Chapter 4 Laminar Boundary Layers

(2) Boundary Layer Theory

Part 1: Integral Methods: Flat Plate; and Boundary Layer Equations

Introduction:

Boundary layer flows: External flows around streamlined bodies at high Re such that viscous (no-slip and shear stress) effects confined close to the body surfaces and its wake but are nearly inviscid far from the body.

Applications of BL theory: *aerodynamics* (airplanes, rockets, projectiles), *hydrodynamics* (ships, submarines, torpedoes), *transportation* (automobiles, trucks, cycles), *wind engineering* (buildings, bridges, water towers), and *ocean engineering* (buoys, breakwaters, cables).

Historical perspective:

1. BL equations: 2D and axisymmetric similarity solutions

2. Momentum integral methods: success 2D and failure 3D due crossflow modeling

3. 3D BL differential codes

4. Separation: viscous/inviscid interaction and thick BL and partially parabolic equations

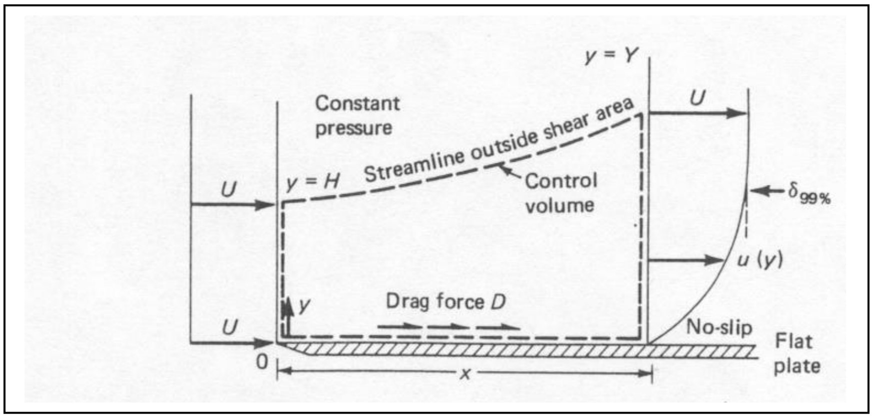
5. CFD: RANS, URANS, LES, Hybrid-RANS/LES, DNS

6. Multi fidelity, ML&AI

7. Fluid-structure interaction, multi-disciplinary

Flat-Plate Momentum Integral Analysis & Laminar approximate solution

Consider flow of a viscous fluid at high Re past a flat plate, i.e., flat plate fixed in a uniform stream of velocity $U\hat{i}$: 2D steady constant property flow, fixed CV, inlet U = constant, outlet u = u(y), no slip y = 0, no shear stress along outer streamline, i.e., at y = H at inlet and y = δ at outer boundary, thickness t = 0 such that p = constant.



Boundary-layer thickness arbitrarily defined by y = $\delta_{99\%}$ (where, $\delta_{99\%}$ is the value of y at u = 0.99U). Streamlines outside $\delta_{99\%}$ will deflect an amount δ^* (**the displacement thickness**). Thus, the streamlines move outward from $y = H$ at $x = 0$ to $y = Y = \delta = H + \delta^*$ at $x = x_1$.

Conservation of mass:

$$\int_{CS} \rho \underline{V} \bullet \underline{n} dA = \mathbf{0} = -\int_0^H \rho U b dy + \int_0^{H+\delta^*} \rho u b dy \qquad \text{b = span width}$$

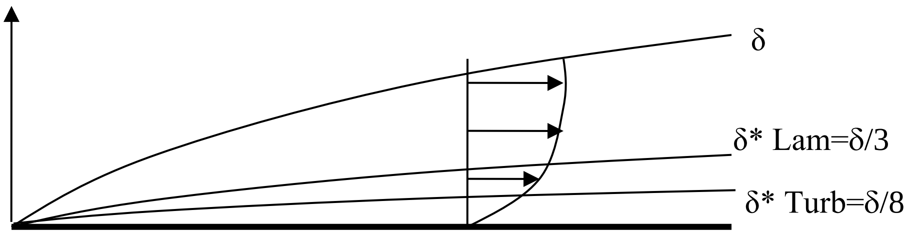
Which simplifies to:

$$UH = \int_0^Y u dy = \int_0^Y (U + u - U) dy = UY + \int_0^Y (u - U) dy$$

Substituting $Y = H + \delta^*$ results in the definition of displacement thickness:

$$\delta^* = \int_0^Y \left(1 - \frac{u}{U}\right) dy$$

δ^* which is only a function of x being an important measure of effect of BL on external flow. To see this more clearly, consider an alternate derivation based on an equivalent discharge/flow rate argument:



$$\underbrace{\int_{\delta^*}^{\delta} U dy}_{\text{Inviscid flow about } \delta^* \text{ body}} = \int_0^{\delta} u dy \qquad \text{Per unit span}$$

