Chapter 6: Viscous Flow in Ducts

6.4 Turbulent Flow in Pipes and Channels using mean-velocity correlations

1. *Smooth circular pipe*

Recall laminar flow exact solution

$$f = \frac{8\tau_w}{\rho u_{ave}^2} = 64 / \mathrm{Re}_d \qquad \mathrm{Re}_d = \frac{u_{ave} d}{\upsilon} \le 2000$$

A turbulent-flow “approximate” solution can be obtained simply by computing u_{ave} based on log law.

$$\frac{u}{u^*} = \frac{1}{\kappa}\ln\frac{yu^*}{\nu} + B$$

Where

$$u = u(y); \kappa = 0.41; B = 5; \; u^* = \sqrt{\tau_w / \rho}; y = R - r$$

$$V = u_{ave} = \frac{Q}{A} = \frac{1}{\pi R^2}\int_0^R u^*\left[\frac{1}{\kappa}\ln\frac{yu^*}{\nu} + B\right] 2\pi r \; dr$$

$$= \frac{1}{2}u^*\left(\frac{2}{\kappa}\ln\frac{Ru^*}{\nu} + 2B - \frac{3}{\kappa}\right)$$

Or:

$$\frac{V}{u^*} = 2.44 \ln \frac{Ru^*}{\nu} + 1.34$$

$$\begin{aligned} f^{-1/2} &= 1.99 \log[\mathrm{Re}_d\, f^{1/2}] - 1.02 \\ &= 2 \log[\mathrm{Re}_d\, f^{1/2}] - 0.8 \end{aligned}$$

EFD Adjusted constants

f only drops by a factor of 5 over $10^4 < \mathrm{Re} < 10^8$

Since f equation is implicit, it is not easy to see dependency on ρ, μ, V, and D

$$f\,(pipe) = 0.316\,\mathrm{Re}_D^{-1/4}$$

$4000 < \mathrm{Re}_D < 10^5$
Blasius (1911) power law curve fit to data

$$h_f = \frac{\Delta p}{\gamma} = f\,\frac{L}{D}\,\frac{V^2}{2g}$$

Turbulent Flow: $\boxed{\Delta p = 0.158 L \rho^{3/4} \mu^{1/4} D^{-5/4} V^{7/4}}$

Nearly linear

Only slightly with μ

Drops weakly pipe size

Near quadratic (as expected)

$$= 0.241 L \rho^{3/4} \mu^{1/4} D^{-4.75} Q^{1.75}$$

Laminar flow: $\boxed{\Delta p = 8 \mu L Q / \pi R^4}$

Δp (turbulent) decreases more sharply with D than Δp (laminar) for same Q; therefore, increase D for smaller Δp. 2D decreases Δp by 27 for same Q.

$$\frac{u_{\max}}{u^*} = \frac{u(r=0)}{u^*} = \frac{1}{\kappa}\ln\frac{Ru^*}{\upsilon} + B$$

Combine with

$$\frac{V}{u^*} = \frac{1}{\kappa}\ln\frac{Ru^*}{\upsilon} + B - \frac{3}{2\kappa}$$

$$\Rightarrow \frac{V}{u^*} = \frac{u_{\max}}{u^*} - \frac{3}{2\kappa} \Rightarrow V = u_{\max} - \frac{3u^*}{2\kappa} \Rightarrow \frac{u_{\max}}{V} = 1 + \frac{3u^*}{2\kappa V}$$

Also

$$\tau_w = \rho u^{*2} \ \ and \ \ f = \frac{\tau_w}{1/8\rho V^2} \Rightarrow f = \frac{\rho u^{*2}}{1/8\rho V^2} \Rightarrow \frac{u^*}{V} = \sqrt{f/8}$$

$$\Rightarrow \frac{u_{\max}}{V} = 1 + \frac{3u^*}{2\kappa V} = 1 + \frac{3}{2\kappa}\sqrt{f/8} = 1 + 1.3\sqrt{f}$$

Or:

For Turbulent Flow: $\boxed{\frac{V}{u_{\max}} = \left(1 + 1.3\sqrt{f}\right)^{-1}}$

Re_D	4K	10^4	10^6	10^8
	.794	.814	.877	.909

Recall laminar flow:

$\boxed{V/u_{\max} = 0.5}$

TABLE 10.1 EXPONENTS FOR POWER-LAW EQUATION AND RATIO OF MEAN TO MAXIMUM VELOCITY

Re →	4×10^3	2.3×10^4	1.1×10^5	1.1×10^6	3.2×10^6
m →	$\frac{1}{6.0}$	$\frac{1}{6.6}$	$\frac{1}{7.0}$	$\frac{1}{8.8}$	$\frac{1}{10.0}$
$\overline{V}/V_{max}$ →	0.791	0.807	0.817	0.850	0.865

SOURCE: Schlichting (36). Used with permission of the McGraw-Hill Companies.

Power law fit to velocity profile:

$$\frac{\overline{u}}{\overline{u}_{\max}} = \left(1 - \frac{r}{r_o}\right)^m \qquad \text{m} = \text{m(Re)}$$

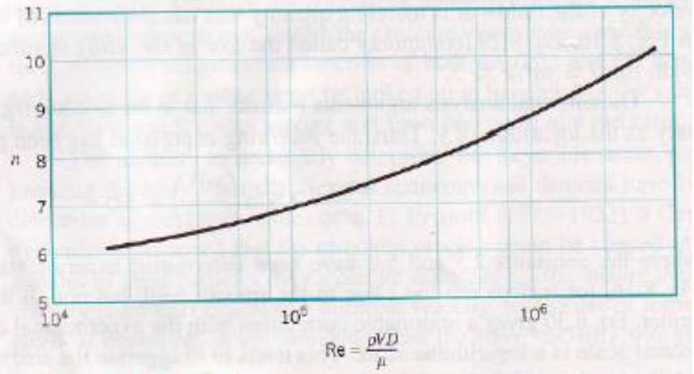


FIGURE 8.17 Exponent, n, for power-law velocity profiles. (Adapted from Ref. 1.)

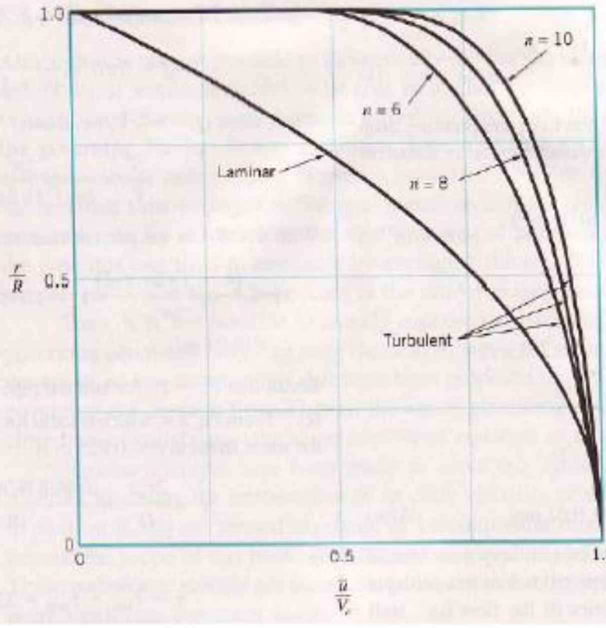


FIGURE 8.18 Typical laminar flow and turbulent flow velocity profiles.

