Bernard : Fluid Dynamics

$\nabla \times \underline{u} = \underline{\omega}$ (1) $\underline{u} = u_r \hat{e}_r + u_\theta \hat{e}_\theta = \nabla \times \underline{A}$ (2)

where $\underline{A} = (0, 0, \frac{\psi}{r\sin\theta}) =$ vector potential

$\nabla \times (\nabla \times \underline{A}) = \underline{\omega}$ (3) For axisymmetric flow in spherical coordinates. Note $\nabla \cdot \underline{A} = 0$ and

$\nabla \times (\nabla \times \frac{\psi}{r\sin\theta} \hat{e}_\theta) =$ (4) used in Helmholtz decomposition

ie $\underline{V}^\omega = \nabla \times \underline{A}$.

$-\frac{1}{r\sin\theta} L\psi \hat{e}_\theta = \omega_\theta \hat{e}_\theta$

$L\psi = \psi_{rr} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\psi_\theta\right)$ (5) using definition curl in spherical coordinates

or $\nabla^2\left(\frac{\psi}{r\sin\theta}\right)\hat{e}_\theta$

Stokes Flow : $\nabla^2 \underline{\omega} = 0$ $\quad \nabla \times \nabla \times \underline{\omega} = \nabla(\nabla \cdot \underline{\omega}) - \nabla^2 \underline{\omega}$

$$\nabla \times (\nabla \times \underline{\omega}) = 0 = -\nabla^2 \underline{\omega}$$

$$\nabla \times (\nabla \times \omega_\theta \hat{e}_\theta) = 0 = -\nabla^2 \omega_\theta$$

$$\nabla \times (\nabla \times \frac{-L\psi}{r\sin\theta} \hat{e}_\theta) = 0 = -\nabla^2 \omega_\theta$$

(4) true any value ψ including $L\psi$ $\therefore$

$\nabla \times (\nabla \times \frac{-L(L\psi)}{r\sin\theta}) = -\frac{1}{r\sin\theta} L(L\psi)\hat{e}_\theta = 0$ Since 3 BC need $L(L\psi) = 0$ equation

ie $L(L\psi) = 0$ (6)

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0$$

$u_r = \frac{\psi_\theta}{r^2\sin\theta}$

BC : $(u_r, u_\theta) = (0, 0)$ $r = a$ no slip $\quad u_\theta = \frac{-\psi_r}{r\sin\theta}$

ie $\psi_\theta(a, \theta)$ and $\psi_r(a, \theta) = 0$

$\psi = \frac{1}{2} U r^2 \sin^2\theta$ uniform flow $r \to \infty$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{\chi}{r\sin\theta}\right)\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\frac{\chi}{r\sin\theta}}{\partial\theta}\right)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{r^2\chi}{\sin\theta}\left(-r^{-2}\right)\right] + \frac{1}{r^3\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta + \frac{\chi}{r}\frac{\partial\csc}{\partial\theta}\right)$$

$\frac{r}{\sin\theta}\chi_r$

$\csc\cot = \frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-\chi}{\sin\theta} + \frac{r}{\sin\theta}\chi_r\right)$$

$$\frac{1}{r^2}\left(\frac{-1}{\sin\theta}\chi_r + \frac{1}{\sin\theta}\chi_r + \frac{r}{\sin\theta}\chi_{rr}\right)$$

$\hookrightarrow$

$$\frac{1}{r\sin\theta}\chi_{rr}$$

$$\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\chi_\theta\right)$$

$$= \left[\csc\cot\chi_\theta + \frac{1}{\sin\theta}\chi_{\theta\theta}\right]$$

$$\frac{1}{r^3}\left[\frac{\chi_{\theta\theta}}{\sin\theta} + \chi_\theta\csc\cot\right] \text{ yes!}$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\frac{\chi}{r\sin\theta}\right)$$

$$\left(\frac{\partial}{\partial\theta}\frac{\chi}{r\sin\theta}\right) = \frac{1}{r\sin\theta}\chi_\theta + \frac{\chi}{r}\csc\cot$$

$$\sin\theta(\downarrow) = \frac{1}{r}\chi_\theta + \frac{\chi}{r}\cot$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\chi_\theta + \frac{\chi}{r}\cot\right)$$

$$\frac{1}{r^2\sin\theta}\left[\frac{1}{r}\chi_{\theta\theta} + \frac{\chi_\theta}{r}\cot + \frac{\chi}{r}\csc^2\right]$$

$$\left[\frac{1}{r^3\sin\theta}\chi_{\theta\theta} + \frac{\chi_\theta}{r^3}\frac{\cot}{\sin\theta} + \frac{\chi}{r^3}\frac{1}{\sin\theta}\right]$$