Flowrate between δ^* and δ of inviscid flow=actual flowrate, i.e., inviscid flow rate about displacement body = equivalent viscous flow rate about actual body

$$\int_0^{\delta} U dy - \int_0^{\delta^*} U dy = \int_0^{\delta} u dy \Rightarrow \delta^* = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$

w/o BL - displacement effect=actual discharge

For 3D flow, in addition it must also be explicitly required that δ^* is a stream surface of the inviscid flow continued from outside of the BL.

<u>Conservation of x-momentum</u>:

$$\sum F_x = -D = \int_{CS} \rho u \underline{V} \bullet \underline{n} dA = -\int_0^H \rho U(Ubdy) + \int_0^Y \rho u(ubdy)$$

$Drag = D = \rho U^2 Hb - \int_0^Y \rho u^2\, bdy$ = Fluid force on plate = - Plate force on CV (fluid)

Using continuity: $H = \int_0^Y \frac{u}{U} dy$

$$D(x) = \rho b U^2 \int_0^Y u/U dy - \int_0^Y u^2\, b dy = b \int_0^x \tau_w\, dx$$

$$\frac{D}{\rho b U^2} = \theta = \int_0^{Y=\delta} \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

θ is the **momentum thickness (function of x only)**, an important measure of the drag.

$$\frac{dD}{dx} = b\tau_w = \rho b U^2 \frac{d\theta}{dx}$$

$$\tau_w = \rho U^2 \frac{d\theta}{dx}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

$$\frac{C_f}{2} = \frac{d\theta}{dx}$$

Special case 2D momentum integral equation for dp/dx = 0

$$C_D = \frac{1}{L}\int_0^L C_f(x)dx = \frac{1}{L}\int_0^L 2\frac{d\theta}{dx}dx = \frac{2}{L}\theta(L)$$

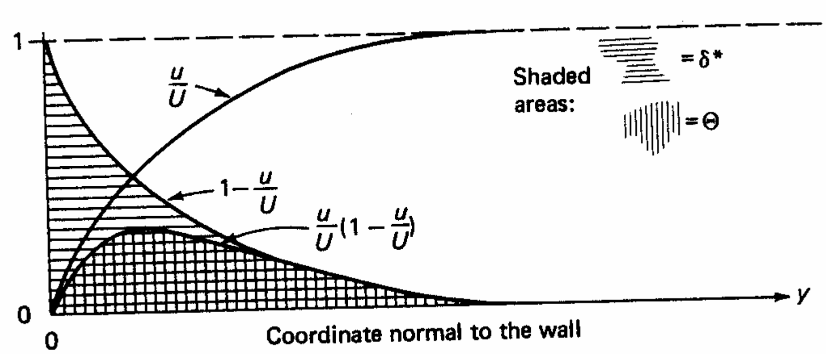


FIGURE 4-2
Momentum and displacement thicknesses.

Simple example solution momentum integral method flat plate BL with assumed velocity profile

Assume polynomial profile in y with degree determined number boundary conditions which can be satisfied

$$\frac{u}{U} = a_0 + a_1 y/\delta + a_2 y^2/\delta^2 \qquad u(x,0) = 0 \Rightarrow a_0 = 0 \quad (1)$$

$u_y = \left(\frac{2}{\delta} - \frac{2y}{\delta^2}\right)U$

$$= 2y/\delta - (y/\delta)^2 \qquad u(x,\delta) = U \Rightarrow 1 = a_0 + a_1 + a_2 \quad (2)$$

$u_{yy} = (-2/\delta^2)U$

$$\frac{\partial u}{\partial y}(x,\delta) = 0 \Rightarrow 0 = a_1 + 2a_2 \quad (3)$$

a1=2, a2=-1

$$\tau_w = \rho U^2 \frac{d\theta}{dx}$$

$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy = \int_0^\delta \left[\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right]\left[1 - \frac{2y}{\delta} + \left(\frac{y}{\delta}\right)^2\right]dy$$

$$= \delta \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta \qquad \eta = y/\delta, \quad d\eta = \delta^{-1} dy$$

