Chapter 3 Solutions of the Newtonian Viscous-Flow Equations

Summary GDE for incompressible constant property fluid flow ($c_v \sim c_p$)

$$\nabla \cdot \underline{V} = 0$$

$$\rho \frac{D\underline{V}}{Dt} = -\rho g\hat{k} - \nabla p + \mu \nabla^2 \underline{V}$$ "elliptic"

$$\rho \frac{D\underline{V}}{Dt} = \underbrace{-\rho g\hat{k} - \nabla p}_{\substack{-\nabla \hat{p} \text{ where } \hat{p} = p + \gamma z \\ \text{piezometric pressure}}} + \mu \nabla^2 \underline{V}$$

$$\left[\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right] = -\frac{1}{\rho} \nabla \hat{p} + \nu \nabla^2 \underline{V} \qquad \nu = \frac{\mu}{\rho}$$ *kinematic viscosity*

$$\rho c_p \frac{DT}{Dt} = k \nabla^2 T + \Phi$$ where $\Phi = \tau_{ij} \frac{\partial u_i}{\partial x_j}$

Continuity and momentum uncoupled from energy; therefore, solve separately and use solution post facto to get T.

Combine NS with $\nabla \cdot \underline{V}$ for 4 equations for 4 unknowns $\underline{V}$, p and can be, albeit difficult, solved subject to initial and boundary conditions for $\underline{V}$, p at t = t_0 and on all boundaries i.e. "well posed" IBVP.

Types of exact analytical solutions incompressible constant property fluid flow are broadly separated by linear $\underline{V} \cdot \nabla \underline{V} = 0$ or nonlinear $\underline{V} \cdot \nabla \underline{V} \neq 0$. No unique classification. Herein, a summary of the topics of exact analytical solutions as follows:

1. Couette (wall-driven) steady flows
    a. Channel flows
    b. Cylindrical flows
2. Poiseuille (pressure-driven) steady flows
    a. Channel flows
    b. Duct flows
3. Combined Couette and Poiseuille steady flows
4. Gravity and free-surface steady flows
5. Unsteady flows
6. Suction and injection flows
7. Wind-driven (Ekman) flows
8. Non-Linear Similarity solutions
9. Low Re linearized creeping motion Stokes flows
10. Lubrication Theory

Note: 1 – 4 All have parallel streamlines either straight or circular and $\underline{V} \cdot \nabla \underline{V} = 0$.

Navier-Stokes Equations in Cartesian and Cylindrical Coordinates. [A and B]

Topics of exact analytical solutions:

1. Couette (wall-driven) steady flows
    a. Channel flows
        i. Fixed/moving walls
        ii. Two fluid fixed/moving walls
    b. Cylindrical flows
        i. Axial inner moving annulus
        ii. Rotating rod
        iii. Outer rotating concentric cylinders
        iv. Inner/outer rotating concentric cylinders
2. Poiseuille (pressure-driven) steady flows
    a. Channel flows
        i. Parallel plates
    b. Duct flows
        i. Pipe and non-circular cross sections
        ii. Pipe flow (2)
        iii. Axial annulus
3. Combined Couette and Poiseuille steady flows
    a. Fixed/moving walls with constant pressure gradient
        i. Lower wall
        ii. Upper wall
    b. Fixed/moving walls with constant pressure gradient and heat transfer

   c. Axial inner moving annulus with constant pressure gradient
4. Gravity and free-surface steady flows
   a. Vertical parallel plates
   b. Vertical parallel plates with pressure gradient
   c. Fixed/moving vertical parallel plates
   d. Inclined fixed/moving parallel plates with constant pressure gradient
   e. Vertical wall with free surface
   f. Inclined plane with free surface (Typed version)
   g. Two fluid inclined plane with free surface
   h. Vertical rod with free surface
5. Unsteady flows
   a. Stokes 1st problem: sudden acceleration
   b. Diffusion vortex sheet
   c. Decay of a Line Vortex
   d. Burgers Vortex
   e. Stokes 2nd problem: steady oscillations
   f. Starting flow circular pipe
   g. Oscillating pressure gradient pipe flow
   h. Starting flow fixed/moving parallel walls.
6. Suction and injection flows

7. Wind-driven (Ekman) flows **Chapters 1 & 2 (6.1 Part 2)**
    a. Impulsive Wind-Driven Free Surface Flow
    b. Ekman Layer
8. Similarity solutions
    a. Stagnation point
    b. Rotating disc
    c. Wedge flows
9. Low Re linearized creeping motion Stokes flows
    a. Types of low Re flows
    b. Immersed bodies
10. Lubrication theory
    a. Reynolds Equation for Bearing Theory
    b. Slider and Squeeze Bearings
        i. P4.83
    c. Journal Bearing
    d. Hele-Shaw Flow

Exact solutions NS, Stokes, lubrication, and boundary layer equations for elemental problems help for understanding fluid mechanics (1) mathematics and (2) physics. Separating or combining elementally both (1) and (2) for the effects of:

a. Inertia: local and convective
b. Pressure gradient
c. Body force $\rho\mathbf{g}$
d. Diffusion
e. Scaling (l, u, ω, ω^{-1}, etc.) and non-dimensional equations and boundary conditions
f. Perturbation and matched asymptotic expansions
g. Coordinate systems
h. Reference Frames

Chapter 3 includes Stokes and lubrication, as more limited discussion compared boundary layer theory, which therefore given in its own chapter.

Perturbation and matched asymptotic expansions need more attention.