2. *Turbulent Flow in Rough circular pipe*

$$U^+ = f(y^+, k^+) \qquad\qquad f = f(\mathrm{Re}_d, k/d)$$

$$U^+ = \frac{1}{\kappa}\ln y^+ + B - \Delta B(k^+) \longleftarrow$$ Log law shifts downward

which leads to three roughness regimes:

1. $k^+ < 4$ hydraulically smooth
2. $4 < k^+ < 60$ transitional roughness (Re dependence)
3. $k^+ > 60$ full rough (no Re dependence)

$$f^{-1/2} = -2\log\left[\frac{k/d}{3.7} + \frac{2.51}{\mathrm{Re}_d\ f^{-1/2}}\right]$$ Moody diagram

$$\sim -1.8\log\left[\frac{6.9}{\mathrm{Re}_d} + \left(\frac{k/d}{3.7}\right)^{1.11}\right]$$ Approximate explicit formula

There are basically four types of problems involved with uniform flow in a single pipe:

1. Determine the head loss, given the kind and size of pipe along with the flow rate, Q = A*V
2. Determine the flow rate, given the head, kind, and size of pipe
3. Determine the pipe diameter, given the type of pipe, head, and flow rate
4. Determine the pipe length, given Q, d, h_f, k_s, μ, g

1. Determine the head loss
   The first problem of head loss is solved readily by obtaining f from the Moody diagram, using values of Re and k_s/D computed from the given data. The head loss h_f is then computed from the Darcy-Weisbach equation.

   $$f = f(Re_D, k_s/D)$$

   $$h_f = f\frac{L}{D}\frac{V^2}{2g} = \Delta h \qquad \Delta h = (z_1 - z_2) + \left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma}\right)$$

   $$= \Delta\left(\frac{p}{\gamma} + z\right)$$

   $$Re_D = Re_D(V, D)$$

2. Determine the flow rate
   The second problem of flow rate is solved by trial, using a successive approximation procedure. This is because both Re and f(Re) depend on the unknown velocity, V. The solution is as follows:
   1) solve for V using an assumed value for f and the Darcy-Weisbach equation

   $$V = \underbrace{\left[\frac{2gh_f}{L/D}\right]^{1/2}}_{\text{known from given data}} \cdot f^{-1/2}$$

   note sign

2) using V compute Re
3) obtain a new value for f = f(Re, k_s/D) and reapeat as above until convergence

Or can use Re $f^{1/2} = \frac{D^{3/2}}{\nu}\left(\frac{2gh_f}{L}\right)^{1/2}$

scale on Moody Diagram

1) compute $\mathrm{Re}\, f^{1/2}$ and k_s/D
2) read f
3) solve V from $h_f = f\,\frac{L}{D}\frac{V^2}{2g}$
4) Q = VA

3. Determine the size of the pipe
   The third problem of pipe size is solved by trial, using a successive approximation procedure. This is because h_f, f, and Q all depend on the unknown diameter D. The solution procedure is as follows:

   1) solve for D using an assumed value for f and the Darcy-Weisbach equation along with the definition of Q

$$D = \underbrace{\left[\frac{8LQ^2}{\pi^2 g h_f}\right]^{1/5}}_{\text{known from given data}} \cdot f^{1/5}$$

2) using D compute Re and k_s/D

3) obtain a new value of $f = f(Re, k_s/D)$ and repeat as above until convergence

4. Determine the pipe length
   The four problem of pipe length is solved by obtaining f from the Moody diagram, using values of Re and k_s/D computed from the given data. Then using given h_f ,V, D, and calculated f to solve L from $L = \frac{2g}{V^2}\frac{Dh_f}{f}$.

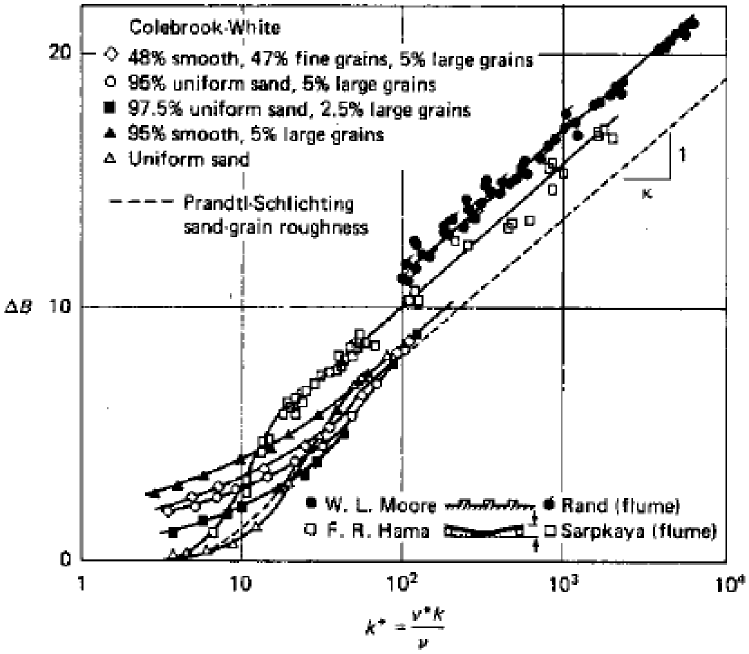


FIGURE 6-18
Composite plot of the profile-shift parameter $\Delta B(k^+)$ for various roughness geometries, as compiled by Clauser (1956).

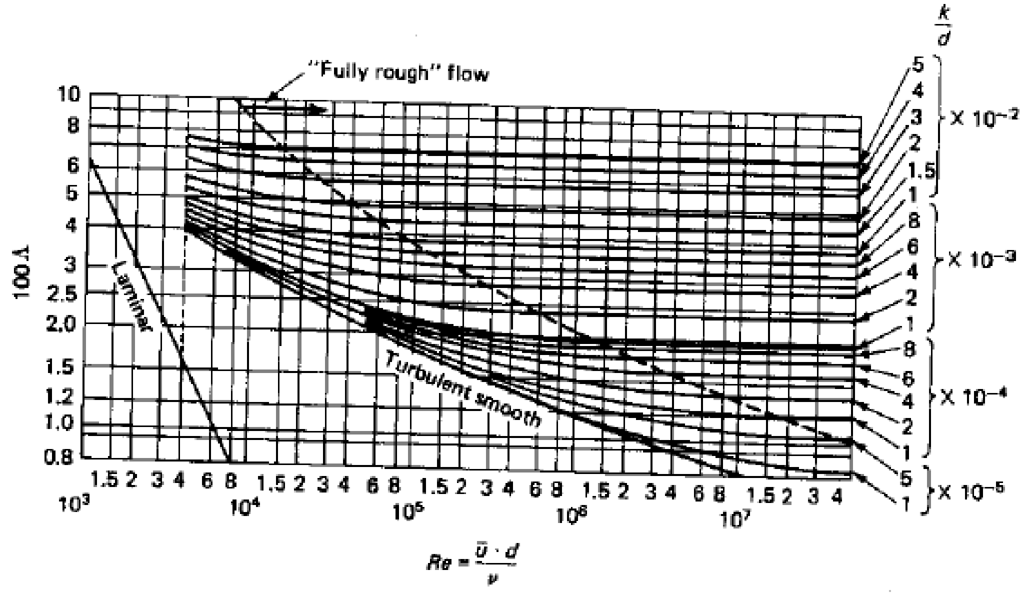


FIGURE 6-19
Friction factors for turbulent flow in rough pipes. [*After Moody (1944)*.]

3. *Concept of hydraulic diameter for noncircular ducts*

For noncircular ducts, τ_w= f(perimeter); thus, new definitions of $f = \frac{8\tau_w}{\rho V^2}$ and $C_f = \frac{2\tau_w}{\rho V^2}$ are required.

Define average wall shear stress

$$\bar{\tau}_w = \frac{1}{P}\int_0^P \tau_w \, ds \qquad \text{ds = arc length, P = perimeter}$$

Momentum:

$$\Delta pA - \bar{\tau}_w PL + \underbrace{\gamma AL}_{W}\left(\frac{\Delta z}{L}\right) = 0$$

$$\Delta h = \Delta\left(p/\gamma + z\right) = \frac{\bar{\tau}_w L}{\gamma A/P}$$

A/P =R_h= Hydraulic radius (=R/2 for circular pipe and $\Delta h = \frac{\tau_w L}{\gamma R/2}$)

Energy:

$$\Delta h = h_L = \frac{\overline{\tau}_w L}{\gamma A / P}$$

$$\overline{\tau}_w = \frac{A}{P}\frac{\Delta h \gamma}{L} = \frac{-A\gamma}{P}\frac{dh}{dx} = \frac{-A}{P}\frac{d(p+\gamma z)}{dx} = \frac{A}{P}\left(-\frac{d\hat{p}}{dx}\right)$$ non-circular duct

Recall for circular pipe:

$$\tau_w = -\frac{R}{2}\frac{d\hat{p}}{dx} = -\frac{D}{4}\frac{d\hat{p}}{dx}$$

In analogy to circular pipe:

$$\overline{\tau}_w = \frac{A}{P}\left(-\frac{d\hat{p}}{dx}\right) = \frac{D_h}{4}\left(-\frac{d\hat{p}}{dx}\right) \Rightarrow \frac{A}{P} = \frac{D_h}{4} \Rightarrow D_h = \frac{4A}{P}$$ Hydraulic diameter