$$= 2/15\, \delta$$

$$\tau_w = \mu \frac{du}{dy}\Big|_{y=0} = \mu U \frac{\partial}{\partial y}\left[\frac{2y}{\delta} - (y/\delta)^2\right]_{y=0} = \frac{\mu U}{\delta}\frac{\partial}{\partial \eta}(2\eta - \eta^2)_{\eta=0}$$

$$= 2\mu U/\delta$$

$$= \rho U^2 \frac{d}{dx}\left(\frac{2}{15}\delta\right)$$

$$\frac{2\rho U^2}{15}\frac{d\delta}{dx} = \frac{2\mu U}{\delta} \Rightarrow \delta\, d\delta = 15\frac{\nu}{U}dx \Rightarrow \delta = \sqrt{30}\sqrt{\frac{\nu x}{U}}$$

$$\delta/x = 5.48\, Re_x^{-1/2}$$

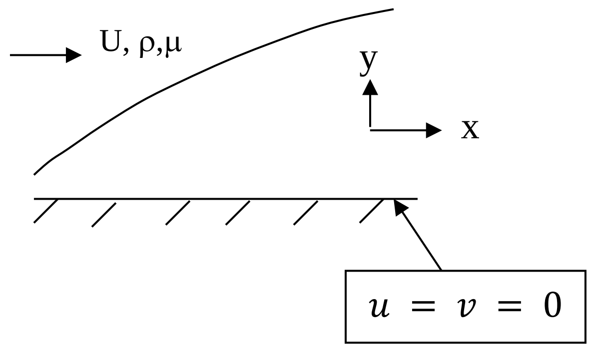
$$\tau_w / \tfrac{1}{2}\rho U^2 = c_f = .73\, Re_x^{-1/2} = \theta/x \qquad Re_x = Ux/\nu$$

$$\delta^*/x = 1.83\, Re_x^{-1/2} \qquad C_D = 1.46\, Re_L^{-1/2} = 2c_f(L)$$

10% error Blasius all $\propto Re_x^{-1/2}$

If $u/U = \frac{2y}{\delta}$ = linear only (1) & (2) can be satisfied & poorer result, whereas if 3rd or higher polynomial more accurate results. 2nd gives $u_{yy}(x,\delta) = -2U/\delta^2 \neq 0$ ie incorrect curvature

Boundary layer approximations, equations, and comments



2D NS, ρ=constant, neglect g (subscript indicates derivative)

$$u_x + v_y = 0$$
$$u_t + uu_x + vu_y = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(u_{xx} + u_{yy})$$
$$v_t + uv_x + vv_y = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(v_{xx} + v_{yy})$$

Introduce non-dimensional variables that includes scales such that all variables are of order magnitude O(1):

$$x^* = x / L$$
$$y^* = \frac{y}{L}\sqrt{\mathrm{Re}}$$
$$t^* = tU / L$$
$$u^* = u / U$$
$$v^* = \frac{v}{U}\sqrt{Re}$$
$$p^* = \frac{p - p_0}{\rho U^2}$$
$$\mathrm{Re} = UL / \nu$$

The NS equations become (drop *)

$$u_x + v_y = 0$$

$$u_t + uu_x + vu_y = -p_x + \underline{\frac{1}{Re} u_{xx}} + u_{yy}$$

$$\frac{1}{Re}\left(v_t + uv_x + vv_y\right) = -p_y + \underline{\frac{1}{Re^2} v_{xx}} + \underline{\frac{1}{Re} v_{yy}}$$

For large Re (BL assumptions) the underlined terms drop out and the BL equations are obtained.

Therefore, y-momentum equation reduces to

$$p_y = 0$$

$$i.e., p = p(x, t)$$

$$\Rightarrow p_x = -\rho(U_t + UU_x)$$

External flow:
unsteady Euler equation or
steady Bernoulli equation
p+$\frac{1}{2}\rho U^2 = B$
p_x =- ρUU_x

2D BL equations:

$$u_x + v_y = 0$$

$$u_t + uu_x + vu_y = (U_t + UU_x) + \nu u_{yy}$$

Note at y=0: $\frac{\partial p}{\partial x} = \nu u_{yy}$

$x^* = x/L \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} = L^{-1}\frac{\partial}{\partial x^+}$

$y^+ = y/L\,\sqrt{Re} \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}\frac{\partial y'}{\partial y} = L^{-1}\sqrt{Re}\,\frac{\partial}{\partial y^+}$

$t' = t\,U/L \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t} = L^{-1}U\frac{\partial}{\partial t^+}$

