4.75 Given the following steady axisymmetric stream function:

\[ \psi = \frac{B}{2} \left( r^2 - \frac{r^4}{2R^2} \right), \]  
where \( B \) and \( R \) are constants

valid in the region \( 0 \leq r \leq R \) and \( 0 \leq z \leq L \). (a) What are the dimensions of the constant \( B \)?

(b) Show whether this flow possesses a velocity potential and, if so, find it. (c) What might this flow represent? [HINT: Examine the axial velocity \( v_z \)]

Solution: (a) From the definition of \( \psi(r, z) \) in Eqs. (4.105), the dimensions of \( \psi \) are \( \{L^3/T\} \). Thus \( B \) has velocity dimensions, \( \{B\} = \{L/T\} \). Ans.(a)

(b) To test for irrotationality, first find the velocity components from Eqs. (4.106):

\[ v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} = 0; \quad v_z = \frac{1}{r} \frac{\partial \psi}{\partial r} = \frac{1}{r} \frac{B}{2} \left( 2r - \frac{4r^3}{2R^2} \right) = \frac{B}{r} \left( 1 - \frac{r^2}{R^2} \right) \]

Now evaluate the curl of the velocity, which has only one possible non-zero component. From Appendix D, Eq. (D.11),

\[ 2\omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} = 0 - \frac{2Br}{R^2} \neq 0 \quad \text{Rotational, } \phi \text{ does not exist. Ans.(b)} \]

(c) The interpretation of the flow follows immediately from the velocity components. The velocity profile is a paraboloid of revolution and represents Poiseuille pipe flow, Eq. (1.137). Ans.(c)