Stokes Stream Function for Axisymmetric Flow

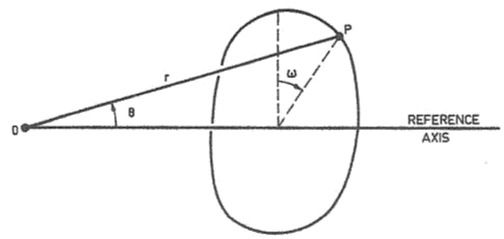


FIGURE 5.1
Definition sketch of spherical coordinates.

$\omega = \alpha$

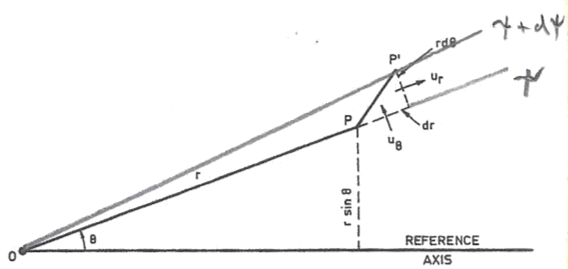


FIGURE 5.2
Velocity components and flow areas defined by a reference point P and neighboring point P'.

$d\theta > 0$
$dr > 0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta \sin\theta\right) = 0$$ Continuity equation $\nabla\cdot\underline{u}$ for $\frac{\partial}{\partial\alpha} = 0$

for $u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$ and $u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$

$\nabla\cdot\underline{u}$ identically satisfied

Let OP rotate around reference axis such that ω varies by 2π while r, θ fixed. The amount of fluid that crosses the surface of revolution formed by OP is

πD = circumference circle generated OP rotated 2π ; quantity fluid per unit area

$$\underbrace{2\pi}\, d\psi = \underbrace{2\pi r \sin\theta}\left(u_r r\,d\theta - u_\theta\,dr\right) = dQ$$ outflow − inflow

$$d\psi = u_r r^2 \sin\theta\, d\theta - u_\theta r \sin\theta\, dr = dQ/2\pi$$

$$= \frac{\partial\psi}{\partial\theta}d\theta + \frac{\partial\psi}{\partial r}dr$$ = difference in flow rate per unit radian with units m^3/s

ie $\frac{\partial\psi}{\partial\theta} = u_r r^2 \sin\theta$ and $\frac{\partial\psi}{\partial r} = -u_\theta r\sin\theta$

note in 2D plane flows ψ has units $\frac{m^2}{s}$

B3. Spherical Polar Coordinates

The spherical polar coordinates used are (r, θ, φ), where φ is the azimuthal angle (Figure 3.1c). Equations are presented assuming ψ is a scalar, and

$$\mathbf{u} = \mathbf{i}_r u_r + \mathbf{i}_\theta u_\theta + \mathbf{i}_\varphi u_\varphi,$$

where $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\varphi$ are the local unit vectors at a point.

Gradient of a scalar

$$\nabla\psi = \mathbf{i}_r \frac{\partial\psi}{\partial r} + \mathbf{i}_\theta \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{i}_\varphi \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}.$$

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

Laplacian of a scalar

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}.$$

Divergence of a vector

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial \cancel{u_\theta}\, u_\varphi}{\partial\varphi}.$$

Spherical coordinates

Curl of a vector

$$\nabla\times\mathbf{u} = \frac{\mathbf{i}_r}{r\sin\theta}\left[\frac{\partial(u_\varphi\sin\theta)}{\partial\theta} - \frac{\partial u_\theta}{\partial\varphi}\right] + \frac{\mathbf{i}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{\partial(ru_\varphi)}{\partial r}\right] + \frac{\mathbf{i}_\varphi}{r}\left[\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial\theta}\right].$$

Laplacian of a vector

$$\begin{aligned}\nabla^2\mathbf{u} = {} & \mathbf{i}_r\left[\nabla^2u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right] \\ & + \mathbf{i}_\theta\left[\nabla^2u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right] \\ & + \mathbf{i}_\varphi\left[\nabla^2u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right].\end{aligned}$$