$u' = u/U \qquad u = u^+U$

$v^+ = v/U\,\sqrt{Re} \qquad v = v^+U/\sqrt{Re}$

$p^+ = p - p_0/\rho U^2 \qquad \Delta p = \rho U^2 p^+$

$Re = UL/\nu$

$u_x + v_y = 0 \qquad \frac{U}{L}\frac{\partial u^+}{\partial x^+} + \frac{\sqrt{Re}}{L}\frac{U}{\sqrt{Re}}\frac{\partial v'}{\partial y^+} = 0$

$UL^{-1}\left(u^+_{x^+} + v^+_{y^+}\right) = 0$

$u_t + uu_x + vu_y = -p_x/\rho + \nu\left(u_{xx} + u_{yy}\right)$

$\frac{U^2}{L}\frac{\partial u^+}{\partial t^+} + u^+U\frac{U}{L}\frac{\partial u^+}{\partial x^+} + \frac{v^+U}{\sqrt{Re}}\frac{\sqrt{Re}}{L}\frac{\partial}{\partial y^+}u^+U = L^{-1}\frac{\partial}{\partial x^+}\left(U^2p^+\right)$

$U^2/L\left[\frac{\partial u^+}{\partial t^+} + u^+\frac{\partial u^+}{\partial x^+} + v^+\frac{\partial u^+}{\partial y^+}\right] = \qquad + \nu\left(L^{-2}\frac{\partial^2(u^+U)}{\partial x^{+2}} + L^{-2}Re\frac{\partial^2(u^+U)}{\partial y^{+2}}\right)$

$\frac{\rho U^2}{\rho L}\frac{\partial p^+}{\partial x^+} + \nu\left(\frac{U}{L^2}\frac{\partial^2 u^+}{\partial x^{+2}} + \frac{U}{L^2}Re\frac{\partial^2 u^+}{\partial y^{+2}}\right)$

$\frac{\partial u^+}{\partial t^+} + u^+\frac{\partial u^+}{\partial x^+} + v^+\frac{\partial u^+}{\partial y^+} = \frac{\partial p^+}{\partial x^+} + Re^{-1}\cancel{\frac{\partial^2 u^+}{\partial x^{+2}}} + \frac{\partial^2 u^+}{\partial y^{+2}} \qquad \frac{U\nu}{L^2}\left(\frac{\partial^2 u^+}{\partial x^{+2}} + \frac{UL}{\nu}\frac{\partial^2 u^+}{\partial y^{+2}}\right)$

$v_t + uv_x + vv_y = -p_y/\rho + \nu\left(v_{xx} + v_{yy}\right) \qquad \frac{\nu}{UL}\left(\frac{\partial^2 u^+}{\partial x'} + \frac{UL}{\nu}\frac{\partial^2 u^+}{\partial y^{+2}}\right)$

$L^{-1}U\frac{\partial}{\partial t'}v^+U/\sqrt{Re} + \frac{u^+U}{L}\frac{\partial}{\partial x'}\frac{v^+U}{\sqrt{Re}} + \frac{v^+U}{\sqrt{Re}}\frac{\sqrt{Re}}{L}\frac{\partial}{\partial y^+}\frac{v^+U}{\sqrt{Re}} = \frac{\sqrt{Re}}{\rho L}\frac{\partial}{\partial y^+}\rho U^2p^+$

$+\nu\left(L^{-2}\frac{\partial^2}{\partial x^+}\frac{v^+U}{\sqrt{Re}} + L^{-2}Re\frac{\partial^2}{\partial y^{+2}}\frac{v^+U}{\sqrt{Re}}\right)$

$\frac{U^2}{L\sqrt{Re}}\underbrace{\left[\frac{\partial v'}{\partial t^+} + u^+\frac{\partial v'}{\partial x^+} + v^+\frac{\partial v'}{\partial y'}\right]}_{} = \frac{\sqrt{Re}\,\rho U^2}{L\,\rho}\frac{\partial p^+}{\partial y^+} + \nu\left(\frac{U}{L^2\sqrt{Re}}\frac{\partial^2 v'}{\partial x'^2} + \frac{U}{L^2}\sqrt{Re}\frac{\partial^2 v^+}{\partial y^{+2}}\right)$

$Re^{-1}\left[\quad\right] = \frac{\partial p'}{\partial y^+} + Re^{-2}\frac{\partial^2 v^+}{\partial x^+} + Re^{-1}\frac{\partial^2 v'}{\partial y^{+2}} \qquad \underbrace{\frac{\nu L}{\sqrt{Re}\,U^2}\left(\frac{U}{L^2\sqrt{Re}}\right.}_{Re^{-2}} \qquad \underbrace{\frac{U}{L^2}\sqrt{Re}}_{Re^{-1}}$

