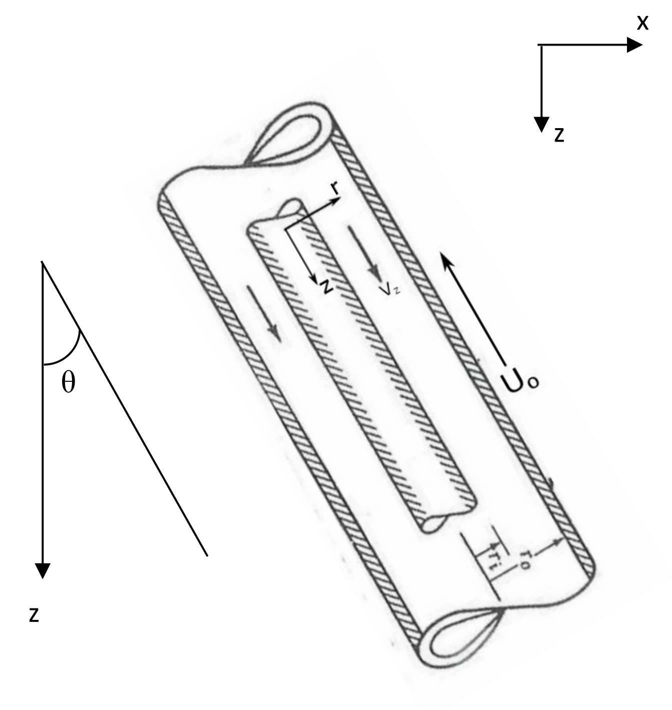
Name: ------------------ **Exam 2** **Time: 50 minutes**

ME:5160 **Fall 2023**

The exam is closed book and closed notes.

1. An incompressible Newtonian fluid flows steadily between two infinitely long, concentric cylinders, as shown in the Figure. The cylinder is inclined with an angle θ with respect to the z axis. The outer cylinder moves with velocity $-U_o$, and the inner cylinder is fixed. The pressure gradient in the axial direction is constant and the effect of gravity cannot be neglected. Assume that the velocities in the radial and tangential directions are zero. (a) Simplify the continuity and momentum equation and determine the velocity field. (b) Determine the shear stress at the inner wall.



The equations of motion of an incompressible Newtonian fluid with constant density and viscosity in cylindrical coordinates (r, θ, z) with velocity components (v_r, v_θ, v_z):

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

r-momentum:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]$$

θ-momentum:

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

z-momentum:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

Shear stress: $\tau_w = \mu\frac{\partial v_z}{\partial r}$

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2. The thrust F of a propeller is generally thought to be a function of its diameter D and angular velocity Ω, the forward speed V, and the density ρ and viscosity μ of the fluid. (a) Find suitable dimensionless groups for this problem and write the functional relationship between them.
(b) Some experiments indicated for D=0.2m, V=20 m/s, and Ω=100rps the thrust is 100N. Estimate the thrust and rps on a propeller with D_m=0.6m, V_m=30m/s if the flow conditions are the same. ($\rho = 1kg/m^3, \mu = 10^{-5}kg/ms$)

Hint: Use ρ, V, D as repeated variables.

Quantity	Symbol	Dimensions $MLT\Theta$	Dimensions $FLT\Theta$
Length	L	L	L
Area	A	L^2	L^2
Volume	$\mathcal{V}$	L^3	L^3
Velocity	V	LT^{-1}	LT^{-1}
Acceleration	dV/dt	LT^{-2}	LT^{-2}
Speed of sound	a	LT^{-1}	LT^{-1}
Volume flow	Q	L^3T^{-1}	L^3T^{-1}
Mass flow	$\dot{m}$	MT^{-1}	FTL^{-1}
Pressure, stress	p, σ, τ	$ML^{-1}T^{-2}$	FL^{-2}
Strain rate	$\dot{\epsilon}$	T^{-1}	T^{-1}
Angle	θ	None	None
Angular velocity	ω, Ω	T^{-1}	T^{-1}
Viscosity	μ	$ML^{-1}T^{-1}$	FTL^{-2}
Kinematic viscosity	ν	L^2T^{-1}	L^2T^{-1}
Surface tension	Υ	MT^{-2}	FL^{-1}
Force	F	MLT^{-2}	F
Moment, torque	M	ML^2T^{-2}	FL
Power	P	ML^2T^{-3}	FLT^{-1}
Work, energy	W, E	ML^2T^{-2}	FL
Density	ρ	ML^{-3}	FT^2L^{-4}
Temperature	T	Θ	Θ
Specific heat	c_p, c_v	$L^2T^{-2}\Theta^{-1}$	$L^2T^{-2}\Theta^{-1}$
Specific weight	γ	$ML^{-2}T^{-2}$	FL^{-3}
Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$	$FT^{-1}\Theta^{-1}$
Thermal expansion coefficient	β	Θ^{-1}	Θ^{-1}