Strain rate and viscous stress (for incompressible form $\sigma_{ij} = 2\mu e_{ij}$)

$$\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{2\mu}\sigma_{rr}, \\
e_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} = \frac{1}{2\mu}\sigma_{\theta\theta}, \\
e_{\varphi\varphi} &= \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r} = \frac{1}{2\mu}\sigma_{\varphi\varphi}, \\
e_{\theta\varphi} &= \frac{\sin\theta}{2r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) + \frac{1}{2r\sin\theta}\frac{\partial u_\theta}{\partial\varphi} = \frac{1}{2\mu}\sigma_{\theta\varphi}, \\
e_{\varphi r} &= \frac{1}{2r\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right) = \frac{1}{2\mu}\sigma_{\varphi r}, \\
e_{r\theta} &= \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial\theta} = \frac{1}{2\mu}\sigma_{r\theta}.
\end{aligned}$$

Vorticity

$$\omega_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(u_\varphi \sin\theta) - \frac{\partial u_\theta}{\partial\varphi}\right],$$

$$\omega_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{\partial(r u_\varphi)}{\partial r}\right],$$

$$\omega_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right].$$

Equation of continuity

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\rho u_\varphi) = 0.$$

Navier–Stokes equations with constant ρ and ν, and no body force

$$\frac{\partial u_r}{\partial t} + (\mathbf{u}\cdot\nabla)u_r - \frac{u_\theta^2 + u_\varphi^2}{r}$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right],$$

$$\frac{\partial u_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2\cot\theta}{r}$$
$$= -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left[\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi}\right],$$

$$\frac{\partial u_\varphi}{\partial t} + (\mathbf{u}\cdot\nabla)u_\varphi + \frac{u_\varphi u_r}{r} + \frac{u_\theta u_\varphi\cot\theta}{r}$$
$$= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\varphi} + \nu\left[\nabla^2 u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right],$$

where

$$\mathbf{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi},$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

Stokes sphere solution using separation of separation of variables

$$\nabla p = \mu \nabla^2 \underline{v}$$

$$0 = \nabla^2 \underline{\omega} \qquad \text{note: } -\nabla \times \nabla \times \underline{\omega} = -\nabla(\nabla \cdot \underline{\omega}) + \nabla^2 \underline{\omega}$$

only component of vorticity is

$$\omega_\phi = \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)$$

$r\,\theta\,\phi$ or $r\,\theta\,\phi$

$$\underline{v} = -\nabla \phi \times \nabla \chi = -\frac{\hat{\imath}_\phi}{r\sin\theta} \times \left(\hat{\imath}_r \chi_r + \hat{\imath}_\theta \frac{\chi_\theta}{r} + \frac{\hat{\imath}_\phi}{r\sin\theta}\chi_\phi\right)$$

$$= \frac{-\hat{\imath}_\theta}{r\sin\theta}\chi_r + \frac{\hat{\imath}_r}{r\sin\theta}\frac{\chi_\theta}{r}$$

$$= \underbrace{\frac{\chi_\theta}{r^2\sin\theta}}_{u_r}\hat{\imath}_r - \underbrace{\frac{\chi_r}{r\sin\theta}}_{u_\theta}\hat{\imath}_\theta$$

$$\omega_\phi = -\frac{1}{r}\left[\frac{\chi_{rr}}{\sin\theta} + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\chi_\theta\right)\right]$$

combining with $\nabla^2 \omega_\phi = 0 \Rightarrow \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \chi = 0 \qquad (1)$

$$\chi_\theta(a,\theta) = 0 \qquad u_r = 0$$

$$\chi_r(a,\theta) = 0 \qquad u_\theta = 0 \quad \Big\} \; r=a \text{ no slip}$$

$$\chi(\infty,\theta) = \tfrac{1}{2} U r^2 \sin^2\theta \qquad \text{uniform flow @ } \infty$$