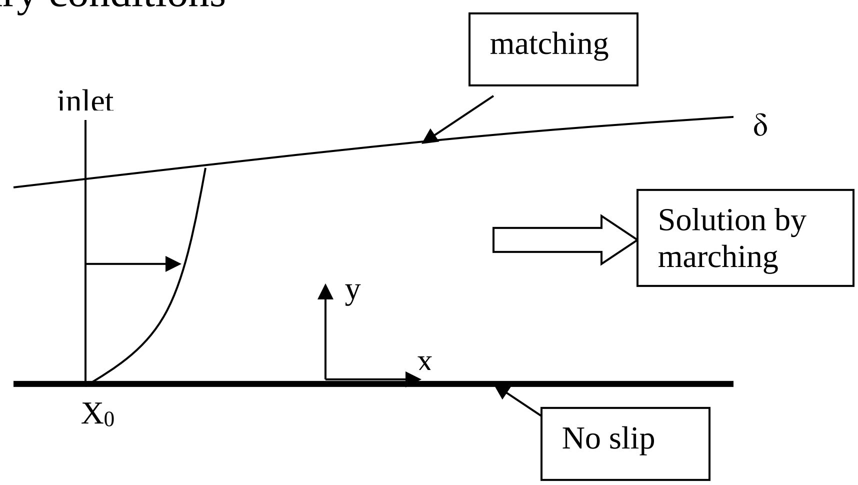
Note:

(1) U(x,t) and $\frac{\partial p(x,t)}{\partial x}$ impressed on BL by the external flow.

(2) $\frac{\partial^2}{\partial x^2} = 0$: i.e. longitudinal (stream-wise) diffusion is neglected.

(3) Due to (2), the equations are parabolic in x. Physically, this means all downstream influences are lost other than that contained in external flow. A marching solution is possible.

(4) Boundary conditions

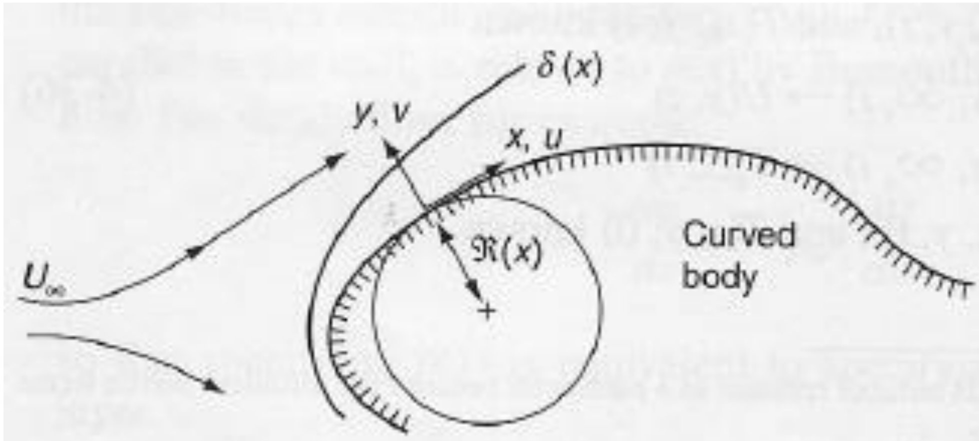


No slip: $u(x,0,t) = v(x,0,t) = 0$
Initial condition: $u(x,y,0)$ known.
Inlet condition: $u(x_0,y,t)$ given at x_0
Matching with outer flow: $u(x,\infty,t) = U(x,t)$

(5) When applying the boundary layer equations, one must keep in mind the restrictions imposed on them due to the basic BL assumptions.

→ not applicable for thick BL or separated flows (although they can be used to estimate occurrence of separation).

(6) Curvilinear coordinates



Although BL equations have been written in Cartesian Coordinates, they apply to curved surfaces provided $\delta << R$ and x, y are curvilinear coordinates measured along and normal to the surface, respectively. In such a system under the BL assumptions:

$$p_y = \frac{\rho u^2}{R}$$

Assume *u* is a linear function of *y*: $u = Uy/\delta$

$$\frac{dp}{dy} = \frac{\rho U^2 y^2}{R\delta^2}$$

$$p(\delta) - p(0) \propto \frac{\rho U^2 \delta}{3R}$$

Or

$\frac{\Delta p}{\rho U^2} \propto \frac{\delta}{3R}$; therefore, we require $\delta << R$.