For multiple surfaces such as concentric annulus P and A based on wetted perimeter and area

$$\overline{f} = \frac{8\overline{\tau}_w}{\rho V^2} = \overline{f}(\mathrm{Re}_{D_h}, \varepsilon / D_h) \qquad \mathrm{Re}_{D_h} = \frac{VD_h}{\upsilon}$$

$$\Delta h = h_L = \frac{\overline{\tau}_w L}{\gamma R_h} = \frac{\rho V^2 \overline{f}}{8}\frac{L}{\gamma R_h} = \overline{f}\,\frac{L}{D_h}\frac{V^2}{2g}$$

However, accuracy not good for laminar flow (40%) and marginal turbulent flow (15%).

a. *Accuracy for laminar flow (smooth non-circular pipe)*

Recall for pipe flow:

$$\boxed{Poiseuille\#\ (P_0)\begin{cases} P_{0c_f} = C_f \operatorname{Re} = 16 \\ P_{0f} = f \operatorname{Re} = 64 \end{cases}}$$

Recall for channel flow:

$$\boxed{f = \frac{24\mu}{\rho V h} = \frac{48}{\operatorname{Re}_{2h}} = \frac{96}{\underbrace{\operatorname{Re}_{4h}}_{\operatorname{Re}_{D_h}}}}$$

$$\boxed{\begin{array}{l} C_f = f/4 \Rightarrow \\ C_f = \dfrac{6\mu}{\rho V h} = \dfrac{12}{\operatorname{Re}_{2h}} = \dfrac{24}{\underbrace{\operatorname{Re}_{4h}}_{\operatorname{Re}_{D_h}}} \end{array}}$$

$$\boxed{Poiseuille\#\ (P_0)\begin{cases} P_{0c_f} = C_f \operatorname{Re}_{D_h} = 24 \\ P_{0f} = f \operatorname{Re}_{D_h} = 96 \end{cases}}$$

Therefore:

$$\boxed{\frac{P_{0c_f\ pipe}}{P_{0c_f\ channel\ based\ on\ D_h}} = \frac{P_{0\ f\ pipe}}{P_{0\ f\ channel\ based\ on\ D_h}} = \frac{16}{24} = \frac{64}{96} = \frac{2}{3}}$$

Thus, if we could not work out the laminar theory and chose to use the approximation $f\,\mathrm{Re}_{D_h} \approx 64$ or $C_f\,\mathrm{Re}_{D_h} \approx 16$, we would be 33 percent low for channel flow.

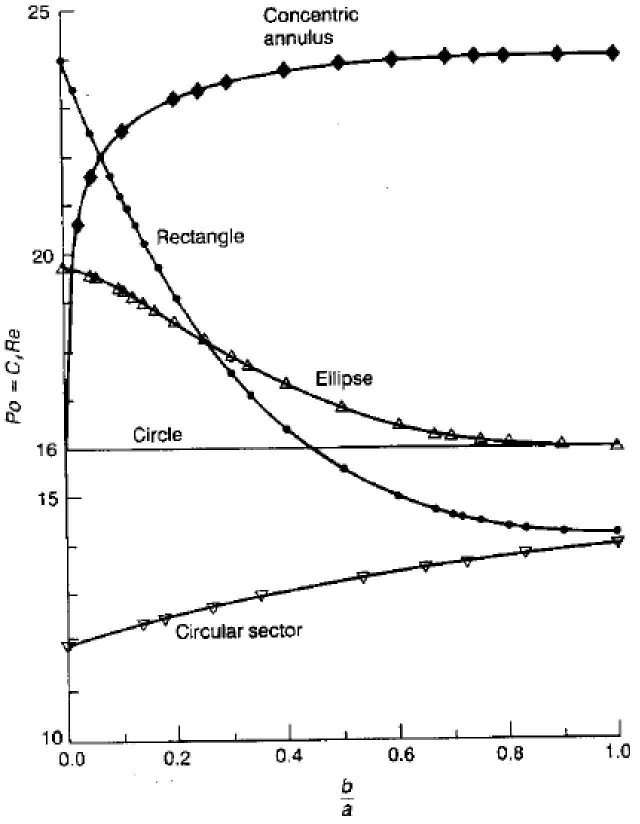
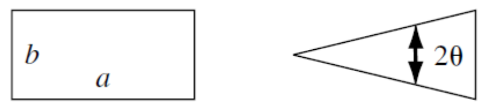


FIGURE 3-13
Comparison of Poiseuille numbers for various duct cross sections when Reynolds number is scaled by the hydraulic diameter. [*Numerical data taken from Shah and London (1978).*]

For laminar flow, $\overline{P}_0$ varies greatly, therefore it is better to use the exact solution

Table 6.4 Laminar Friction Constants $f\,\mathrm{Re}$ for Rectangular and Triangular Ducts



Rectangular		**Isosceles triangle**	
b/a	$f\mathrm{Re}_{D_h}$	θ, deg	$f\mathrm{Re}_{D_h}$
0.0	96.00	0	48.0
0.05	89.91	10	51.6
0.1	84.68	20	52.9
0.125	82.34	30	53.3
0.167	78.81	40	52.9
0.25	72.93	50	52.0
0.4	65.47	60	51.1
0.5	62.19	70	49.5
0.75	57.89	80	48.3
1.0	56.91	90	48.0

b. *Accuracy for turbulent flow(smooth non-circular pipe)*

For turbulent flow, D_h works much better especially if combined with "effective diameter" concept based on ratio of exact laminar circular and noncircular duct P_0 numbers, i.e. $16/\overline{P}_{0C_f}$ or $64/\overline{P}_{0f}$.

First recall turbulent circular pipe solution and compare with turbulent channel flow solution using log-law in both cases

Channel Flow

$$V = \frac{1}{h}\int_0^h u^*\left[\frac{1}{\kappa}\ln\frac{(h-y)u^*}{\upsilon} + B\right]dY$$ Y=h-y wall coordinate

$$= u^*\left(\frac{1}{\kappa}\ln\frac{hu^*}{\upsilon} + B - \frac{1}{\kappa}\right)$$

$$D_h = \frac{4A}{P} = \lim_{B\to\infty}\frac{4(2hB)}{2B+4h} = 4h$$ h= half width

Define $\mathrm{Re}_{D_h} = \frac{VD_h}{\upsilon} = \frac{V4h}{\upsilon}$

$$f^{-1/2} = 2\log\left(\mathrm{Re}_{D_h}\ f^{1/2}\right) - 1.19$$

Very nearly the same as circular pipe
7% to large at Re = 10^5
4% to large at Re = 10^8

Therefore error in D_h concept relatively smaller for turbulent flow.

Note $f^{-1/2}(channel) = 2\log\left(0.64\,\mathrm{Re}_{D_h}\ f^{1/2}\right) - 0.8$

Define $D_{\text{effective}} = 0.64 D_h \sim \underbrace{\frac{P_{0f}(circle) = 16}{P_{0f}(channel) = 24}}_{\boxed{\text{Laminar solution}}} D_h$

(therefore, improvement on D_h is)

$$\mathrm{Re}_{D_{eff}} = \frac{VD_{eff}}{\upsilon}$$

$$D_{eff} = \frac{P_{0f}(circle)}{P_{0f}(non-circular)} D_h = \frac{P_{0C_f}(circle)}{P_{0C_f}(non-circular)} D_h$$

Or

$$\boxed{D_{eff} = \frac{64}{P_{0f}(non-circular)} D_h = \frac{16}{P_{0C_f}(non-circular)} D_h}$$

From exact laminar solution

10.5 Flow at Pipe Inlets and Losses From Fittings

For real pipe systems in addition to friction head loss these are additional so called minor losses due to

$$\left.\begin{array}{l}\text{1. entrance and exit effects}\\ \text{2. expansions and contractions}\\ \text{3. bends, elbows, tees, and other fittings}\\ \text{4. valves (open or partially closed)}\end{array}\right\}\begin{array}{l}\text{can be}\\ \text{large}\\ \text{effect}\end{array}$$

For such complex geometries we must rely on experimental data to obtain a loss coefficient

$$K = \cfrac{h_m}{\cfrac{V^2}{2g}} \longleftarrow \text{head loss due to minor losses}$$

In general,

$$K = K(\text{geometry}, \underbrace{\text{Re}, \varepsilon/D}_{\substack{\text{dependence usually}\\ \text{not known}}})$$

Loss coefficient data is supplied by manufacturers and also listed in handbooks. The data are for turbulent flow conditions but seldom given in terms of Re.