$$\frac{1}{r^3}\left[\frac{\chi_{\theta\theta}}{\sin\theta} + \chi_\theta\cot\csc\right]$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(\chi/r\sin\theta\right)\right) - \frac{\chi/r\sin\theta}{r^2\sin^2\theta}$$

$$\frac{\chi}{r^3\sin^3\theta}$$

$\psi = f(r) \sin^2\theta$

$\psi_r = f' \sin^2\theta \qquad \psi_\theta = f\, 2 \sin\theta \cos\theta$

$\lim_{r\to\infty} f(r)/r^2 = U/2$

$f(a) = 0 \quad f_r(a) = 0$

$L^2\psi = (f^{iv} - 4/r^2 f'' + 8/r^3 f' - 8/r^4 f) \sin^2\theta = 0$

general solution homogeneous equation

$f(r) = c_1 r^4 + c_2 r^2 + c_3 r + c_4/r$ equidimensional equation

$c_1 = 0 \quad c_2 = U/2 \quad c_3 = -3Ua/4 \quad c_4 = \frac{Ua^3}{4}$ Solution r^β

$f(r) = U\left(r^2/2 - 3ar/4 + a^3/4r\right)$

$\psi(r,\theta) = \frac{1}{2} U r^2 \left(1 - \frac{3}{2}\left(\frac{a}{r}\right) + \frac{1}{2}\left(\frac{a}{r}\right)^3\right) \sin^2\theta$

Comparison ψ inviscid and Stokes with same contour levels shows viscous spread wider from sphere and shows presence at further distance. Forward and aft symmetry due to outward viscous diffusion due vortex flux from $r = a$. Convection would cause asymmetry

Recall $\omega_\varphi = -L\psi / r\sin\theta = -\frac{\sin\theta}{r}\left(f'' - 2f/r^2\right)$

$f' = U\left(r - 3a/4 - a^3/4r^2\right)$

$f'' = U\left(1 + a^3/2r^3\right)$

$U\left(1 + a^3/2r^3\right) - \frac{2U}{r^2}\left(\frac{r^2}{2} - \frac{3ar}{4} + a^3/4r\right)$

$\cancel{U} + Ua^3/2r^3 - \cancel{U} + 3Ua/2r - Ua^3/2r^3$

$\omega_\theta = -\frac{\sin\theta}{r}\left(\frac{3Ua}{2r}\right) = -\frac{3}{2}\frac{Ua}{r^2}\sin\theta$

[Flow past a sphere up to a Reynolds number of 300]

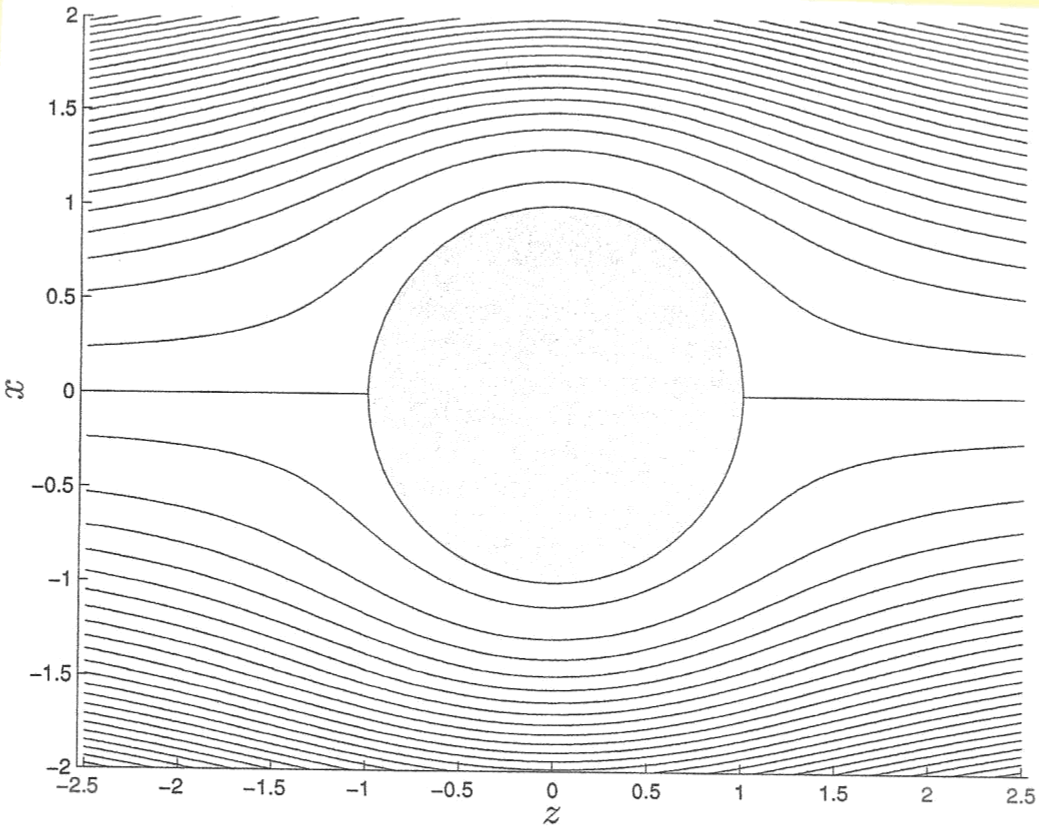

Figure 19.4. Streamlines of Stokes flow past a sphere.