(7) Practical use of the BL theory

For a given body geometry:

(a) Inviscid theory gives p(x) → integration gives Lift and Drag = 0.

(b) BL theory gives → $\delta^*(x)$, $\tau_w(x)$, $\theta(x)$, etc. and predicts separation if any.

(c) If separation present then no further information → must use inviscid models, BL equation in inverse mode, or NS equations.

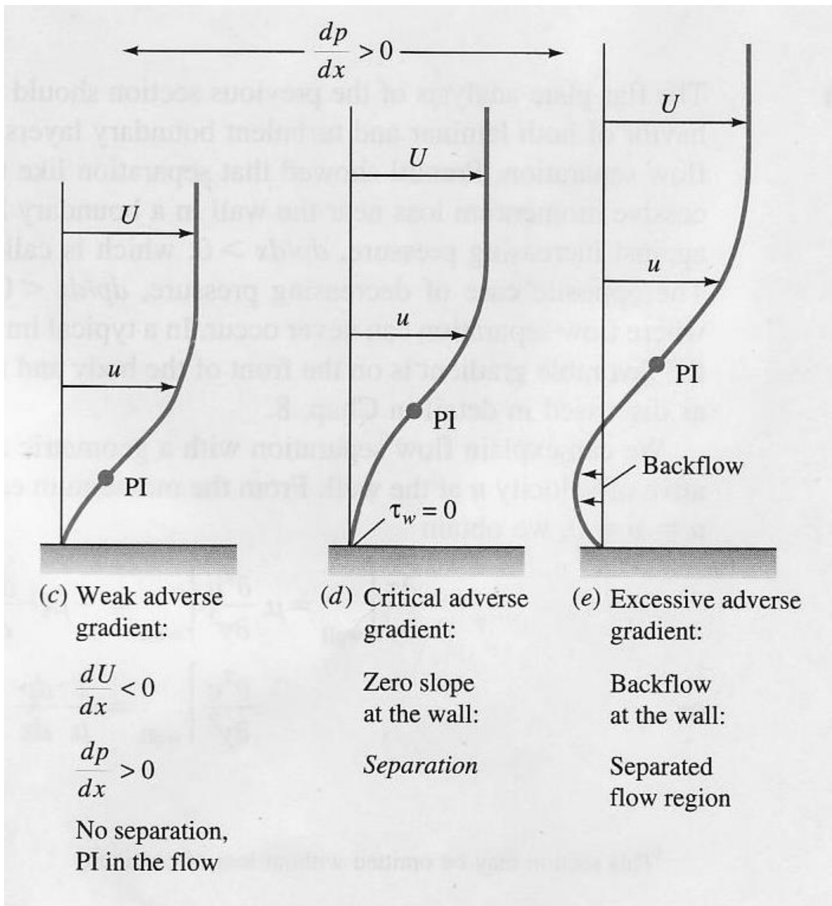
(d) If separation is absent, integration of $\tau_w(x)$ provides frictional resistance; displacement body (including δ^*) inviscid theory gives new p(x); and for displacement body drag go back to (2) for more accurate BL calculation including viscous – inviscid interaction.

(8) Separation and shear stress

At the wall,

$u = v = 0 \rightarrow u_{yy} = \frac{1}{\mu} p_x$ 2nd derivative u depends on p_x

1st derivative u gives $\tau_w \rightarrow \tau_w = \mu u_y \big|_w$ $\tau_w = 0$ separation



Bernoulli: $p_x = -\rho U U_x$

Adverse pressure gradient $p_x > 0$ and $U_x < 0$:

H = shape parameter = $\frac{\delta^*}{\theta}$ depends shape velocity profile provides indicator for separation = 3.5 laminar = 2.4 turbulent flow

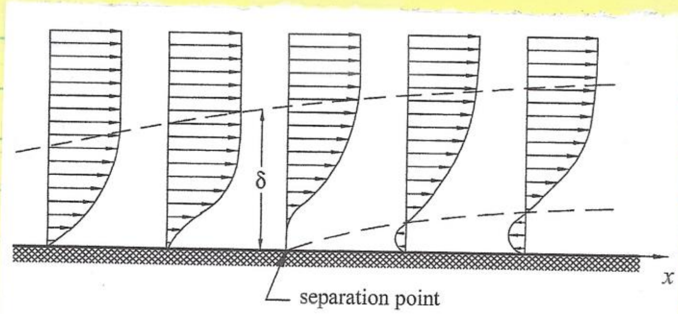


FIGURE 9.8 Velocity profiles in a boundary layer in the vicinity of separation.

note BL theory $p_y = 0$ across δ

Low momentum fluid near wall responds first to adverse pressure gradient resulting in reverse flow, ie, BL separates from surface and is deflected over reverse flow region.