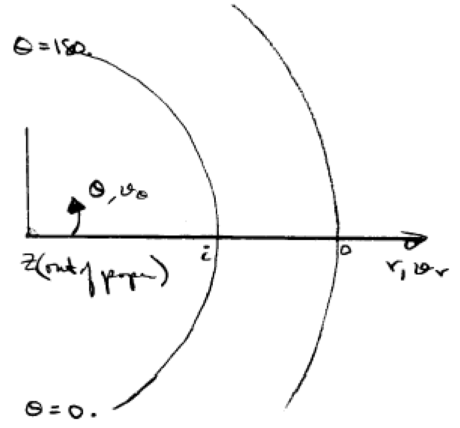
Modified Energy Equation to Include Minor Losses:

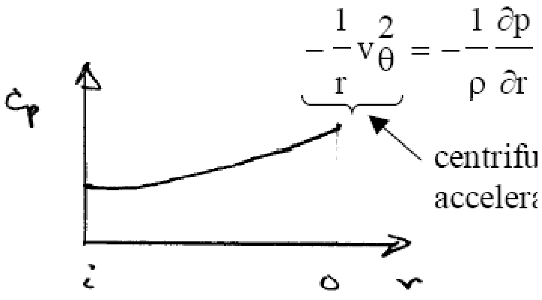
$$\frac{p_1}{\gamma}+z_1+\frac{1}{2g}\alpha_1 V_1^2+h_p=\frac{p_2}{\gamma}+z_2+\frac{1}{2g}\alpha_2 V_2^2+h_t+h_f+\sum h_n$$

$$h_m = K\frac{V^2}{2g}$$

Note: Σh_m does not include pipe friction and e.g. in elbows and tees, this must be added to h_f.

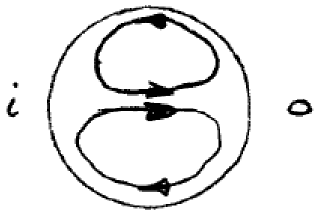
1. Flow in a bend:





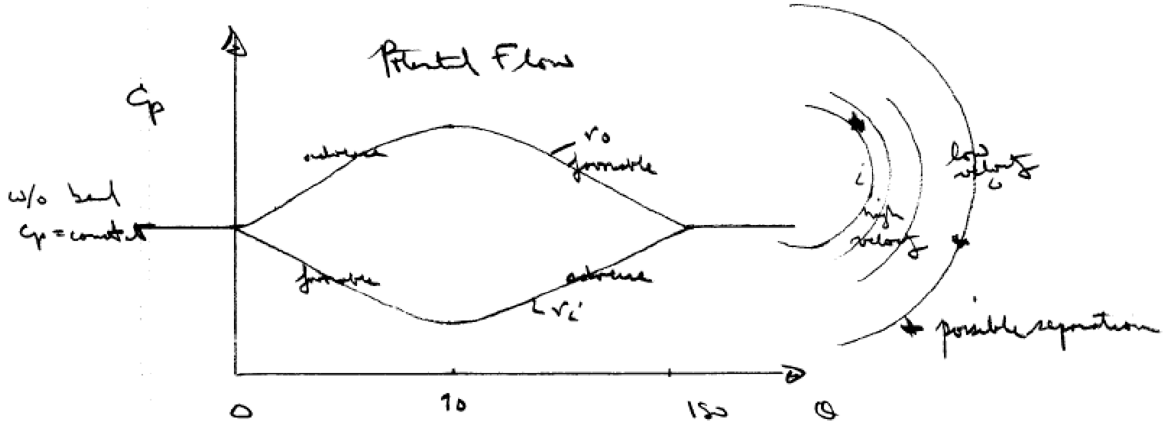
$$\underbrace{-\frac{1}{r}v_\theta^2}= -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

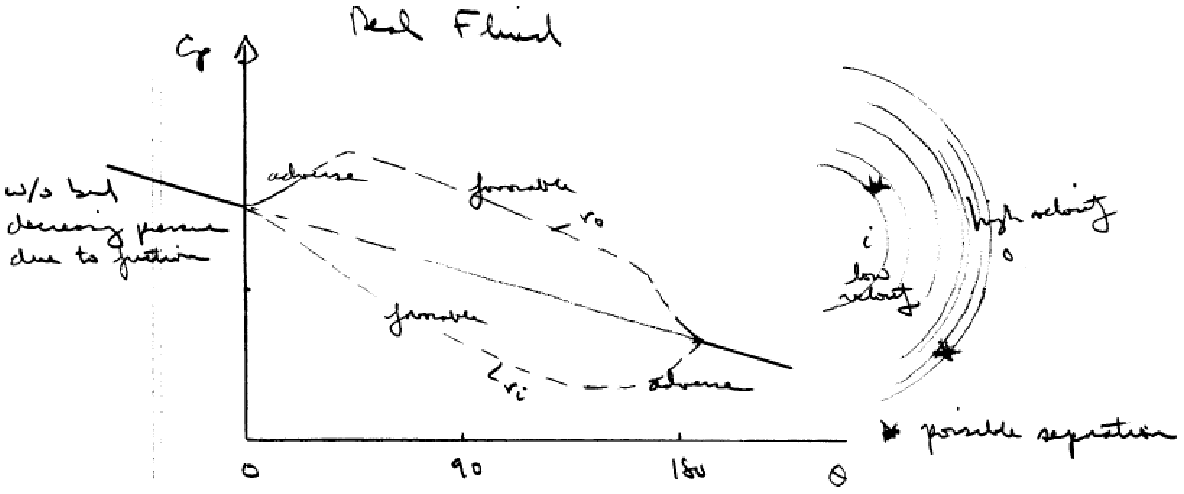
i.e. $\frac{\partial p}{\partial r} > 0$ which is an adverse pressure gradient in r direction. The slower moving fluid near wall responds first and a swirling flow pattern results.



This swirling flow represents an energy loss which must be added to the h_L.

Also, flow separation can result due to adverse longitudinal pressure gradients which will result in additional losses.





This shows potential flow is not a good approximate in internal flows (except possibly near entrance)

2. Valves: enormous losses

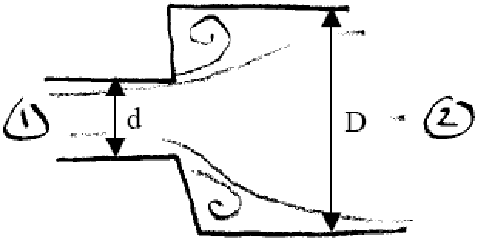
3. Entrances: depends on rounding of entrance

4. Exit (to a large reservoir): K = 1
i.e., all velocity head is lost

5. Contractions and Expansions
$\underbrace{\text{sudden}}_{\downarrow}$ or gradual

theory for expansion:

$$h_L = \frac{(V_1 - V_2)^2}{2g}$$

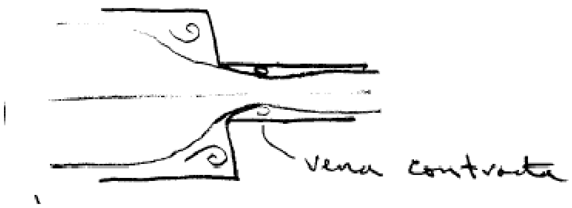


from continuity, momentum, and energy
(assuming p = p_1 in separation pockets)

$$\Rightarrow \quad K_{SE} = \left(1 - \frac{d^2}{D^2}\right)^2 = \frac{h_m}{V_1^2 / 2g}$$

no theory for contraction:

$$K_{SC} = .42\left(1 - \frac{d^2}{D^2}\right)$$



from experiment

If the contraction or expansion is gradual the losses are quite different. A gradual expansion is called a diffuser. Diffusers are designed with the intent of raising the static pressure.

$$C_p = \frac{p_2 - p_1}{\frac{1}{2}\rho V_1^2}$$

$$C_{p_{ideal}} = 1 - \left(\frac{A_1}{A_2}\right)^2$$ Bernoulli and continuity equation

$$K = \frac{h_m}{V^2 / 2g} = C_{p_{ideal}} - C_p$$ Energy equation

Actually very complex flow and

$$C_p = C_p(\text{geometry}, \underbrace{\text{inlet flow conditions}})$$

i.e., fully developed (long pipe) reduces C_p
thin boundary layer (short pipe) high C_p
(more uniform inlet profile)

FIGURE 10.10
Flow characteristics at a pipe inlet (not to scale).