Assume $\chi = f(r)\sin^2\theta$ ie separation of variables based far field solution substitution in (1)

$$f^{IV} - \frac{4f''}{r^2} + \frac{8f'}{r^3} - \frac{8f}{r^4} = 0$$

$$f = Ar^4 + Br^2 + Cr + D/r$$

∞ condition : $A = 0$ & $B = U/2$

$r = a$ conditions : $C = -3Ua/4$ & $D = Ua^3/4$

$$\chi = Ur^2 \sin^2\theta \left[\frac{1}{2} - \frac{3a}{4r} + \frac{a^3}{4r^3}\right]$$

$$= \frac{U \sin^2\theta}{4}\left[\underbrace{2r^2}_{\text{uniform flow}} - \underbrace{3ar}_{\text{Stokeslet}} + \underbrace{\frac{a^3}{r}}_{\text{dipole}}\right]$$

$$u_r = U\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)$$

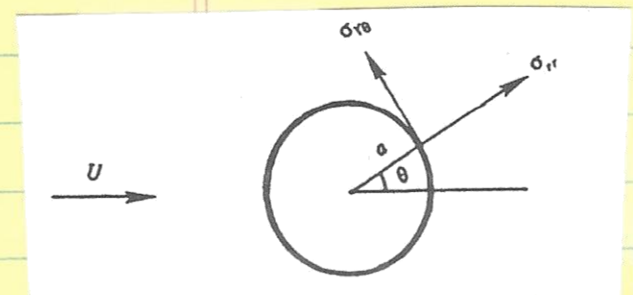
$$u_\theta = -U\sin\theta\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)$$

$$\nabla p = \mu \nabla^2 \underline{V}$$

$p \propto \mu$ anti symmetric
+ front − back
creates pressure drag

$$p = -\frac{3a\mu\cos\theta}{2r^2} + p_\infty$$

p_{max}/p_{min} at fwd/aft stagnation points $= \pm 3\mu U/2a$



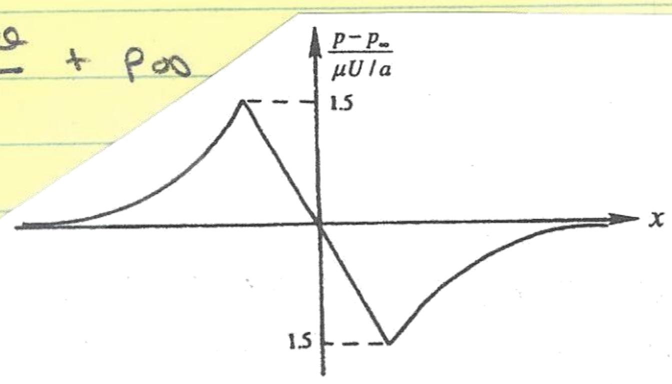


Figure 9.14 Creeping flow over a sphere. The upper panel shows the viscous stress components at the surface. The lower panel shows the pressure distribution in an axial (φ = const.) plane.

Viscous shear stress $\tau_{rr} = 2\mu \frac{\partial u_r}{\partial r} = 2\mu U \cos\theta \left[\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right]$

σ in F_D

$$\tau_{r\theta} = \mu \left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \underbrace{\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}}_{r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)}\right] = \frac{-\mu U \sin\theta}{r}\left(\frac{3a^3}{2r^3}\right)$$

$$F = -\int_0^{2\pi} \tau_{r\theta}\Big|_{r=a} \sin\theta \, dA - \int_0^{2\pi} p\Big|_{r=a} \cos\theta \, dA$$

Surface area : $4\pi a^2$

$\forall$: $\frac{4}{3}\pi a^3$

$$dA = 2\pi a^2 \sin\theta \, d\theta$$

$$F = \underbrace{4\pi\mu U a}_{2/3 \text{ viscous}} + \underbrace{2\pi\mu U a}_{1/3 \text{ pressure}} = 6\pi\mu U a \quad \propto U$$

$\propto \mu$

$Re \ll 1$, but agrees EFD up to $Re = 1$

$\frac{F}{\mu U a} = 6\pi$ = constant since $\underline{V}\cdot\nabla\underline{V} = 0$