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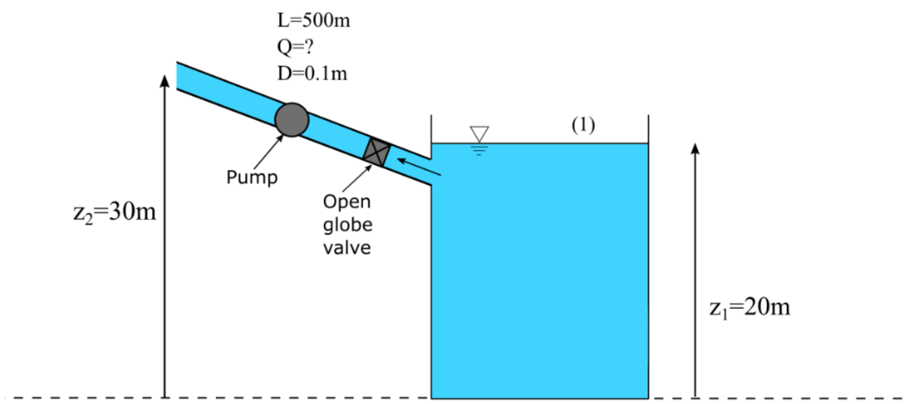
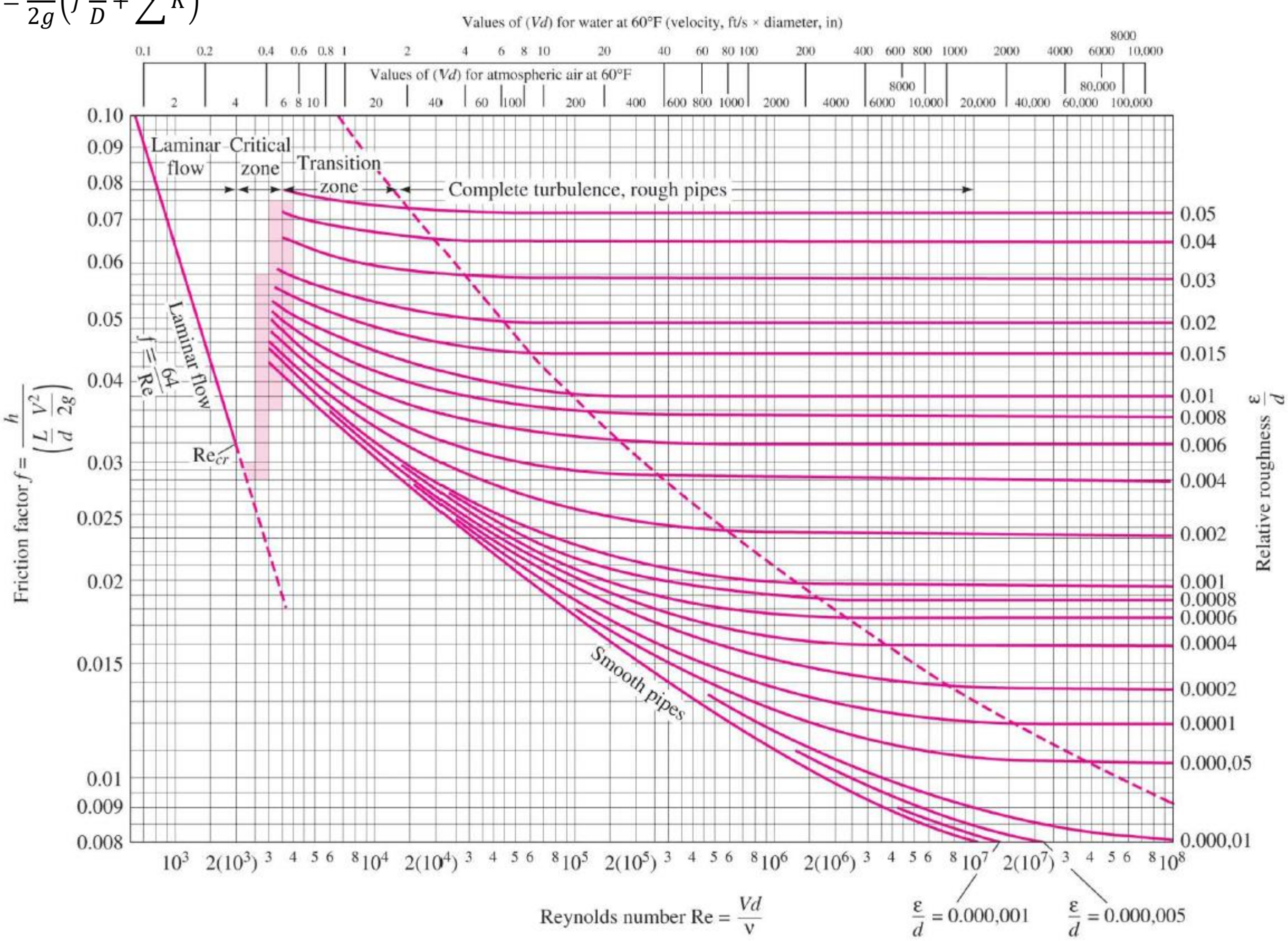
3. Water (ρ=1000kg/m^3, μ=0.001kg/ms) needs to be pumped using a commercial pump. The iron pipe (ε=0.01mm) is 500m long, with a diameter of 0.1m. (a) Determine the flowrate in the pipe if the pump is providing h_p =20m and consider the minor losses due to the open globe valve ($K = 6.9$) and the sharp entrance ($K = 0.5$) starting with an initial value of f =0.015. (b) Assuming the same friction factor obtained in a), what would be the value of the pump head required to achieve a flow rate ten times larger than that obtained in in a)?



