**Name: ------------------** **Quiz 5** **Time: 20 minutes**

**ME:5160** **Fall 2022**

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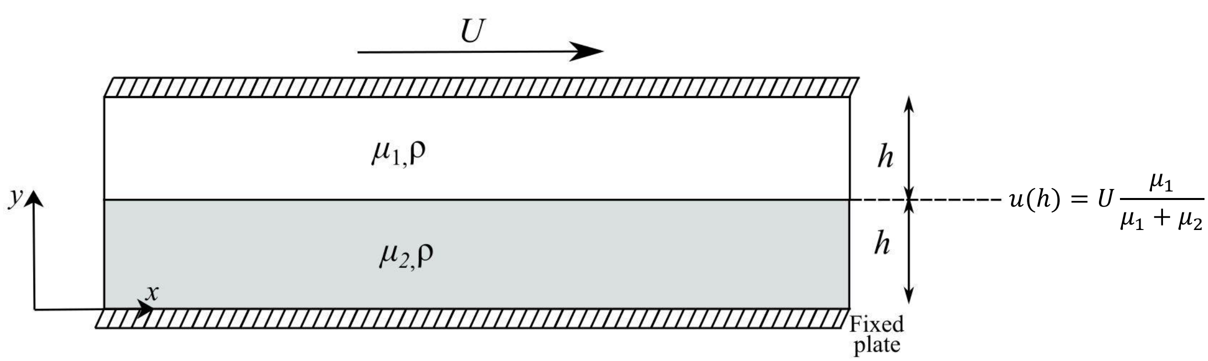
**The exam is closed book and closed notes.**

Two immiscible, incompressible, viscous fluids having the same densities, but different viscosities are contained between two infinite, horizontal, parallel plates, as shown in the Figure. Pressure gradient is negligible, and the flow is fully developed ($\frac{\partial}{\partial x} = 0$). The bottom plate is fixed and the upper plate moves with a constant velocity $U$.

The velocity at the interface is equal to $u(h) = U\frac{\mu_1}{\mu_1+\mu_2}$ for both fluids.

Determine (a) the velocity field in both fluids and (b) evaluate the shear stress in both fluids at y=h. (c) What can you notice for the shear stress value?

**Hint**: Solve NS for each fluid, separately, using appropriate BCs for the velocity at the upper, lower wall and fluid interface.




**Incompressible continuity equation:**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

**Incompressible Navier-Stokes Equations in Cartesian Coordinates:**

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(u\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\left(v\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\left(w\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

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## Solution:

Assumptions:

1. Steady flow ($\frac{\partial}{\partial t} = 0$)
2. 2D flow (w=0)
3. Incompressible flow ($\rho$ =constant)
4. Fully developed flow ($\frac{\partial u}{\partial x} = 0$)
5. Pressure gradient is negligible (∇P=0)

(a) x-momentum for fluid 2:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}\right) = \rho\left(u_2\frac{\partial u_2}{\partial t} + u_2\frac{\partial u_2}{\partial x} + v_2\frac{\partial u_2}{\partial y}\right) \quad \textbf{(+1)}$$

$$\frac{\partial^2 u_2}{\partial y^2} = 0 \quad \textbf{(+1.5)}$$

$$u_2(y) = Ay + B \quad \textbf{(+1)}$$

Apply BCs:

$$u_2(0) = 0 \quad \textbf{(+0.5)}$$

$$u_2(h) = U\frac{\mu_1}{\mu_1 + \mu_2} \quad \textbf{(+0.5)}$$

$$A = \frac{U}{h}\frac{\mu_1}{\mu_1 + \mu_2} \quad \textbf{(+0.5)}$$

$$B = 0 \quad \textbf{(+0.5)}$$

$$u_2(y) = \frac{U}{h}\frac{\mu_1}{\mu_1 + \mu_2}y \quad \textbf{(+0.5)}$$

x-momentum for fluid 1:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right) = \rho\left(u_1\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y}\right)$$

$$\frac{\partial^2 u_1}{\partial y^2} = 0$$

$$u_1(y) = Cy + D$$

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Apply BCs:

$$u_1(h) = U\frac{\mu_1}{\mu_1 + \mu_2}$$ **(+0.5)**

$$u_1(2h) = U$$ **(+0.5)**

$$C = \frac{U}{h}\frac{\mu_2}{\mu_1 + \mu_2}$$ **(+0.5)**

$$D = U\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}$$ **(+0.5)**

$$u_1(y) = \frac{U}{h}\frac{\mu_2}{\mu_1 + \mu_2}y + U\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}$$ **(+0.5)**

(b) Evaluate shear stress at y=h:

$$\tau_1(h) = \mu_1\frac{\partial u_1}{\partial y} = \mu_1\frac{U}{h}\frac{\mu_2}{\mu_1 + \mu_2}$$ **(+0.5)**

$$\tau_2(h) = \mu_2\frac{\partial u_2}{\partial y} = \mu_2\frac{U}{h}\frac{\mu_1}{\mu_1 + \mu_2}$$ **(+0.5)**

The shear stress value is the same. Therefore, shear stress is continuous across the two fluids. **(+0.5)**