ρ not important

$C_D = \frac{2F}{\rho U^2 \pi a^2} = 24/Re \qquad Re = \frac{2a\rho U}{\mu}$

$Re > 20$ separation p drag $\uparrow$

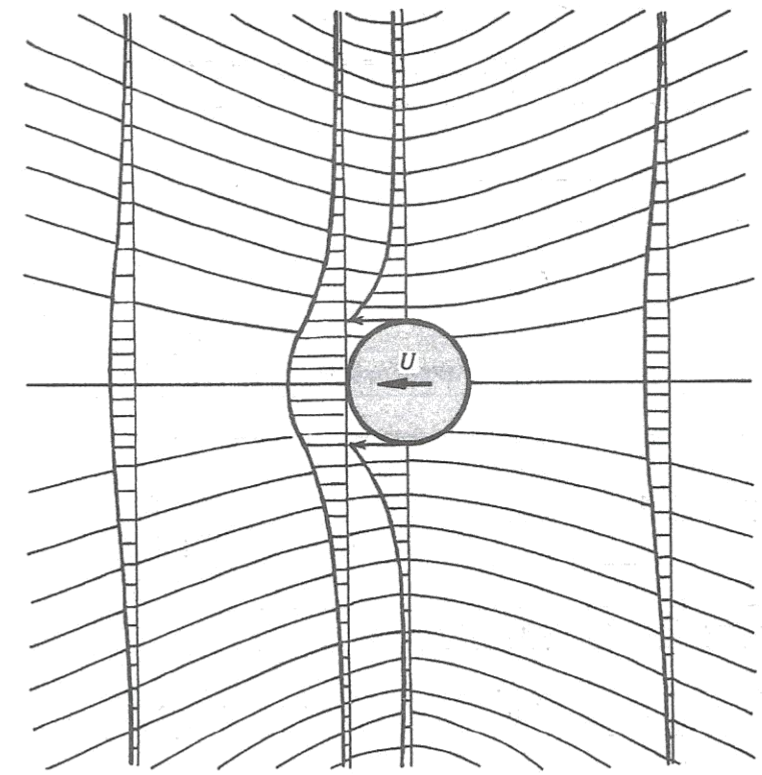


Figure 9.16 Streamlines and velocity distributions in Stokes' solution of creeping flow due to a moving sphere. Note the upstream and downstream symmetry, which is a result of complete neglect of nonlinearity.

inertial reference frame

ψ − uniform stream $= Ur^2\sin^2\theta\left(-\frac{3a}{4r}+\frac{a^3}{4r^3}\right)$

Symmetric fore and aft due $\nabla\cdot\underline{v}=0$

and no wake ie change in direction

just sign $-\underline{v}$ and $-p$

Non uniformity Stokes Solution: Oseen Approximation

Viscous force/$\forall$ = shear gradient $\sim \frac{\mu U a}{r^3}$ $\quad r\to\infty$

inertia force/$\forall$ $\sim \rho u_r \frac{\partial u_\theta}{\partial r} \sim \frac{\rho U^2 a}{r^2}$ $\quad r\to\infty$

$\therefore \frac{\text{inertia force}}{\text{viscous force}} \sim \frac{\rho U a}{\mu}\frac{r}{a} = Re\frac{r}{a}$ $\quad r\to\infty$

inertia not negligible for $\frac{r}{a}\sim\frac{1}{Re}$ no matter how small Re, which occurs at distances of order ν/U

It can be shown that much of 1st order term $\underline{v}$ is infinite at large distances, which is called Whitehead's paradox, as was the case for the 0th order 2D solution, which was called Stokes paradox : singular perturbation problems

Oseen improvement:

$$u = U + u' \qquad v = v' \qquad w = w' \qquad u', v', w' = \text{Cartesian components}$$

e.g. x-momentum $u u_x + v u_y + w u_z = U u'_x + [u' u'_x + v' u'_y + w' u'_z]$ perturbation $\underline{v} \ll U$ at $r \to \infty$

Oseen Equations

$$\rho U \frac{\partial u'_i}{\partial x} = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u'_i$$

$u'_i = (u', v', w')$ ie $\underline{V} \cdot \nabla \underline{V} \sim U \frac{\partial \underline{u}}{\partial x}$