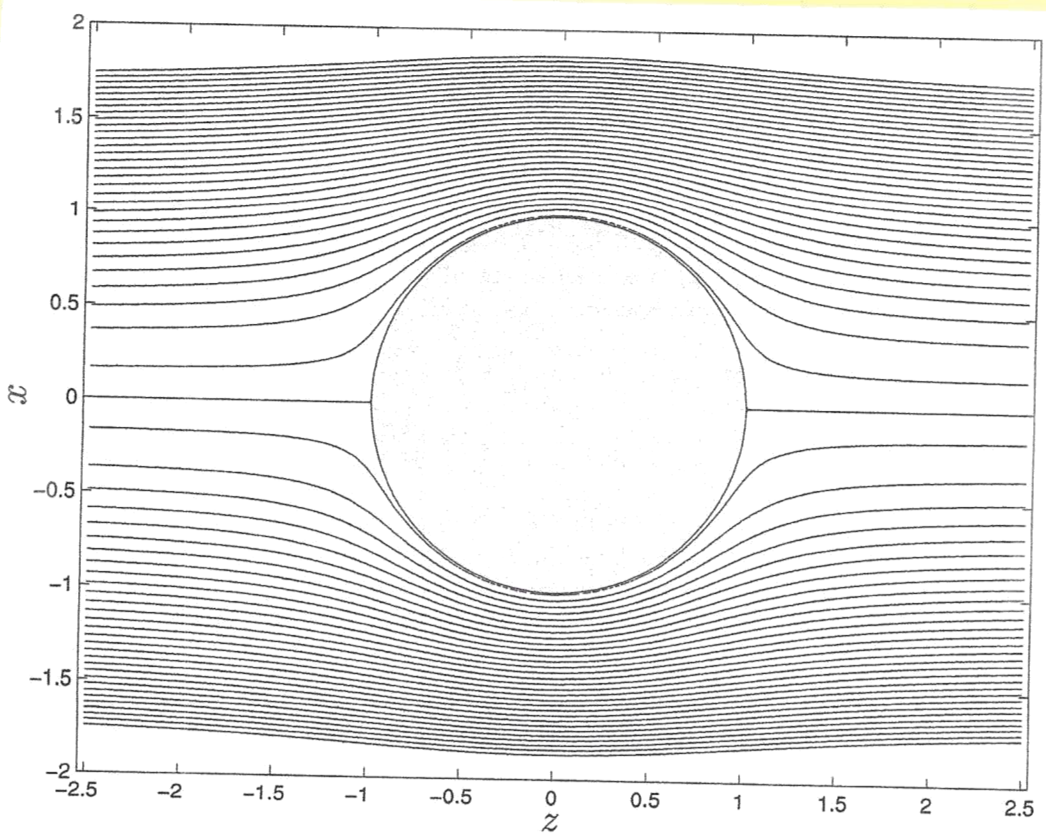

Figure 6.4. Streamlines corresponding to Eq. (6.11) in inviscid sphere flow.

re-arrange

$$\psi(r,\theta) = \underbrace{\tfrac{1}{2} U r^2 \sin^2\theta \left(1 + \tfrac{1}{2}\left(\tfrac{a}{r}\right)^3\right)}_{①} - \underbrace{\tfrac{3}{4} U a r \sin^2\theta}_{②}$$

① modifies ψ_I ② responsible ω_θ

compare potential flow: $\psi(r,\theta) = \frac{1}{2} U r^2 \sin^2\theta \left(1 - \left(\frac{a}{r}\right)^3\right)$

doublet + uniform flow $u_r = U \cos\theta \left(1 - \left(\frac{a}{r}\right)^3\right)$

$u_\theta = -U \sin\theta \left(1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right)$

$u_r(a,\theta) = 0 \quad u_\theta(a,\theta) = -\frac{3}{2} U \sin\theta$ ie $u_\theta = 0$ at $0, \pi$

$$\psi_\sigma - \psi_I = \tfrac{1}{2} U r^2 \sin^2\theta \left(\tfrac{3}{2}\left(\tfrac{a}{r}\right)^3\right) - \tfrac{3}{4} U a r \sin^2\theta$$

forces on sphere

$\nabla p = \mu \nabla^2 \underline{u}$

$= -\mu \nabla \times \underline{\omega}$

$\nabla \times \nabla \times \underline{u} = \nabla(\nabla \cdot \underline{u}) - \nabla^2 \underline{u}$

