y dy$

$\psi = 0$ at $y = 0 \Rightarrow f(0) = 0 \qquad = \delta u_e \int_0^\eta \frac{u}{u_e} d\left(\frac{y}{\delta}\right)$

and

$\psi/\delta u_e = f(\eta) \qquad = \delta u_e \int_0^\eta f' d\eta = \delta u_e f(\eta)$

ie $\quad \psi = \delta u_e f(\eta)$

$\eta = y\delta^{-1}$

$\eta_x = -y\delta^{-2}\delta_x$

$= -\eta\delta^{-1}\delta_x$

$\underbrace{u = \psi_y \quad v = -\psi_x}_{\text{continuity automatically satisfied}} = -f(\delta u_e)_x - (\delta u_e) f' \eta_x$

$= -f(\delta_x u_e + \delta u_{e_x}) + (\delta u_e) f' \eta \delta^{-1} \delta_x$

$= \underbrace{-f(\delta_x u_e + \delta u_{e_x}) + \eta u_e f' \delta_x}$

$u_y = u_e f'' \delta^{-1}$

$u_{yy} = u_e f''' \delta^{-2}$

$u u_x + v u_y = u_e u_{e_x} + \nu u_{yy}$

$u_e f'(u_{e_x} f' - u_e f'' \eta \delta^{-1} \delta_x) + [\quad] u_e f'' \delta^{-1} = u_e u_{e_x} + \nu u_e f''' \delta^{-2}$

$u_e u_{e_x} f'^2 - \cancel{u_e^2 \eta \frac{\delta_x}{\delta} f'' f'} - (\delta_x u_e^2 + \delta u_e u_{e_x}) \frac{f f''}{\delta} + \cancel{u_e^2 \eta \frac{\delta_x}{\delta} f' f''} = RHS$

$u_e u_{e_x} f'^2 - (\delta_x u_e^2 + \delta u_e u_{e_x}) \frac{f f''}{\delta} - u_e u_{e_x} = \nu \frac{u_e}{\delta^2} f'''$

$f''' + \frac{u_e u_{e_x} \delta^2}{\nu u_e} (1 - f'^2) + \frac{\delta}{\nu u_e} (\delta_x u_e^2 + \delta u_e u_{e_x}) f f'' = 0$

$$f''' + \frac{u_{ex}\delta^2}{\nu}(1-f'^2) + \frac{\delta}{\nu}\underbrace{(\delta_x u_e + \delta u_{e_x})}_{\frac{d}{dx}(\delta u_e)} ff'' = 0$$

$$f''' + \alpha f f'' + \beta(1-f'^2) = 0 \qquad \beta = \frac{u_{e_x}\delta^2}{\nu}$$

ODE for f assuming $\alpha, \beta \neq f(x)$ $\quad \alpha = \frac{\delta}{\nu}(\delta u_e)_x$ $= \text{constant}$

note: $2\alpha - \beta = \frac{1}{\nu}\frac{\partial}{\partial x}(\delta^2 u_e)$

$$(2\alpha-\beta)(x-x_0) = \frac{1}{\nu}\delta^2 u_e$$

Assume $x_0 = 0$ & $\delta(0) = 0$: $\delta = \left[\frac{\nu(2\alpha-\beta)x}{u_e}\right]^{1/2}$

usually $\underbrace{u_e, x > 0}_{2\alpha-\beta=1}$; however, may have $\underbrace{\text{opposite signs}}_{2\alpha-\beta=-1}$

$2\alpha - \beta$ any value acceptable since y/δ similarity variable $y/c\delta$ also OK

$$\delta = \left[\pm\frac{\nu x}{u_e}\right]^{1/2}$$

Original definition, whereas c = constant
often $\alpha = 1$ used such that $\delta = \left[\frac{\nu(2-\beta)x}{u_e}\right]^{1/2}$

$u_e(x)$ from $\beta = \frac{\pm\frac{\nu x}{u_e}}{\nu} u_{e_x} = \pm\frac{x}{u_e}u_{e_x} = \mp\frac{x}{\rho u_e^2}p_x$

$$\beta = \pm\frac{x}{u_e}\frac{du_e}{dx}$$

$$\beta\frac{dx}{x} = \pm\frac{du_e}{u_e}$$

$$\ln L^{-1} + \beta \ln x = \pm \ln u_e + \ln u_0^{-1}$$

$$\ln\left(\frac{x}{L}\right)^\beta = \pm \ln u_e/u_0 \qquad L, u_0 \text{ same sign } u_e, x$$

Self similar BL u_e simple power law

$$u_e = u_0\left(\frac{x}{L}\right)^m \qquad m = \beta \quad u_e, x \text{ same sign}$$
$$= -\beta \quad u_e, x \text{ opposite sign}$$

$$\eta = y/\delta = y/[\pm \nu x/u_e]^{1/2} = y/L \,/\, \sqrt{Re}\left(\frac{x}{L}\right)^{\frac{1-m}{2}} \qquad Re = \frac{u_e L}{\nu}$$

