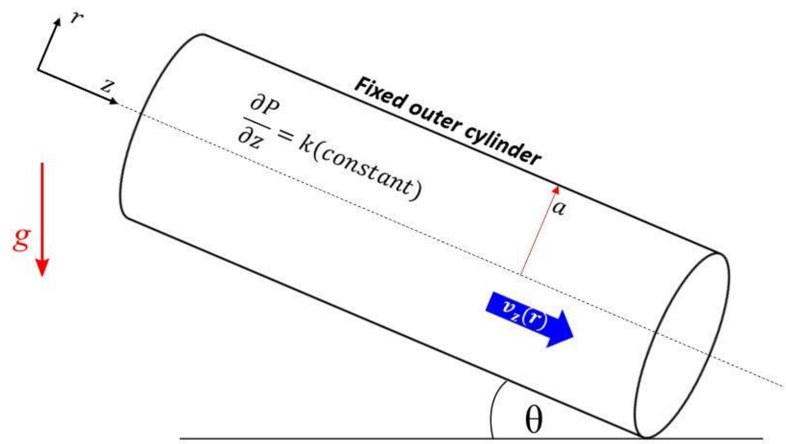
Name: ------------------ **Quiz 7** **Time: 20 minutes**

ME:5160 **Fall 2022**

The exam is closed book and closed notes.

The viscous oil shown in figure is set into steady motion by a constant pressure gradient $\partial p/\partial z$. The radius of the pipe is a. Assume fully developed flow, constant density, circumferentially symmetric flow, and a purely axial fluid motion. (a) Simplify the governing equations with these given conditions and show that v_z does is independent from z. (b) Apply appropriate boundary conditions and derive the fluid velocity distribution $v_z(r)$. (c) Calculate the wall shear stress at the pipe wall.



Boundary condition Hint

- At the pipe wall, the velocity is zero
- At the pipe center, the velocity gradient should be zero

Wall shear stress Hint

- $\tau_{wall} = \mu \frac{\partial v_z}{\partial y}\Big|_{y=0} = -\mu \frac{\partial v_z}{\partial r}\Big|_{r=a}, \quad y = a - r$ where $a: Radius\ of\ pipe$

The equations of motion of an incompressible Newtonian fluid with constant density and viscosity in cylindrical coordinates (r, θ, z) with velocity components (v_r, v_θ, v_z):

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

r-momentum:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]$$

θ-momentum:

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

z-momentum:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

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Solution:

ASSUMPTIONS:
1. Steady flow ($\frac{\partial}{\partial t}$=0)
2. Incompressible flow (ρ=constant)
3. Purely axial flow (vr=vθ=0)
4. Circumferentially symmetric flow, so properties do not vary with θ ($\frac{\partial}{\partial \theta}$=0)
5. Constant pressure gradient ($\partial p/\ \partial$z =k)
6. gz = $g \sin\theta$

(a) Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

$$0(3) + 0(3) + \frac{\partial v_z}{\partial z} = 0$$ **(+1.5)**

(b) z-momentum:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$ **(+0.5)**

$$\rho(0(1) + 0(3) + 0(3{,}4) + 0(cont)) = \rho g\sin\theta - k(5) + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + 0(4) + 0(cont)\right]$$ **(+1.5)**

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = k - \rho g\sin\theta$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{k - \rho g\sin\theta}{\mu}r$$ **(+1)**

$$r\frac{\partial v_z}{\partial r} = \frac{k - \rho g\sin\theta}{2\mu}r^2 + c_1$$

$$\frac{\partial v_z}{\partial r} = \frac{k - \rho g\sin\theta}{2\mu}r + \frac{c_1}{r}$$

$$v_z(r) = \frac{k - \rho g\sin\theta}{4\mu}r^2 + c_1\ln r + c_2$$ **(+2)**

Boundary conditions:

$$v_z(a) = 0, \qquad \left.\frac{\partial v_z}{\partial r}\right|_{r=0} = 0$$ **(+1.5)**

Apply BCs:

$$v_z(a) = 0 = \frac{k - \rho g \sin\theta}{4\mu} a^2 + c_1 \ln a + c_2$$

$$\left.\frac{\partial v_z}{\partial r}\right|_{r=0} = 0 = \frac{k - \rho g \sin\theta}{2\mu}(0) + \frac{c_1}{(0)}$$

$$c_1 = 0$$ **(+0.5)**

$$c_2 = \frac{\rho g \sin\theta - k}{4\mu} a^2$$ **(+0.5)**

Velocity distribution:

$$v_z(r) = \frac{k - \rho g \sin\theta}{4\mu} r^2 + \frac{\rho g \sin\theta - k}{4\mu} a^2 = \frac{k - \rho g \sin\theta}{4\mu}(r^2 - a^2)$$ **(+0.5)**

(c) Shear stress:

$$\tau_{wall} = -\mu \left.\frac{\partial v_z}{\partial r}\right|_{r=a} = \frac{\rho g \sin\theta - k}{2} a$$ **(+0.5)**