$\nabla p = p_r \hat{e}_r + r^{-1} p_\theta \hat{e}_\theta$

$\underline{\omega} = \omega_\theta \hat{e}_\theta \quad \omega_\theta = -\frac{3}{2} \frac{U a}{r^2} \sin\theta$

$\nabla = \frac{\partial}{\partial r} \hat{e}_r + r^{-1} \frac{\partial}{\partial \theta} \hat{e}_\theta$

$\nabla \times \omega_\theta \hat{e}_\theta = \frac{1}{r \sin\theta}\left(\frac{\partial}{\partial \theta}(\omega_\theta \sin\theta)\right) \hat{e}_r$

$- \frac{1}{r}\left(\frac{\partial (r\omega_\theta)}{\partial r}\right) \hat{e}_\theta$

$\frac{1}{r\sin\theta}\left[\cos\theta\, \omega_\theta + \sin\theta \frac{\partial \omega_\theta}{\partial \theta}\right]$

$\frac{1}{r\sin\theta}\left[-\frac{3}{2}\frac{U a}{r^2} \cos\theta \sin\theta\right.$

$\left. + \sin\theta \left(-\frac{3}{2}\frac{U a}{r^2} \cos\theta\right)\right]$

$-3 \frac{U a}{r^3} \cos\theta$ ie $p_r = 3\mu U a \cos\theta / r^3$

$p_\theta = \frac{3 \mu U a}{2 r^3} \sin\theta$

$\frac{\partial \omega_\theta}{\partial \theta} = -\frac{3}{2} \frac{U a}{r^2} \cos\theta$

$\frac{\partial \omega_\theta}{\partial r} = -\frac{3}{2} U a \sin\theta (-2 r^{-3})$

$= 3 U a \sin\theta / r^3$

$r\omega_\theta = -\frac{3}{2} \frac{U a}{r} \sin\theta$

$(r\omega_\theta)_r = \frac{3}{2} \frac{U a}{r^2} \sin\theta$

$-\frac{1}{r}(r\omega_\theta)_r = -\frac{3}{2} \frac{U a}{r^3} \sin\theta$

$$p - p_\infty = \int_0^r p_r\, dr + \int_0^\theta p_\theta\, r\, d\theta = -\frac{3}{2}\mu U a \frac{\cos\theta}{r^2}$$

$$3\mu U a \cos\theta \int_0^r r^{-3}\, dr = 3\mu U a \cos\theta \left(\frac{r^{-2}}{-2}\right) = -\frac{3}{2}\mu U a \cos\theta / r^2 + f(\theta)$$

$$\frac{3\mu U a}{2r^3} \int_0^\theta \sin\theta\, r\, d\theta = -\frac{3}{2}\mu U a \cos\theta / r^2 + f(r)$$

$$p(r,\theta) = p_\infty - \frac{3}{2}\mu U a \frac{\cos\theta}{r^2}$$ asymmetric ∴ unlike potential flow there will be drag

$$C_p = \frac{\Delta p}{\rho U^2 / Re} = -\frac{3\cos\theta}{(r/a)^2} \qquad Re = \frac{2aU}{\nu}$$

$$\frac{\rho U^2}{\frac{2aU}{\nu}} = \frac{\mu U}{2a}$$

ie p scales like $\rho U^2/Re$ vs ρU^2
low Re — high Re

$$F_i = \sigma_{ij} n_j = [-p\delta_{ij} + \tau_{ij}] n_j = -p n_i + \tau_{ij} n_j$$ = force per unit area normal surface

component $\hat{z}$ direction $= [-p\cos\theta + \tau_{rr}\cos\theta - \tau_{r\theta}\sin\theta]$

$$\tau_{rr} = 2\mu \frac{\partial u_r}{\partial r} = 2\mu U \cos\theta \left[\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right] \quad \text{on } r = a$$

$$\tau_{r\theta} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] = -\frac{3\mu U a^3}{2r^4}\sin\theta$$

$$\frac{3\mu U}{2a}\cos^2\theta + 0 + \frac{3\mu U}{2a}\sin^2\theta = \frac{3\mu U}{2a}$$

1/3 — 2/3

$$D = \frac{3\mu U}{2a} \times 4\pi a^2 = 6\pi \mu a U$$ Re < .5 excellent
10% error Re = 1

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 \pi a^2} = 24/Re$$ Stokes drag law

(projected area)

$$D/\mu a U = 6\pi$$ proper form Stokes flow