For u_e, x same sign $m = \beta$ $\quad 2\alpha - m = 1$

$\alpha = m+1/2$

$$f''' + \frac{m+1}{2} f f'' + m(1 - f'^2) = 0$$

where for BL flow $u = 0, v = 0 \quad y = 0$

$u = u_e \quad y \to \infty$

$u(x,y) = u_e(x) f'(\eta) \qquad \eta = y/\delta$

$v(x,y) = -f(u_e\delta)_x + \eta u_e f' \delta_x$

so $f(0) = 0 \quad f'(0) = 0 \quad$ and $f'(\infty) = 1$

IC not needed; however, $x_1 = 0$ η singular

FS equation nonlinear ODE so a priori not known which m yield solutions and whether or not unique.

$m \geq 0$ unique solutions for:

$m = 0$:	Blasius flow over a flat plate with a sharp leading edge; also the local flow at any cusp leading edge
$0 < m < 1$:	Flow over a wedge with half-angle $\theta_{1/2} = m/(m+1)$ with $0 < \theta_{1/2} < \pi/2$
$m = 1$:	Hiemenz flow toward a plane stagnation point: Section 11.9
$1 < m < 2$:	Flow into a corner with $\theta_{1/2} > \pi/2$; a flow of this type may be difficult to produce experimentally
$m > 2$:	No corresponding simple ideal flow

Many people have contributed to the classification and computation of Falkner–Skan flows. Rosenhead (1963) contains a good summary by C. W. Jones and E. J. Watson. From these works some of the complicated behavior at negative values of m can be pieced together. When $-0.0904 < m < 0.0$, there are an infinite number of solutions for each value of m (Fig. 20.4). However, not all of these solutions are physically acceptable. One of the main arguments in establishing boundary layer theory is that the viscous effects are confined to a thin region near the wall. In light of this fact, people have proposed that a boundary layer should approach the free stream exponentially:

$$1 - u^* = 1 - f' \sim Ae^{-B\eta} \qquad \text{as} \quad \eta \to \infty$$

If this condition is applied, there are only two known acceptable solutions for each m in the range $-0.0904 < m < 0$. One of the solutions has $u > 0$ for all η, while the other has the interesting characteristic that there is backflow for a small region near the wall [Stewartson's (1954) reverse-flow profiles]. When m is exactly equal to -0.0904, only one solution exists. This profile has zero shear stress at the wall and therefore is on the verge of separating for all x.

For $-1 < m < -0.0904$, all solutions for a given m tend to oscillate about $f' = 1$ as η becomes infinite. At each value of m, one of these solutions has just one region where the velocity $f' > 1$, and then $f \to 1$ exponentially. Because a laminar boundary layer with these super velocities may be difficult to produce experimentally, some workers reject these solutions as physically impossible.

The case $m = -1$ with u_e and x having opposite signs, $u = -a_0/x(u_0 = -a_0)$, represents a solid wall in the flow field of an ideal line sink. When two walls are present, the problem represents the flow into a wedge. The differential equation in this case has an exact closed-form solution (Problem 20.9). On the other hand, the equivalent problem with the sign changed so the flow comes from a source ($m = -1$ and $u = +u_0/x$) has no solution. This means that boundary layer theory does not produce a similarity solution for flows in a flat-wall diffuser. These flows require a nonsimilar solution.

Most of the complicated behavior in Falkner–Skan solutions happens when m is between -1 and 0. When $m < -1$ we again find a unique solution. All solutions in the range $m < -1$ have the flow going from large x toward $x = 0$. Hence, the flows are strongly accelerated with $u = -a_0(x/L)^m, m < -1$.

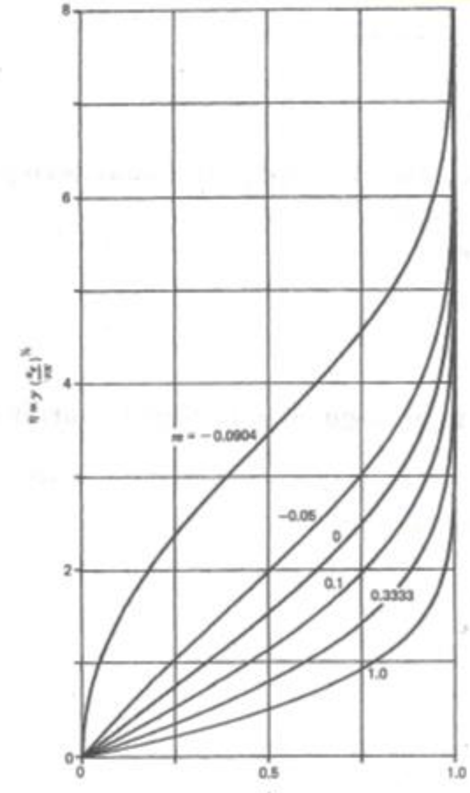


Figure 20.4 Falkner–Skan profiles. The parameter m indicates the external velocity variation through $u_e = u_0 x^m$.

