The Kutta – Joukowski lift theorem:

Since we know the tangential component of velocity at any point on the cylinder (and the radial component of velocity is zero), we can find the pressure field over the surface of the cylinder from Bernoulli’s equation:

\[
\frac{p}{\rho} + \frac{1}{2} \left( v_r^2 + v_\theta^2 \right) = \frac{p_\infty}{\rho} + \frac{1}{2} U_\infty^2
\]

Therefore:

\[
p = p_\infty + \frac{1}{2} \rho \left[ U_\infty^2 - (v_r^2 + v_\theta^2) \right] = p_\infty + \frac{1}{2} \rho \left[ U_\infty^2 - \left( -2U_\infty \sin \theta + \frac{\Gamma}{2\pi R} \right)^2 \right]
\]

\[
= \left( p_\infty + \frac{1}{2} \rho U_\infty^2 - \frac{\rho \Gamma^2}{8\pi^2 R^2} \right) - 2\rho U_\infty^2 \sin^2 \theta + \frac{\rho U_\infty \Gamma}{\pi R} \sin \theta = A + B \sin \theta + C \sin^2 \theta
\]

And

\[
A = \left( p_\infty + \frac{1}{2} \rho U_\infty^2 - \frac{\rho \Gamma^2}{8\pi^2 R^2} \right) \quad B = \frac{\rho U_\infty \Gamma}{\pi R} \quad C = -2\rho U_\infty^2
\]

Calculation of Lift:

\[
L = -RB \int_0^{2\pi} (A + B \sin \theta + C \sin^2 \theta) \sin \theta \, d\theta
\]

\[
= -RB \int_0^{2\pi} (A \sin \theta + B \sin^2 \theta + C \sin^3 \theta) \, d\theta
\]

\[
= -RB \int_0^{2\pi} B \sin^2 \theta \, d\theta = -BRb \left( \frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right) \bigg|_0^{2\pi} = -BRb \pi = -\rho U_\infty \Gamma b
\]

This is an important result. It says that clockwise vortices (negative numerical values of \( \Gamma \)) will produce positive lift that is proportional to \( \Gamma \) and the free stream speed with direction 90 degrees from the stream direction rotating opposite to the circulation. Kutta and Joukowski generalized this result to lifting flow over airfoils. Equation \( L = -\rho U_\infty \Gamma b \) is known as the Kutta-Joukowski theorem.
Drag: \[ L = -Rb \int_0^{2\pi} (A + B \sin \theta + C \sin^2 \theta) \cos \theta \, d\theta \]
\[ = -Rb \int_0^{2\pi} (A \cos \theta + B \sin \theta \cos \theta + C \sin^2 \theta \cos \theta) \, d\theta \]
\[ = -Rb \left( A \sin \theta + \frac{1}{2} B \cos^2 \theta + \frac{1}{3} C \sin^3 \theta \right) \bigg|_0^{2\pi} = 0 \]

This contrast between potential flow theory and realistic drag is the d\text{Á}lembert Paradox, which was solved by Prandtl (1904) with his boundary layer theory, i.e. viscous effects are always important very close to the body where the no slip boundary condition must be satisfied and large shear stress exists which contributes the drag.

**Lift for rotating Cylinder:**

We know that \[ L = -\rho U_\infty \Gamma b, \] so the lift coefficient is

\[ C_L = \frac{L}{\frac{1}{2} \rho U_\infty^2 2bR} = -\frac{\Gamma}{U_\infty R} \]

Experiments have been performed that simulate the previous flow by rotating a circular cylinder in a uniform stream. In this case \( v_\theta = R \omega \) which is due to no slip boundary condition.

- Lift is quite high but not as large as theory (due to viscous effect i.e. flow separation)
- Note drag force is also fairly high

Flettner (1924) used rotating cylinder to produce forward motion.
4 Method of Images

The method of image is used to model “slip” wall effects by constructing appropriate image singularity distributions.

2D Plane Boundaries:

\[ Q = \frac{m}{2} \ln \left[ x^2 + (y - a)^2 \right] + \frac{m}{2} \ln \left[ x^2 + (y + 1)^2 \right]. \]

Similar results can be obtained for dipoles and vortices

Multiple Boundaries:

The method can be extended for multiple boundaries by using successive images.

airfoil in ground effect with image airfoil of opposite circulation