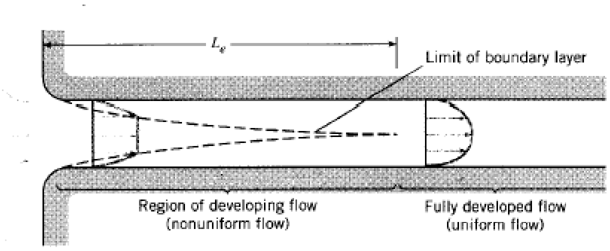
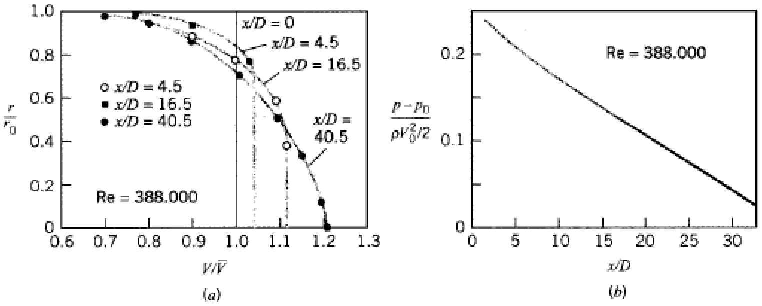
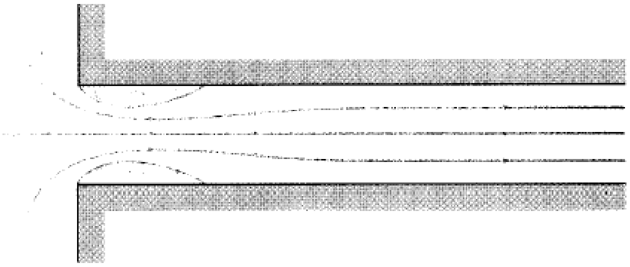


FIGURE 10.11
Distribution of velocity and pressure in the inlet region of a pipe [Barbin and Jones (3)].
(a) Velocity distribution.
(b) Pressure distribution.



Turbulent flow



K = .5

FIGURE 10.12
Flow at a sharp-edged inlet.

FIGURE 10.13
Flow pattern in an elbow.

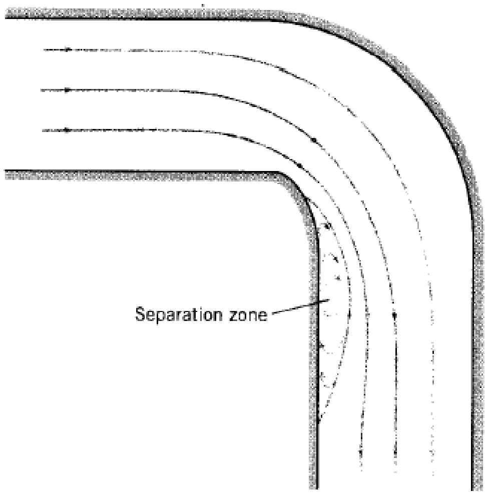
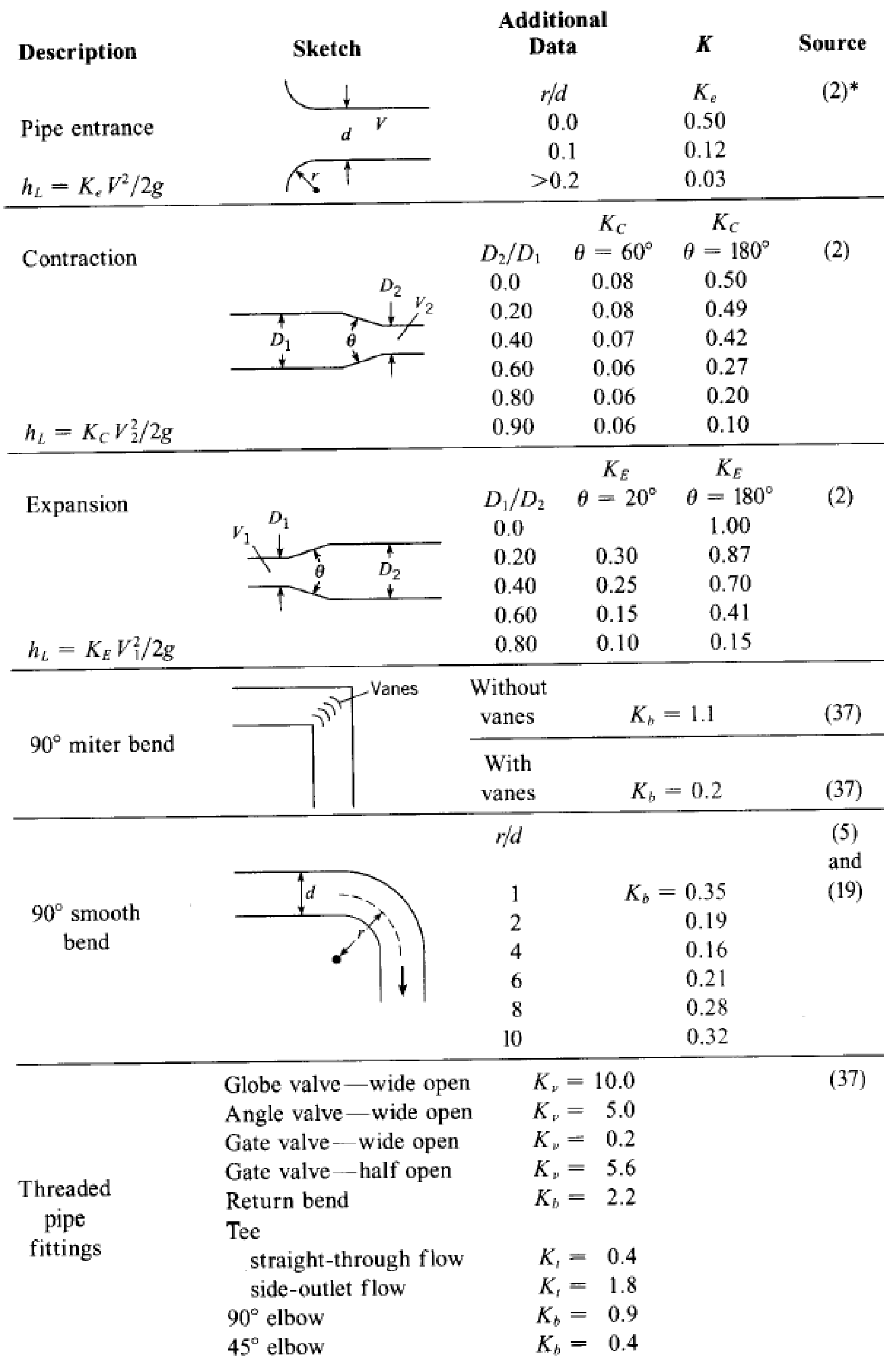


TABLE 10.2 LOSS COEFFICIENTS FOR VARIOUS TRANSITIONS AND FITTINGS



Description	Sketch	Additional Data	K		Source
Pipe entrance		r/d	K_e		(2)*
		0.0	0.50		
		0.1	0.12		
$h_L = K_e V^2/2g$		>0.2	0.03		
			K_C	K_C	
Contraction		D_2/D_1	$\theta = 60°$	$\theta = 180°$	(2)
		0.0	0.08	0.50	
		0.20	0.08	0.49	
		0.40	0.07	0.42	
		0.60	0.06	0.27	
		0.80	0.06	0.20	
$h_L = K_C V_2^2/2g$		0.90	0.06	0.10	
			K_E	K_E	
Expansion		D_1/D_2	$\theta = 20°$	$\theta = 180°$	(2)
		0.0		1.00	
		0.20	0.30	0.87	
		0.40	0.25	0.70	
		0.60	0.15	0.41	
$h_L = K_E V_1^2/2g$		0.80	0.10	0.15	
90° miter bend		Without vanes	$K_b = 1.1$		(37)
		With vanes	$K_b = 0.2$		(37)
90° smooth bend		r/d			(5) and (19)
		1	$K_b = 0.35$		
		2	0.19		
		4	0.16		
		6	0.21		
		8	0.28		
		10	0.32		
Threaded pipe fittings	Globe valve—wide open	$K_v = 10.0$			(37)
	Angle valve—wide open	$K_v = 5.0$			
	Gate valve—wide open	$K_v = 0.2$			
	Gate valve—half open	$K_v = 5.6$			
	Return bend	$K_b = 2.2$			
	Tee				
	straight-through flow	$K_t = 0.4$			
	side-outlet flow	$K_t = 1.8$			
	90° elbow	$K_b = 0.9$			
	45° elbow	$K_b = 0.4$			