Energy equation $\left(\frac{P}{\rho g}+\frac{V^2}{2g}+z\right)_1 = \left(\frac{P}{\rho g}+\frac{V^2}{2g}+z\right)_2 + h_f + h_t - h_p$

$h_f = \frac{V^2}{2g}\left(f\frac{L}{D}+\sum K\right)$



Solution 1:

ASSUMPTIONS:
1. Steady flow ($\frac{\partial}{\partial t}$=0)
2. Incompressible flow (ρ=constant)
3. Purely axial flow (vr=vθ=0)
4. Circumferentially symmetric flow, so properties do not vary with θ ($\frac{\partial}{\partial \theta}$=0)
5. Constant pressure gradient ($\partial p/\partial z$ =k)
6. Gravity (gz=gcodθ)

(a)

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)+\frac{\partial}{\partial z}(v_z)=0$$

$$0(3)+0(3)+\frac{\partial v_z}{\partial z}=0 \qquad \textbf{(+1.5)}$$

z-momentum:

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right] \quad \textbf{(+0.5)}$$

$$\rho(0(1)+0(3)+0(3{,}4)+0(continuity))=\rho g sin\theta-k+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+0(4)+0(continuity)\right]\textbf{(+1.5)}$$

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=-\rho g cos\theta+k$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=\frac{-\rho g cos\theta+k}{\mu}r$$

$$r\frac{\partial v_z}{\partial r}=\frac{-\rho g cos\theta+k}{\mu}\frac{r^2}{2}+c_1$$

$$\frac{\partial v_z}{\partial r}=\frac{-\rho g cos\theta+k}{\mu}\frac{r}{2}+\frac{c_1}{r}$$

$$v_z(r)=\frac{-\rho g cos\theta+k}{\mu}\frac{r^2}{4}+c_1\ln r+c_2 \qquad \textbf{(+2)}$$

