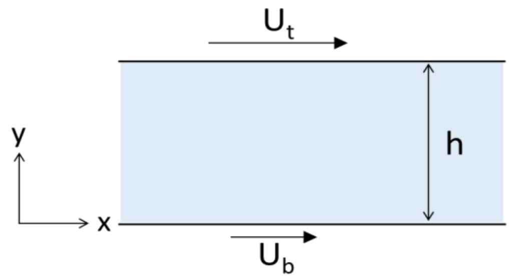
Name: ------------------ **Quiz 5** **Time: 20 minutes**

ME:5160 **Fall 2023**

The exam is closed book and closed notes.

Consider steady, two-dimensional, incompressible viscous flow of a Newtonian fluid between parallel plates a distant h apart, as shown in the Figure below. The upper plate moves at constant velocity U_t while the lower plate moves at constant velocity U_b. The pressure varies linearly in the x direction ($p(x) = Cx$). If the plates are very wide and very long and the flow is parallel ($v = w = 0$), neglect gravity and find the velocity distribution between the plates using continuity and momentum equations.



Incompressible Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Incompressible Navier-Stokes Equations in Cartesian Coordinates:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

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Solution:

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} + \underbrace{0}_{(3)} + \underbrace{0}_{(3)} = 0 \qquad \textbf{(+1.5)}$$

$$\frac{\partial u}{\partial x} = 0: \text{ the flow is fully} - \text{developed } \Longrightarrow u = u(y) \text{ only } \quad (5) \qquad \textbf{(+1.5)}$$

x-momentum:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\underbrace{0}_{(4)} - \frac{\partial p}{\partial x} + \mu\left(\underbrace{0}_{(5)} + \frac{\partial^2 u}{\partial y^2} + \underbrace{0}_{(2)}\right) = \rho\left(\underbrace{0}_{(1)} + \underbrace{0}_{(5)} + \underbrace{0}_{(3)} + \underbrace{0}_{(3)}\right) \qquad \textbf{(+2)}$$

$$\frac{\partial p}{\partial x} = C$$

$$\Longrightarrow \mu\frac{d^2 u}{dy^2} = C \qquad \textbf{(+2)}$$

Integrate twice:

$$u = \frac{1}{\mu}C\frac{y^2}{2} + C_1 y + C_2 \qquad \textbf{(+1)}$$

Boundary conditions:

$$\text{at } y = 0: u(y) = U_b \quad \rightarrow \quad C_2 = U_b$$

$$\text{at } y = h: u(y) = U_t \quad \rightarrow \quad C_1 = \frac{U_t - U_b}{h} - \frac{Ch}{2\mu}$$

Replace and find:

$$u = \frac{1}{\mu}C\frac{y^2}{2} + \left(\frac{U_t - U_b}{h} - \frac{Ch}{2\mu}\right)y + U_b \qquad \textbf{(+2)}$$