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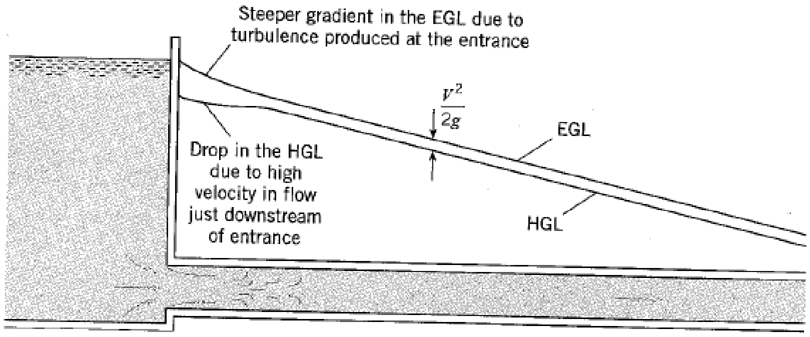
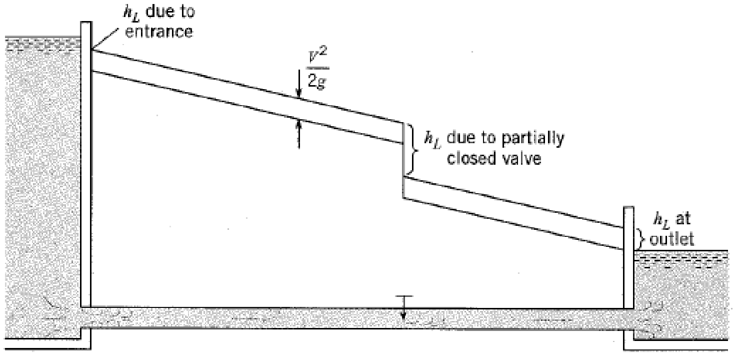
Engineers, Atlanta, Georgia, from the 1981 ASHRAE Handbook-Fundamentals.

FIGURE 10.14

EGL and HGL at a sharp-edged pipe entrance.



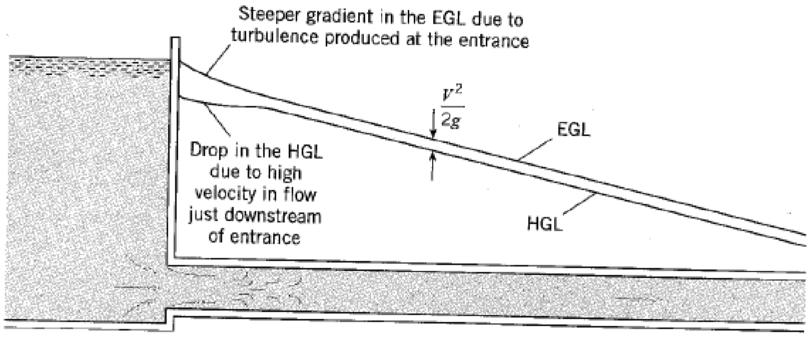


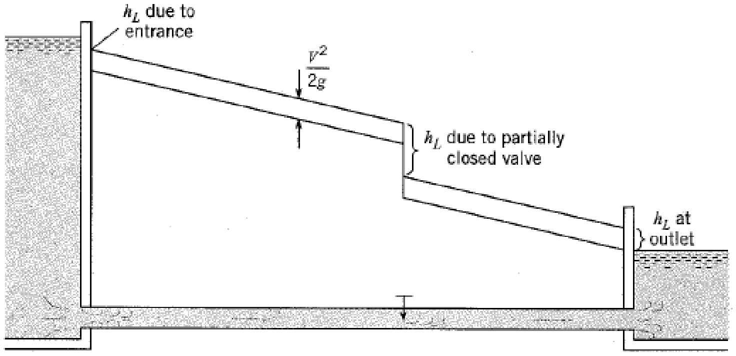
IGURE 10.15

ead losses in a pipe.

FIGURE 10.14

EGL and HGL at a sharp-edged pipe entrance.





IGURE 10.15

ead losses in a pipe.

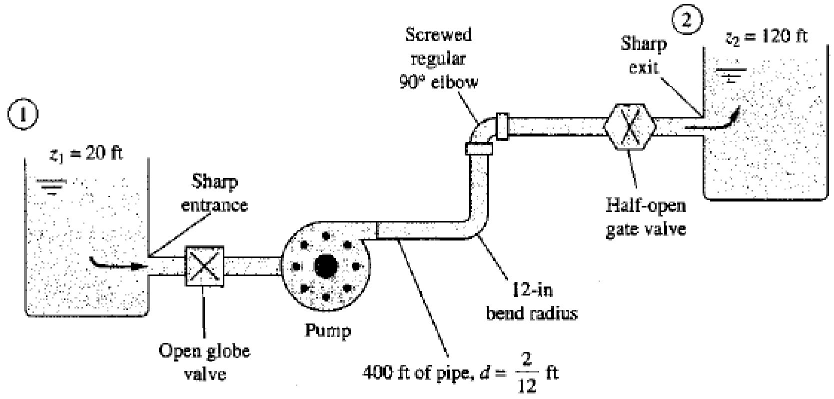
For a gradual *contraction*, the loss is very small, as seen from the following experimental values [15]:

Contraction cone angle 2θ, deg	30	45	60
K for gradual contraction	0.02	0.04	0.07

References 15, 16, 43, and 46 contain additional data on minor losses.

EXAMPLE 6.16

Water, $\rho = 1.94$ slugs/ft^3 and $\nu = 0.000011$ ft^2/s, is pumped between two reservoirs at 0.2 ft^3/s through 400 ft of 2-in-diameter pipe and several minor losses, as shown in Fig. E6.16. The roughness ratio is $\epsilon/d = 0.001$. Compute the pump horsepower required.



E6.16

Solution

Write the steady flow energy equation between sections 1 and 2, the two reservoir surfaces:

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \left(\frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2\right) + h_f + \sum h_m - h_p$$

where h_p is the head increase across the pump. But since $p_1 = p_2$ and $V_1 = V_2 \approx 0$, solve for the pump head:

$$h_p = z_2 - z_1 + h_f + \sum h_m = 120\text{ ft} - 20\text{ ft} + \frac{V^2}{2g}\left(\frac{fL}{d} + \sum K\right) \qquad (1)$$

Now with the flow rate known, calculate

$$V = \frac{Q}{A} = \frac{0.2\text{ ft}^3/\text{s}}{\frac{1}{4}\pi(\frac{2}{12}\text{ ft})^2} = 9.17\text{ ft/s}$$

Now list and sum the minor loss coefficients:

Loss	K
Sharp entrance (Fig. 6.21)	0.5
Open globe valve (2 in, Table 6.5)	6.9
12-in bend (Fig. 6.20)	0.25
Regular 90° elbow (Table 6.5)	0.95
Half-closed gate valve (from Fig. 6.18*b*)	2.7
Sharp exit (Fig. 6.21)	1.0
	$\Sigma K = 12.3$

Calculate the Reynolds number and pipe friction factor:

$$\mathrm{Re}_d = \frac{Vd}{\nu} = \frac{9.17(\frac{2}{12})}{0.000011} = 139{,}000$$

For $\epsilon/d = 0.001$, from the Moody chart read $f = 0.0216$. Substitute into Eq. (1):

$$h_p = 100 \text{ ft} + \frac{(9.17 \text{ ft/s})^2}{2(32.2 \text{ ft/s}^2)}\left[\frac{0.0216(400)}{\frac{2}{12}} + 12.3\right]$$

$$= 100 \text{ ft} + 84 \text{ ft} = 184 \text{ ft} \quad \text{pump head}$$

The pump must provide a power to the water of

$$P = \rho g Q h_p = [1.94(32.2) \text{ lbf/ft}^3](0.2 \text{ ft}^3\text{/s})(184 \text{ ft}) = 2300 \text{ ft} \cdot \text{lbf/s}$$

The conversion factor is 1 hp = 550 ft · lbf/s. Therefore

$$P = \frac{2300}{550} = 4.2 \text{ hp} \qquad \textit{Ans.}$$

Allowing for an efficiency of 70 to 80 percent, a pump is needed with an input of about 6 hp.