Same order Stokes near body; however, in far field provides better approximation where solution $\sim U$

$$u' = v' = w' = 0 \quad r \to \infty$$
$$u' = -U \quad v' = w' = 0 \quad r = 0$$

$$\frac{\psi}{U a^2} = \left[\frac{r^2}{2a^2} + \frac{a}{4r}\right]\sin^2\theta - \frac{3}{Re}(1+\cos\theta)\left\{1 - \exp\left[-\frac{Re\, r}{4\, a}(1-\cos\theta)\right]\right\}$$

$Re = 2aU/\nu \quad r/a \sim 1$ recover Stokes solution

$$C_D = \frac{24}{Re}\left(1 + \frac{3}{16} Re\right)$$

Lowest order solution uniformly valid near & far field

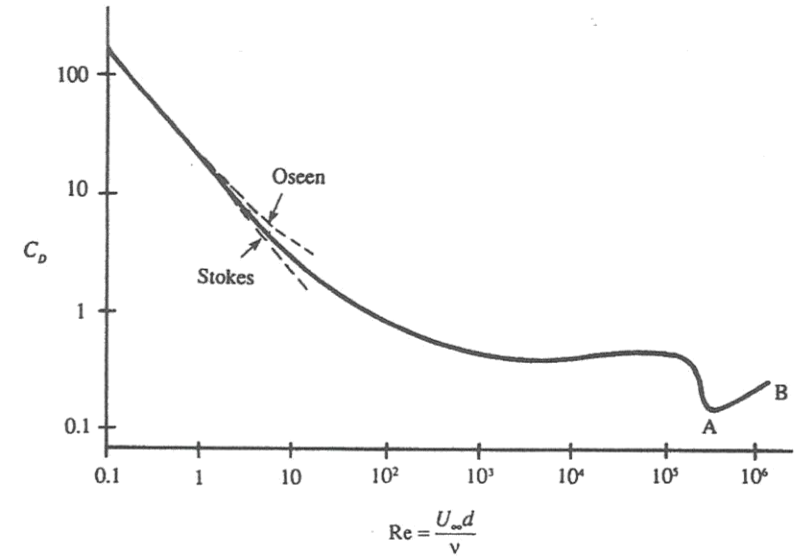


Figure 10.23 Measured drag coefficient of a smooth sphere. The Stokes solution is $C_D = 24/\text{Re}$, and the Oseen solution is $C_D = (24/\text{Re})(1 + 3\text{Re}/16)$; these two solutions are discussed in Chapter 9, Sections 12 and 13. The increase of drag coefficient in the range AB has relevance in explaining why the flight paths of sports balls bend in the air.

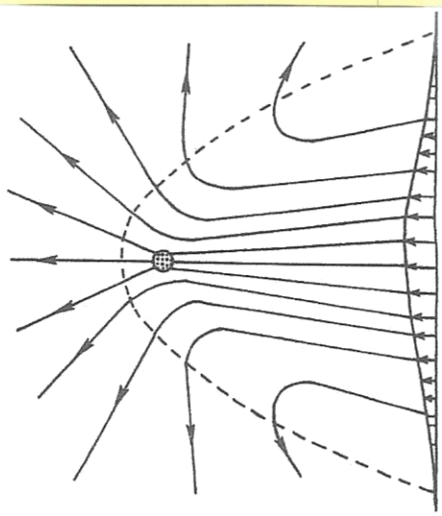


Figure 9.17 Streamlines and velocity distribution in Oseen's solution of creeping flow due to a moving sphere. Note the upstream and downstream asymmetry, which is a result of partial accounting for advection in the far field.

Oseen solution for inertial reference frame. Flow no longer symmetric, but has a wake where ψ closer together, $\underline{v}$ larger in wake than front, whereas in moving reference frame flow slower in wake than in front. Advanced methods use matched asymptotic expansion techniques

Exercise 9.45. Using the velocity field (8.49), determine the drag on Stokes' sphere from the surface pressure and the viscous surface stresses σ_{rr} and $\sigma_{r\theta}$.