Boundary conditions:

$$v_z(r_i)=0,\qquad v_z(r_o)=-U_o \qquad \textbf{(+1)}$$

Apply BCs:

$$v_z(r_i) = 0 = \frac{-\rho g cos\theta + k}{\mu}\frac{r_i^2}{4} + c_1 \ln r_i + c_2$$

$$v_z(r_o) = -U_o = \frac{-\rho g cos\theta + k}{\mu}\frac{r_o^2}{4} + c_1 \ln r_o + c_2$$

Subtract:

$$U_o = \frac{\rho g cos\theta - k}{4\mu}\left(r_o^2 - r_i^2\right) + c_1 \ln\frac{r_i}{r_o}$$

$$c_1 = \frac{U_o - \frac{\rho g cos\theta - k}{4\mu}\left(r_o^2 - r_i^2\right)}{\ln\frac{r_i}{r_o}}$$

Add:

$$-U_0 = \frac{-\rho g cos\theta + k}{4\mu}\left(r_o^2 + r_i^2\right) + c_1 \ln r_i \ln r_o + 2c_2$$

$$c_2 = \frac{-U_0 - c_1 \ln r_i \ln r_o + \frac{\rho g cos\theta - k}{4\mu}\left(r_o^2 + r_i^2\right)}{2}$$ **(+1.5)**

Velocity distribution:

$$v_z(r) = \frac{-\rho g cos\theta + k}{\mu}\frac{r^2}{4} + \frac{U_o - \frac{\rho g cos\theta - k}{4\mu}\left(r_o^2 - r_i^2\right)}{\ln\frac{r_i}{r_o}}\ln r + \frac{-U_0 - c_1 \ln r_i \ln r_o + \frac{\rho g cos\theta - k}{4\mu}\left(r_o^2 + r_i^2\right)}{2}$$ **(+1.5)**

Shear stress:

$$\tau_w(r) = \mu\frac{\partial v_z}{\partial r} = \frac{-\rho g cos\theta + k}{2}r + \mu\frac{U_o - \frac{\rho g cos\theta - k}{4\mu}\left(r_o^2 - r_i^2\right)}{\ln\frac{r_i}{r_o}}\frac{1}{r}$$

(+0.5)

$$\tau_w(r)|_{r=r_i} = \mu\left.\frac{\partial v_z}{\partial r}\right|_{r=r_i} = \frac{-\rho g cos\theta + k}{2}r_i + \mu\frac{U_o - \frac{\rho g cos\theta - k}{4\mu}\left(r_o^2 - r_i^2\right)}{\ln\frac{r_i}{r_o}}\frac{1}{r_i}$$

Solution 2:

(a)

$$F = f(D, \Omega, V, \rho, \mu) \quad \textbf{(+1)}$$

$$n = 6$$

$$F = \{MLT^{-2}\}; \quad D = \{L\}; \quad \Omega = \{T^{-1}\}$$

$$V = \{LT^{-1}\}; \quad \rho = \{ML^{-3}\}; \quad \mu = \{ML^{-1}T^{-1}\}$$

$$j = 3 \rightarrow k = n - j = 6 - 3 = 3 \quad \textbf{(+1)}$$

The repeating variables are ρ, *V*, *D*. Using the Pi theorem, we find the three Pi groups:

$$\Pi_1 = \rho^a V^b D^c F = \{(ML^{-3})^a (LT^{-1})^b (L)^c (MLT^{-2})\} = \{M^0 L^0 T^0\} \quad \textbf{(+0.5)}$$

a= -1, b= -2, c= -2

$$\Pi_1 = \frac{F}{\rho V^2 D^2} \quad \textbf{(+1)}$$

$$\Pi_2 = \rho^a V^b D^c \Omega = \{(ML^{-3})^a (LT^{-1})^b (L)^c (T^{-1})\} = \{M^0 L^0 T^0\} \quad \textbf{(+0.5)}$$

a= 0, b= -1, c= 1

$$\Pi_2 = \frac{\Omega D}{V} \quad \textbf{(+1)}$$

$$\Pi_3 = \rho^a V^b D^c \mu = \{(ML^{-3})^a (LT^{-1})^b (L)^c (ML^{-1}T^{-1})\} = \{M^0 L^0 T^0\} \quad \textbf{(+0.5)}$$

a= -1, b= -1, c= 1

$$\Pi_3 = \frac{\mu}{\rho V D} \rightarrow \frac{\rho V D}{\mu} = Re \quad \textbf{(+1)}$$