6.10 Multiple-Pipe Systems[8]

If you can solve the equations for one-pipe systems, you can solve them all; but when systems contain two or more pipes, certain basic rules make the calculations very smooth. Any resemblance between these rules and the rules for handling electric circuits is not coincidental.

Figure 6.24 shows three examples of multiple-pipe systems.

Pipes in Series

The first is a set of three (or more) pipes in series. Rule 1 is that the flow rate is the same in all pipes:

$$Q_1 = Q_2 = Q_3 = \text{const}$$

or

$$V_1 d_1^2 = V_2 d_2^2 = V_3 d_3^2 \tag{6.84}$$

Rule 2 is that the total head loss through the system equals the sum of the head loss in each pipe:

$$\Delta h_{A \to B} = \Delta h_1 + \Delta h_2 + \Delta h_3 \tag{6.85}$$

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[8] This section may be omitted without loss of continuity.

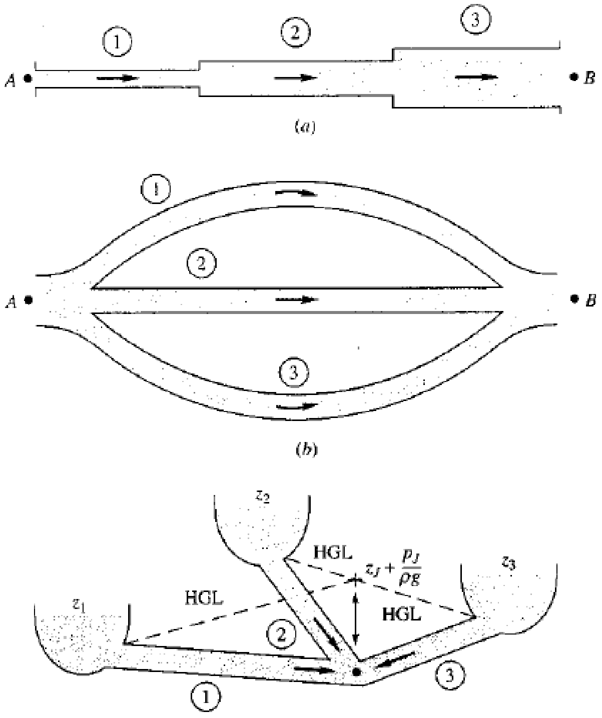

Fig. 6.24 Examples of multiple-pipe systems: (*a*) pipes in series; (*b*) pipes in parallel; (*c*) the three-reservoir junction problem.

In terms of the friction and minor losses in each pipe, we could rewrite this as

$$\Delta h_{A\to B} = \frac{V_1^2}{2g}\left(\frac{f_1 L_1}{d_1} + \sum K_1\right) + \frac{V_2^2}{2g}\left(\frac{f_2 L_2}{d_2} + \sum K_2\right) + \frac{V_3^2}{2g}\left(\frac{f_3 L_3}{d_3} + \sum K_3\right) \tag{6.86}$$

and so on for any number of pipes in the series. Since V_2 and V_3 are proportional to V_1 from Eq. (6.84), Eq. (6.86) is of the form

$$\Delta h_{A\to B} = \frac{V_1^2}{2g}(\alpha_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3) \tag{6.87}$$

where the α_i are dimensionless constants. If the flow rate is given, we can evaluate the right-hand side and hence the total head loss. If the head loss is given, a little iteration is needed, since f_1, f_2, and f_3 all depend on V_1 through the Reynolds number. Begin by calculating f_1, f_2, and f_3, assuming fully rough flow, and the solution for V_1 will converge with one or two iterations. EES is ideal for this purpose.

EXAMPLE 6.17

Given is a three-pipe series system, as in Fig. 6.24*a*. The total pressure drop is $p_A - p_B =$ 150,000 Pa, and the elevation drop is $z_A - z_B = 5$ m. The pipe data are

Pipe	L, m	d, cm	ϵ, mm	ϵ/d
1	100	8	0.24	0.003
2	150	6	0.12	0.002
3	80	4	0.20	0.005

The fluid is water, $\rho = 1000$ kg/m^3 and $\nu = 1.02 \times 10^{-6}$ m^2/s. Calculate the flow rate Q in m^3/h through the system.

Solution

The total head loss across the system is

$$\Delta h_{A\to B} = \frac{p_A - p_B}{\rho g} + z_A - z_B = \frac{150{,}000}{1000(9.81)} + 5\text{ m} = 20.3\text{ m}$$

From the continuity relation (6.84) the velocities are

$$V_2 = \frac{d_1^2}{d_2^2}V_1 = \frac{16}{9}V_1 \qquad V_3 = \frac{d_1^2}{d_3^2}V_1 = 4V_1$$

and
$$\text{Re}_2 = \frac{V_2 d_2}{V_1 d_1}\text{Re}_1 = \frac{4}{3}\text{Re}_1 \qquad \text{Re}_3 = 2\text{Re}_1$$

Neglecting minor losses and substituting into Eq. (6.86), we obtain

$$\Delta h_{A\to B} = \frac{V_1^2}{2g}\left[1250f_1 + 2500\left(\frac{16}{9}\right)^2 f_2 + 2000(4)^2 f_3\right]$$

or
$$20.3\text{ m} = \frac{V_1^2}{2g}(1250f_1 + 7900f_2 + 32{,}000f_3) \qquad (1)$$

This is the form that was hinted at in Eq. (6.87). It seems to be dominated by the third pipe loss $32{,}000f_3$. Begin by estimating f_1, f_2, and f_3 from the Moody-chart fully rough regime:

$$f_1 = 0.0262 \qquad f_2 = 0.0234 \qquad f_3 = 0.0304$$

Substitute in Eq. (1) to find $V_1^2 \approx 2g(20.3)/(33 + 185 + 973)$. The first estimate thus is $V_1 =$ 0.58 m/s, from which

$$\text{Re}_1 \approx 45{,}400 \qquad \text{Re}_2 = 60{,}500 \qquad \text{Re}_3 = 90{,}800$$

Hence, from the Moody chart,

$$f_1 = 0.0288 \qquad f_2 = 0.0260 \qquad f_3 = 0.0314$$

Substitution into Eq. (1) gives the better estimate

$$V_1 = 0.565 \text{ m/s} \qquad Q = \tfrac{1}{4}\pi d_1^2 V_1 = 2.84 \times 10^{-3} \text{ m}^3\text{/s}$$

or

$$Q = 10.2 \text{ m}^3\text{/h} \qquad \textit{Ans.}$$

A second iteration gives $Q = 10.22$ m^3/h, a negligible change.

Pipes in Parallel

The second multiple-pipe system is the *parallel* flow case shown in Fig. 6.24*b*. Here the pressure drop is the same in each pipe, and the total flow is the sum of the individual flows:

$$\Delta h_{A\rightarrow B} = \Delta h_1 = \Delta h_2 = \Delta h_3 \tag{6.88a}$$

$$Q = Q_1 + Q_2 + Q_3 \tag{6.88b}$$

If the total head loss is known, it is straightforward to solve for Q_i in each pipe and sum them, as will be seen in Example 6.18. The reverse problem, of determining ΣQ_i when h_f is known, requires iteration. Each pipe is related to h_f by the Moody relation $h_f = f(L/d)(V^2/2g) = fQ^2/C$, where $C = \pi^2 g d^5/8L$. Thus each pipe has nearly quadratic nonlinear parallel resistance, and head loss is related to total flow rate by

$$h_f = \frac{Q^2}{(\Sigma\sqrt{C_i/f_i})^2} \qquad \text{where } C_i = \frac{\pi^2 g d_i^{\,5}}{8L_i} \tag{6.89}$$

Since the f_i vary with Reynolds number and roughness ratio, one begins Eq. (6.89) by guessing values of f_i (fully rough values are recommended) and calculating a first estimate of h_f. Then each pipe yields a flow-rate estimate $Q_i \approx (C_i h_f/f_i)^{1/2}$ and hence a new Reynolds number and a better estimate of f_i. Then repeat Eq. (6.89) to convergence.

It should be noted that both of these parallel-pipe cases—finding either ΣQ or h_f—are easily solved by EES if reasonable initial guesses are given.

EXAMPLE 6.18

Assume that the same three pipes in Example 6.17 are now in parallel with the same total head loss of 20.3 m. Compute the total flow rate Q, neglecting minor losses.