Solution 9.45. There are pressure and shear stress contributions to the drag on a moving sphere at low Reynolds number. The pressure distribution is given by (8.50):

$$p(r,\theta) - p_\infty = -\frac{3\mu a U}{2r^2}\cos\theta .$$

The pressure drag can be obtained by integrating this result:

$$F_{pressure} = -\int_{surface} p(r=a,\theta)\mathbf{e}_r\cdot\mathbf{e}_z dS = -2\pi a^2\int_{\theta=0}^{\theta=\pi}\mu\left(\frac{3U}{2a}\right)\cos^2\theta\sin\theta d\theta = 2\pi\mu U a$$

The viscous drag can be obtained from surface integrals of the viscous stresses:

$$F_{vicous} = -\int_{surface}\sigma_{r\theta}(r=a,\theta)\mathbf{e}_\theta\cdot\mathbf{e}_z dS + \int_{surface}\sigma_{rr}(r=a,\theta)\mathbf{e}_r\cdot\mathbf{e}_z dS$$

$$= -2\pi a^2\int_{\theta=0}^{\theta=\pi}\sigma_{r\theta}(r=a,\theta)\sin^2\theta d\theta + 2\pi a^2\int_{\theta=0}^{\theta=\pi}\sigma_{rr}(r=a,\theta)\cos\theta\sin\theta d\theta,$$

where $\sigma_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = -\frac{\mu U\sin\theta}{r}\left(\frac{3a^3}{2r^3}\right)$, and $\sigma_{rr} = 2\mu\frac{\partial u_r}{\partial r} = 2\mu U\cos\theta\left(\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right)$.

Thus, at $r = a$, $\sigma_{r\theta} \neq 0$, but $\sigma_{rr} = 0$, so

$$F_{vicous} = 3\pi\mu U a\int_{\theta=0}^{\theta=\pi}\sin^3\theta d\theta = 4\pi\mu U a .$$

Thus, one third of the drag comes from pressure forces and two thirds come from the shear stress. The total drag is the sum of these two contributions:

$$F_{drag} = F_{pressure} + F_{vicous} = 2\pi\mu U a + 4\pi\mu U a = 6\pi\mu U a .$$

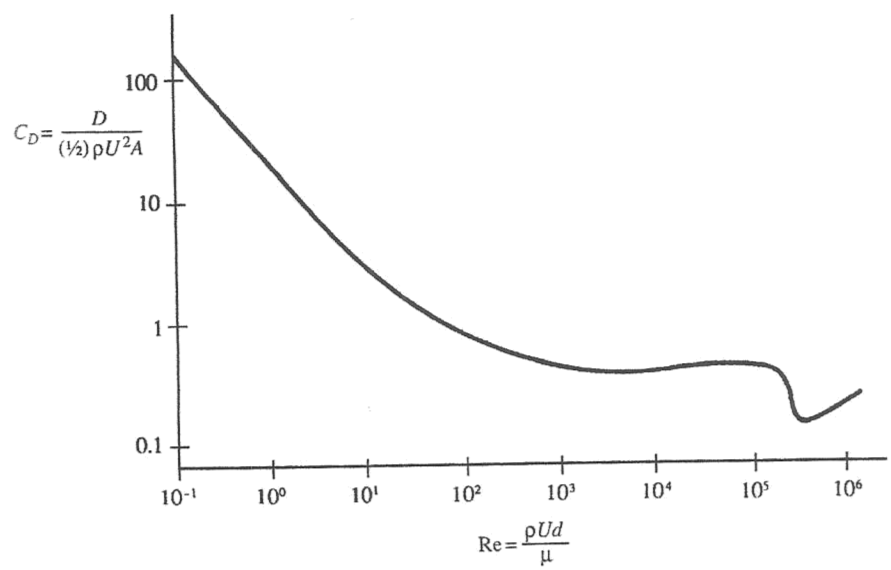


FIGURE 4.23 Coefficient of drag C_D for a sphere vs. the Reynolds number Re based on sphere diameter. At low Reynolds number $C_D \sim 1/Re$, and above $Re \sim 10^3$, $C_D \sim$ constant (except for the dip between $Re = 10^5$ and 10^6). These behaviors (except for the dip) can be explained by simple dimensional reasoning. The reason for the dip is the transition of the laminar boundary layer to a turbulent one, as explained in Chapter 10.