Prior separation $u_y > 0$ ie $\tau_w = \mu u_y > 0$ and opposes outer flow

After separation $u_y < 0$ ie $\tau_w < 0$

Definition 2D separation: $u_y = 0$ ie $\tau_w = \mu u_y = 0$

$\therefore$ Separation can only exist in region $p_x > 0$

$\mu u_{yy} = p_x$ ie curvature $u \propto p_x$ at $y = 0$

Note for $p_x = 0$ ie flat plate BL, PI at $y = 0$

$p_x < 0$ $u_{yy} < 0$ at $y = 0$ and remain < 0 same as at δ
ie no separation will occur and no PI

$p_x > 0$ $u_{yy} > 0$ at $y = 0$ and since < 0 at δ
must have PI $0 < y < \delta$ and separation
may occur for sufficiently large p_x

Or can explain in terms of u_y that all BL have PI when $p_x > 0$

$p_x > 0 \Rightarrow \frac{\partial}{\partial y}(\mu u_y)\Big|_{y=0} > 0$ ie $u_y > 0$ at $y = 0$ & therefore since $\tau = \mu u_y = 0$ for $y \geq \delta$ must have maximum within δ, which implies PI since $\mu u_y = 0\Big|_{\tau_{max}}$ ie $\mu u_{yy} = 0$

Since $U(x) \neq f(Re)$, prediction separation seemingly at least for laminar flow without viscous/inviscid interaction does not depend on Re. Interestingly even for bluff bodies with transition from laminar to turbulent flow (circular cylinder & sphere) with large wake (viscous/inviscid interaction) x_{sep} not very sensitive Re that is except whether BL is laminar or turbulent

circular cylinder θ_s
laminar 82
turbulent 125

Separation types:

① bluff body ie large wake, which changes effective body shape due δ^*

② slender body ie of local perturbation $U(x)$ eg airfoil LE separation bubble & TE separation

for smooth surface separation

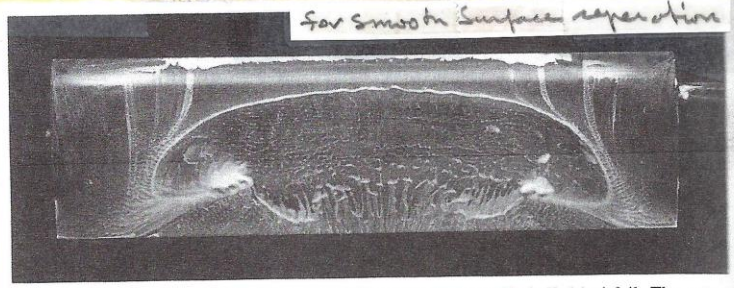


Figure 20.19 Plan view of the trailing-edge stall pattern on a Clark *Y*–14 airfoil. The pattern is made visible by the oil-flow technique. Flow is from top to bottom. Photography courtesy of A. Winkelmann, Department of Aerospace Engineering, University of Maryland. Reprinted with permission.

Separation is usually 3D

Characteristics steady laminar separated flow for large Re focus on how wake grows and interacts potential flow.

Two main concepts:

1) Wake of $\underline{B}$ as $Re \to \infty$ has $\underline{\omega} =$ constant:
Prandtl-Batchelor assumption

2) Wake has eddy length $O(Re)$, width $O(Re^{1/2})$
Small eddies, at constant pressure
free streamline: Kirchhoff assumption
(shear layer separates wake at potential flow)

Both models and/or combination models have defects & do not account for turbulence.

Most important is providing pressure distribution on the body since BL equations with proper $p|_{body}$ can predict separation.

Slender body separation can be predicted with BL theory, eg, separation bubbles using inverse methods, which do not use specified p_x in BL.

BL equations breakdown at separation when p_x specified: singular point where v & $\delta^* \to \infty$ while $\tau_w \to 0$ with $\tau_x \to \infty$: Goldstein singularity. Not due BL theory but specified p_x.