Solution

From Eq. (6.88*a*) we can solve for each V separately:

$$20.3 \text{ m} = \frac{V_1^2}{2g}1250 f_1 = \frac{V_2^2}{2g}2500 f_2 = \frac{V_3^2}{2g}2000 f_3 \tag{1}$$

Guess fully rough flow in pipe 1: $f_1 = 0.0262$, $V_1 = 3.49$ m/s; hence $\text{Re}_1 = V_l d_1/\nu = 273{,}000$. From the Moody chart read $f_1 = 0.0267$; recompute $V_1 = 3.46$ m/s, $Q_1 = 62.5$ m^3/h. [This problem can also be solved from Eq. (6.51).]

Next guess for pipe 2: $f_2 \approx 0.0234$, $V_2 \approx 2.61$ m/s; then $\text{Re}_2 = 153{,}000$, and hence $f_2 = 0.0246$, $V_2 = 2.55$ m/s, $Q_2 = 25.9\ \text{m}^3/\text{h}$.

Finally guess for pipe 3: $f_3 \approx 0.0304$, $V_3 \approx 2.56$ m/s; then $\text{Re}_3 = 100{,}000$, and hence $f_3 = 0.0313$, $V_3 = 2.52$ m/s, $Q_3 = 11.4\ \text{m}^3/\text{h}$.

This is satisfactory convergence. The total flow rate is

$$Q = Q_1 + Q_2 + Q_3 = 62.5 + 25.9 + 11.4 = 99.8\ \text{m}^3/\text{h} \qquad \textit{Ans.}$$

These three pipes carry 10 times more flow in parallel than they do in series.

This example is ideal for EES. One enters the pipe data (L_i, d_i, ϵ_i); the fluid properties (ρ, μ); the definitions $Q_i = (\pi/4)d_i^2V_i$, $\text{Re}_i = \rho V_i d_i/\mu$, and $h_f = f_i\,(L_i/d_i)\,(V_i^2/2g)$; plus the Colebrook formula (6.48) for each friction factor f_i. There is no need to use resistance ideas such as Eq. (6.89). Specify that $f_i > 0$ and $\text{Re}_i > 4000$. Then, if one enters $Q = \Sigma\, Q_i = (99.8/3600)\ \text{m}^3/\text{s}$, EES quickly solves for $h_f = 20.3$ m. Conversely, if one enters $h_f = 20.3$ m, EES solves for $Q = 99.8\ \text{m}^3/\text{h}$.

Three-Reservoir Junction

Consider the third example of a *three-reservoir pipe junction,* as in Fig. 6.24*c*. If all flows are considered positive toward the junction, then

$$Q_1 + Q_2 + Q_3 = 0 \tag{6.90}$$

which obviously implies that one or two of the flows must be away from the junction. The pressure must change through each pipe so as to give the same static pressure p_J at the junction. In other words, let the HGL at the junction have the elevation

$$h_J = z_J + \frac{p_J}{\rho g}$$

where p_J is in gage pressure for simplicity. Then the head loss through each, assuming $p_1 = p_2 = p_3 = 0$ (gage) at each reservoir surface, must be such that

$$\begin{aligned} \Delta h_1 &= \frac{V_1^2}{2g}\frac{f_1L_1}{d_1} = z_1 - h_J \\ \Delta h_2 &= \frac{V_2^2}{2g}\frac{f_2L_2}{d_2} = z_2 - h_J \\ \Delta h_3 &= \frac{V_3^2}{2g}\frac{f_3L_3}{d_3} = z_3 - h_J \end{aligned} \tag{6.91}$$

We guess the position h_J and solve Eqs. (6.91) for V_1, V_2, and V_3 and hence Q_1, Q_2, and Q_3, iterating until the flow rates balance at the junction according to Eq. (6.90). If we guess h_J too *high*, the sum $Q_1 + Q_2 + Q_3$ will be *negative* and the remedy is to reduce h_J, and vice versa.

EXAMPLE 6.19

Take the same three pipes as in Example 6.17, and assume that they connect three reservoirs at these surface elevations

$$z_1 = 20\ \text{m} \qquad z_2 = 100\ \text{m} \qquad z_3 = 40\ \text{m}$$

Find the resulting flow rates in each pipe, neglecting minor losses.

Solution

As a first guess, take h_J equal to the middle reservoir height, $z_3 = h_J = 40$ m. This saves one calculation ($Q_3 = 0$) and enables us to get the lay of the land:

Reservoir	h_J, m	$z_i - h_J$, m	f_i	V_i, m/s	Q_i, m^3/h	L_i/d_i
1	40	−20	0.0267	−3.43	−62.1	1250
2	40	60	0.0241	4.42	45.0	2500
3	40	0		0	0	2000
					$\Sigma Q = -17.1$	

Since the sum of the flow rates toward the junction is negative, we guessed h_J too high. Reduce h_J to 30 m and repeat:

Reservoir	h_J, m	$z_i - h_J$, m	f_i	V_i, m/s	Q_i, m^3/h
1	30	−10	0.0269	−2.42	−43.7
2	30	70	0.0241	4.78	48.6
3	30	10	0.0317	1.76	8.0
					$\Sigma Q = 12.9$

This is positive ΣQ, and so we can linearly interpolate to get an accurate guess: $h_J \approx 34.3$ m. Make one final list:

Reservoir	h_J, m	$z_i - h_J$, m	f_i	V_i, m/s	Q_i, m^3/h
1	34.3	−14.3	0.0268	−2.90	−52.4
2	34.3	65.7	0.0241	4.63	47.1
3	34.3	5.7	0.0321	1.32	6.0
					$\Sigma Q = 0.7$

This is close enough; hence we calculate that the flow rate is 52.4 m^3/h toward reservoir 3, balanced by 47.1 m^3/h away from reservoir 1 and 6.0 m^3/h away from reservoir 3.

One further iteration with this problem would give $h_J = 34.53$ m, resulting in $Q_1 = -52.8$, $Q_2 = 47.0$, and $Q_3 = 5.8$ m^3/h, so that $\Sigma Q = 0$ to three-place accuracy. Pedagogically speaking, we would then be exhausted.

Pipe Networks

The ultimate case of a multipipe system is the *piping network* illustrated in Fig. 6.25. This might represent a water supply system for an apartment or subdivision or even a city. This network is quite complex algebraically but follows the same basic rules:

1. The net flow into any junction must be zero.
2. The net pressure change around any closed loop must be zero. In other words, the HGL at each junction must have one and only one elevation.
3. All pressure changes must satisfy the Moody and minor-loss friction correlations.

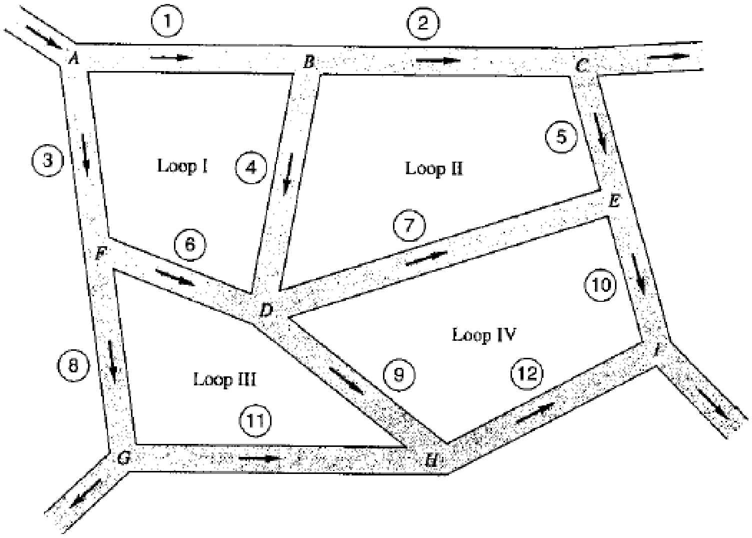

Fig. 6.25 Schematic of a piping network.

By supplying these rules to each junction and independent loop in the network, one obtains a set of simultaneous equations for the flow rates in each pipe leg and the HGL (or pressure) at each junction. Solution may then be obtained by numerical iteration, as first developed in a hand calculation technique by Prof. Hardy Cross in 1936 [17]. Computer solution of pipe network problems is now quite common and covered in at least one specialized text [18]. Network analysis is quite useful for real water distribution systems if well calibrated with the actual system head loss data.