Thus, the arrangement of the dimensionless variables is:

$$\Pi_1 = f(\Pi_2, \Pi_3)$$

$$\frac{F}{\rho V^2 D^2} = f\left(\frac{\Omega D}{V}, \frac{\rho V D}{\mu}\right) \quad \textbf{(+1)}$$

(b) D=0.2m, V=20 m/s, and *Ω*=100rps the thrust is 100N:

$$\frac{\Omega D}{V} = 100 * \frac{0.2}{20} = 1 \quad \textbf{(+1.5)}$$

$$\frac{F}{\rho V^2 D^2} = \frac{100}{1 * 20^2 * 0.2^2} = 6.25$$

D_m=0.6m, V_m=30m/s:

$$\frac{\Omega_m D_m}{V_m} = 1 \rightarrow \Omega_m = \frac{V_m}{D_m} = \frac{30}{0.6} = 50rps$$ **(+1)**

$$\frac{F_m}{\rho V_m^2 D_m^2} = 6.25 \rightarrow F_m = 6.25\rho V_m^2 D_m^2 = 6.25 * 1 * 30^2 * 0.6^2 = 2025N$$

Solution 3:

(a) Apply energy equation between (1) and (2):

$$\left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_1 = \left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_2 + h_f$$ **(+2)**

$$(0 + 0 + z)_1 = \left(0 + \frac{V^2}{2g} + z\right)_2 + h_f - h_p$$

$$z_1 - z_2 = \frac{V^2}{2g}\left(1 + f\frac{L}{D} + \sum K_i\right) - h_p$$

$$-10 = \frac{V^2}{2g}\left(1 + f\frac{500}{0.1} + 6.9 + 0.5\right) - 20$$ **(+1)**

$$\frac{20g}{\left(1 + f\frac{500}{0.1} + 6.9 + 0.5\right)} = V^2$$

$$V = \sqrt{\frac{196.2}{8.4 + 5000f}}$$ **(+1)**

Calculate ε/d:

$$\frac{\varepsilon}{d} = \frac{0.01 \times 10^{-3}}{0.1} = 0.0001$$ **(+1)**

Assume $f = 0.015$

$$V = 1.53m/s$$

Calculate Re_d: **(+1.5)**

$$Re_d = \frac{\rho V d}{\mu} = 153{,}379$$

Enter Moody diagram and select $f = 0.022$

$$V = \sqrt{\frac{196.2}{8.4 + 5000f}} = 1.29m/s$$ **(+1)**

Calculate Re_d:

$$Re_d = \frac{\rho V d}{\mu} = 129{,}000$$

The process has converged.
Calculate the flowrate:

$$Q = VA = 1.29 \times \frac{\pi d^2}{4} = 0.01 m^3/s \quad \textbf{(+1.5)}$$

(b)

$$-h_p = \frac{V^2}{2g}\left(1 + 0.022\frac{500}{0.1} + 6.9 + 0.5\right) - 20$$

$$\sqrt{\frac{(20 - h_p)2g}{\left(1 + 0.022\frac{500}{0.1} + 6.9 + 0.5\right)}} = V \quad \textbf{(+0.5)}$$

$$Q = VA = V \times \frac{\pi d^2}{4} = \frac{\pi d^2}{4}\sqrt{\frac{(20 - h_p)2g}{\left(1 + 0.022\frac{500}{0.1} + 6.9 + 0.5\right)}}$$

$$Q = 10Q_a = 0.1 m^3/s$$

$$0.1 = \frac{\pi d^2}{4}\sqrt{\frac{(20 - h_p)2g}{\left(1 + 0.022\frac{500}{0.1} + 6.9 + 0.5\right)}}$$

$$\left(\frac{0.4}{\pi d^2}\right)^2 = \frac{(20 - h_p)2g}{\left(1 + 0.022\frac{500}{0.1} + 6.9 + 0.5\right)}$$

$$h_p = 20 - \frac{\left(1 + 0.022\frac{500}{0.1} + 6.9 + 0.5\right)\left(\frac{0.4}{\pi d^2}\right)^2}{2g}$$

$$h_p = 20 - \frac{118.4 \times 162.28}{2 \times 9.81} = 959.3m \quad \textbf{(+0.5)}$$