Inverse method: (1) Specify $\delta^* \Rightarrow u_e(x)$ & p_x as part of solution such that BL equations can be integrated ie continued through separation

(2) Specify τ_w & modify p_x such that $\beta_x > 0$ at x_s (β = pressure gradient parameter)

Note: δ^* & τ_w not really known a priori
another problem in reverse flow
uu_x assumed forward marching
∴ must use upwind marching or
neglect uu_x

Alternatively Boundary Layer Interaction Theory

Remove Goldstein singularity via triple deck theory: lower deck shifts middle layer by v (inviscid with $\underline{w}$) $\Rightarrow$ perturbation $u_e(x)$; p obtained using interactive solution. Used for TE separation, bump on wall, initial stage suction, etc. Flow for which separation only small perturbation.

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INCOMPRESSIBLE BOUNDARY-LAYER SEPARATION

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INTRODUCTION

For high-Reynolds-number flow over bodies or in confined channels the effects of viscosity are generally limited to a thin layer, the boundary layer, adjacent to the bounding surface. When the imposed pressure gradient is adverse, however, the thickness of the viscous layer increases as momentum is consumed by both wall shear and pressure gradient, and at some point the viscous layer breaks away from the bounding surface. Downstream of this point (or line) of breakaway the original boundary-layer fluid passes over a region of recirculating flow. The point at which the thin boundary layer breaks away from the surface and which divides the region of downstream-directed flow, in which the viscous effects are quite limited in extent, from the region of recirculating flow is known as the separation point.[1] Two different types of post-separation behavior are known to exist. In some cases the original boundary layer passes over the region of recirculating fluid and reattaches to the body at some point downstream, trapping a bubble of recirculating fluid beneath it. The characteristic length of this separation bubble may be of the same order as the upstream boundary-layer thickness or many times the boundary-layer thickness. In other cases, the original boundary-layer fluid never reattaches to the body but passes downstream, mixing with recirculating fluid, to form a wake. For this wake-type of separation the characteristic dimension of the recirculating region is generally of the same order as the characteristic body dimension. In either case, the recirculating flow alters the effective body shape and hence the inviscid flow about the body.

[1] For two-dimensional flow over fixed walls, the point of vanishing shear coincides with the point of separation. As a result, the point of vanishing shear has been taken for years as a significant indicator of separation. For two-dimensional flow over moving walls, two-dimensional unsteady flow, and three-dimensional steady flow, the point or line of vanishing wall shear does not coincide with separation, and a general definition of separation cannot include vanishing wall shear as a characteristic of separation.

Separation is the controlling, if not dominant, feature of many fluid flows. For the flow over bluff bodies the location of separation determines the pressure drag on the body. For airfoils and blades at high angle of attack the conditions at separation dictate the circulation about the airfoil or blade and hence the lift. For flows in confined passages, such as diffusers, separation frequently drastically alters the flow field and hence the performance of the device.

In spite of its fundamental importance, the complete analytical or experimental description of separation at high Reynolds numbers remains one of the main unsolved problems of fluid mechanics. It is still not possible to calculate the entire flow field including both the boundary layer and the recirculating region when both the recirculating region and the Reynolds number are large. The extent to which the recirculating region alters the pressure distribution upstream of separation is therefore not known. When the flow field is such that separation is followed by a small bubble of recirculating fluid and reattachment of the main boundary layer, it is possible to integrate the steady laminar boundary-layer equations through the points of separation and reattachment if the wall shear or displacement thickness is prescribed, while a similar calculation is terminated by a singularity at separation if the pressure gradient is prescribed. In most practical cases, however, transition occurs after separation and accurate calculations are hampered by a lack of models for turbulent reverse flow. The model for unsteady separation has recently been verified analytically for the case where the separation point moves forward along the body, but the case where the separation point moves aft or oscillates is still in question. These are but a few of the problems that must be resolved before separation is completely understood.

An enlightening and thorough review of the problem of separation was given by Brown & Stewartson (1969). In the intervening eight years there has been considerable progress in understanding the problem of separation, although it may be safely said that the problem has not been solved at this point. In the present review an attempt is made to describe some of the progress made in the past eight years. Attention is limited to incompressible flows, although much of the present knowledge regarding incompressible separation is directly applicable to compressible flows. Two-dimensional steady laminar separation is considered initially and followed by a discussion of two-dimensional unsteady laminar separation and three-dimensional steady laminar separation. No attempt is made to review the theory of asymptotic solutions of the Navier-Stokes equations for the flow in the vicinity of separation. This theory, as applied to the problem of separation, has been reviewed recently in some detail by Stewartson (1974). The emphasis is on laminar boundary-layer separation. As the opportunity arises some comments are made regarding turbulent separation. Our understanding of turbulent separation is, however, so primitive that a complete and concise discussion of this problem is not possible at this time.