Vertical Velocity

From outside BL, as $Re \uparrow \infty$ $v/u_0 \to 0$ since $\delta \to 0$

From inside BL, $v u_y$ some order remaining term.

X-momentum equation since

$v^* = \frac{v}{u_0}\left(\frac{L}{\delta}\right) = \frac{v}{u_0}\sqrt{Re}$ $\quad$ $u_y \to \infty$ as $v \to 0$ as $Re \to \infty$

$\uparrow$ $\frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}$ $\quad$ ie $0 \times \infty$ finite.

$u u_x \sim \frac{U^2}{L} \;\alpha\; \nu u_{yy} \sim \nu \frac{U}{\delta^2}$

$\delta^2 \sim \frac{\nu L}{U}$ ie $\frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}$

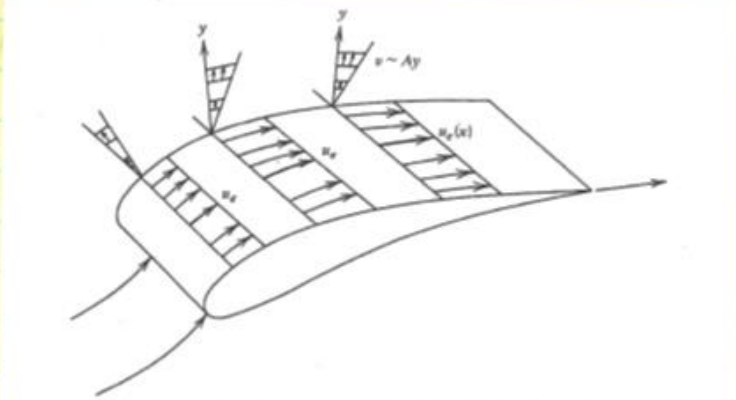


Figure 20.7 View from outside the boundary layer looking at the behavior of u and v in the inviscid flow at the top of the boundary layer.

(1) Inviscid flow near wall continuity equation

$$\frac{du_e}{dx} + \frac{\partial v}{\partial y} = 0 \Rightarrow v\Big|_{\text{inviscid}} = \int_0^y \frac{\partial v}{\partial y}\,dy = -\frac{du_e}{dx}\int_0^y dy$$

Inviscid flow near wall

1. $u_e = u_e(x)$ — assuming $\frac{du_e}{dx}$ constant $\sim -\frac{du_e}{dx}y$ as $y \to 0$
2. $\frac{\partial v}{\partial n} = 0$ at $\delta = 0$
3. $v \sim Ay$ ie grows linearly

(2) Viscous BL flow near wall continuity equation

$$v = \int_0^y \frac{\partial v}{\partial y}\,dy = -\int_0^y \frac{du}{dx}\,dy = \frac{d}{dx}\int_0^y (u_e - u)\,dy - \frac{du_e}{dx}y$$

$$v(y \to \infty) = \underbrace{-\frac{du_e}{dx}y}_{\text{inviscid } v} + \underbrace{\frac{d}{dx}(u_e\delta^*)}_{\text{influence BL profile on } v \text{ ie } v_{BL} \text{ correction inviscid flow}} \quad \text{where } \delta^* = \int_0^\infty \left(1 - \frac{u}{u_e}\right) dy$$

2nd order solution
inviscid flow instead
of $v(\text{wall}) = 0$ use
$v(\text{wall}) = \frac{d}{dx}(u_e\delta^*)$

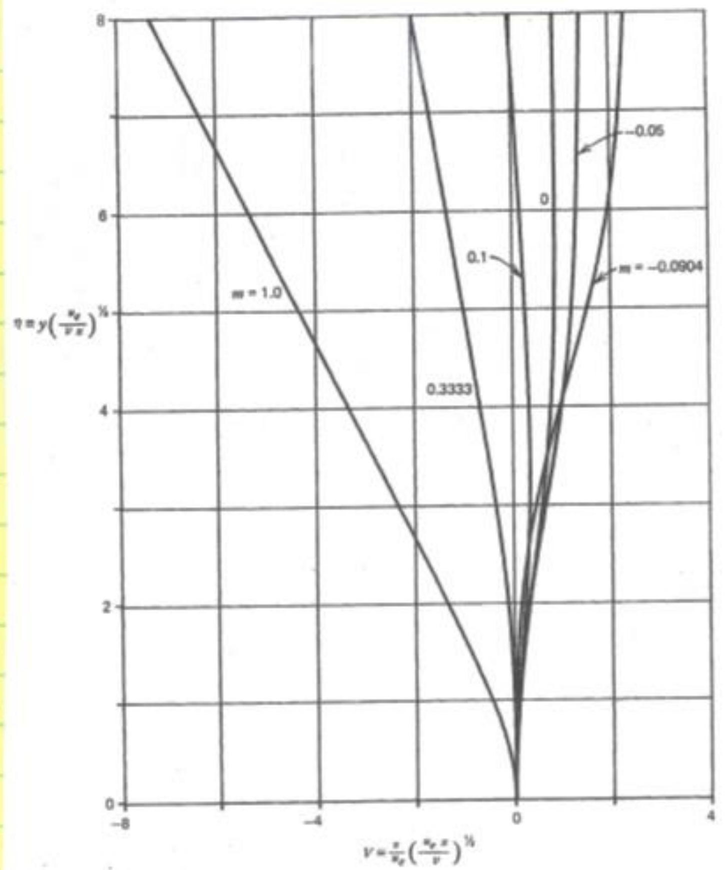


Figure 20.8 Vertical-velocity profiles for Falkner–Skan boundary layers.

$v(\eta)$ for Falkner-Skan BL solutions various m

$v = -\psi_x = -f\frac{d}{dx}(u_e\delta) + \eta u_e f' \frac{d\delta}{dx}$ For $\eta \to \infty$ approaches nondimensional form previous equation

$V(\infty) = \frac{v}{u_e}\left(\frac{u_e x}{\nu}\right)^{1/2} = -\beta\eta + \frac{x}{u_e\delta}\frac{d}{dx}(u_e\delta^*)$

$\beta \neq 0$ $\underbrace{\qquad}_{v \text{ intercept}}$ $\delta = \left[2\frac{\nu x}{u_e}\right]^{1/2}$

$V(\infty) \to \infty$

BL satisfy no slip & match inviscid flow as $y \to \infty$

$u \to u_e$

$v \to$ curve with slope $-\frac{du_e}{dx}$ same inviscid flow v near wall

Flat Plate with Wall Suction or Blowing

Blasius solution with $v_w \ll U \neq 0$ & $u_w = 0$

Similarity requires

$$v_w = -\psi_x = \sqrt{\frac{\nu U}{2x}}\,(\eta f' - f)\Big|_{\eta=0} \qquad \eta = y\sqrt{\frac{U}{2\nu x}}$$

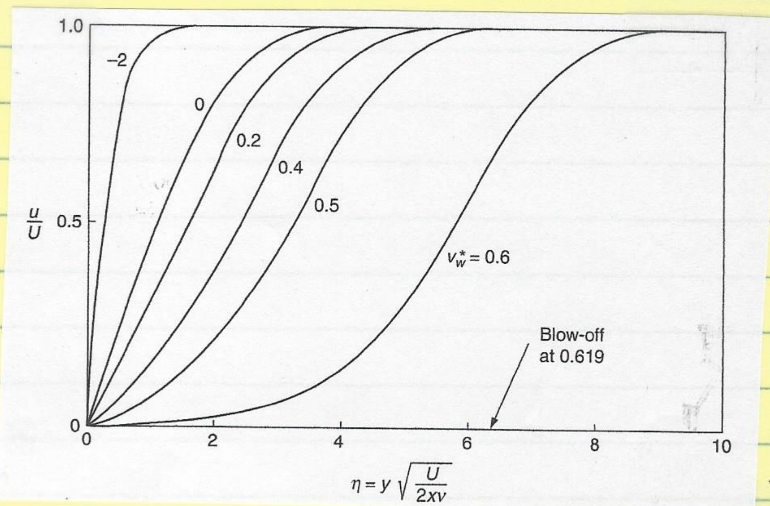
$$= \sqrt{\frac{\nu U}{2x}}\,\big(-f(0)\big) \qquad \therefore f(0) \neq 0$$

with $v_w \propto x^{-1/2}$

Solve Blasius equation with: $f'(0)=0$, $f'(\infty)=1$

& $f(0) \neq 0$

Suction-blowing parameter $\quad v_w^* = \frac{v_w}{U}\sqrt{Re_x} = -f(0)/\sqrt{2}$



$v_w < 0$ suction

thins BL, increases τ_w

similar favorable p_x

& stable transition

$v_w > 0$ blowing

thickens BL, reduces τ_w

similar adverse p_x

S-shaped with inflection point

ie unstable to turbulent transition

$v_w^* = .619 \Rightarrow \frac{\partial u}{\partial y} = 0$ at $y = 0$ BL blown off ie $\tau_w = 0$

$u = 0 \quad y \geq 0$ & BL approximations

not valid!