Absolute inertial, relative inertial and non-inertial coordinates for a moving but non-deforming control volume

Tao Xing[1], Pablo Carrica, and Fred Stern

Objective Derive and correlate the governing equations of motion in integral and differential forms in absolute inertial, relative inertial and non-inertial coordinates for a moving but non-deforming control volume and evaluate the advantages and disadvantages of using the three coordinates.

Approach The governing differential equations of motion are derived and solved in the absolute inertial earth-fixed coordinates (*X*,*Y*,*Z*), non-inertial ship-fixed coordinates (*x*,*y*,*z*), and relative inertial coordinates (*X´*,*Y´*,*Z´*) that translate at a constant velocity in (i.e. with respect to) (*X*,*Y*,*Z*) for an arbitrary moving but non-deforming control volume (CV), as shown in Figure 1. The control volume is for either the ship in red or background in green while the solution domain includes both. The origin of (*x*,*y*,*z*) *o* is located at the center of gravity of the ship. The origin of (*X´*,*Y´*,*Z´*) O´ is located at the intersection of the bow and the static water line. $\underline{r}$ is the instantaneous position vector of any grid point or fluid particle in (*x*,*y*,*z*).

In the absolute inertial earth-fixed coordinates (Fig. 1(a)), the position vector of *o* is $\underline{R}$. $\underline{S} = \underline{R} + \underline{r}$ is the instantaneous position vectors of any grid point or fluid particle. The ship translation velocity is $\dot{\underline{R}} = \partial \underline{R} / \partial t = u_i \hat{i}_0 + v_i \hat{j}_0 + w_i \hat{k}_0$, where $\hat{i}_0$, $\hat{j}_0$, and $\hat{k}_0$ are the unit vectors of *X*, *Y*, and *Z* axes, respectively, and u_i, v_i, and w_i are the surge, sway, and heave velocities of the ship respect to (*X*,*Y*,*Z*). The ship rotation is angular velocity $\underline{\Omega} = \Omega_X \hat{i}_0 + \Omega_Y \hat{j}_0 + \Omega_Z \hat{k}_0$ of (*x*,*y*,*z*) in (*X*,*Y*,*Z*). The velocity of the control volume is defined by:

[1] Ph.D., Assistant Professor, Department of Mechanical Engineering, University of Idaho, EP 324F, 875 Perimeter Drive MS 0902, Moscow, ID 83844-0902, United States; Phone: (208) 629-5518 (H), (208) 885-9032 (O); Fax: (208) 885-9031
E-mail: xing@uidaho.edu Web: http://www.taoxing.net

$$\underline{V}_{CS} = \dot{\underline{R}} + \underline{\Omega} \times \underline{r} \tag{1}$$

In relative inertial coordinates (Fig. 1(b)), the position vector of *o* is $\underline{R}'$ in (*X´*,*Y´*,*Z´*). $\underline{S}' = \underline{R}' + \underline{r}$ and $\underline{S}_O = \underline{R}_O + \underline{S}'$ are the instantaneous position vectors of any grid point or fluid particle in (*X´*,*Y´*,*Z´*) and (*X*,*Y*,*Z*), respectively. The ship translation velocity is $\dot{\underline{R}}' = \partial \underline{R}' / \partial t = u_i' \hat{i}_0 + v_i' \hat{j}_0 + w_i' \hat{k}_0$, where unit vectors of (*X´*,*Y´*,*Z´*) are the same as those for (*X*,*Y*,*Z*). u_i', v_i', and w_i' are the surge, sway, and heave velocities of the ship respect to (*X´*,*Y´*,*Z´*). The ship rotation is angular velocity $\underline{\Omega} = \Omega_{X'} \hat{i}_0 + \Omega_{Y'} \hat{j}_0 + \Omega_{Z'} \hat{k}_0$ of (*x*,*y*,*z*) in (*X´*,*Y´*,*Z´*). The velocities of the control volume in (*X´*,*Y´*,*Z´*) and (*X*,*Y*,*Z*) are:

$$\underline{V}'_{CS} = \dot{\underline{R}}' + \underline{\Omega} \times \underline{r} \tag{2}$$

$$\underline{V}_{CS} = \dot{\underline{R}}_O + \dot{R}' + \underline{\Omega} \times \underline{r} \tag{3}$$

where the relative inertial coordinates are moving at a constant speed V_C respect to (*X*,*Y*,*Z*), $\dot{\underline{R}}_O = V_C \hat{i}_0$. Comparison between equations (2) and (3) shows:

$$\underline{V}_{CS} = \dot{\underline{R}}_O + \underline{V}_{CS}' \tag{4}$$

The red control volume performs up to 6 degrees of freedom (6DOF) motions (surge, sway, heave, roll, pitch, and yaw). The green control volume performs up to 3DOF (surge, sway, yaw) motions copied from the corresponding degrees of freedom of the ship's motions. Any number of degrees of freedom can be imposed and the rest is predicted by 6DOF solvers, which results in captive, free, or semi-captive motions. $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the unit vectors of *x*, *y*, and *z* axes, respectively.

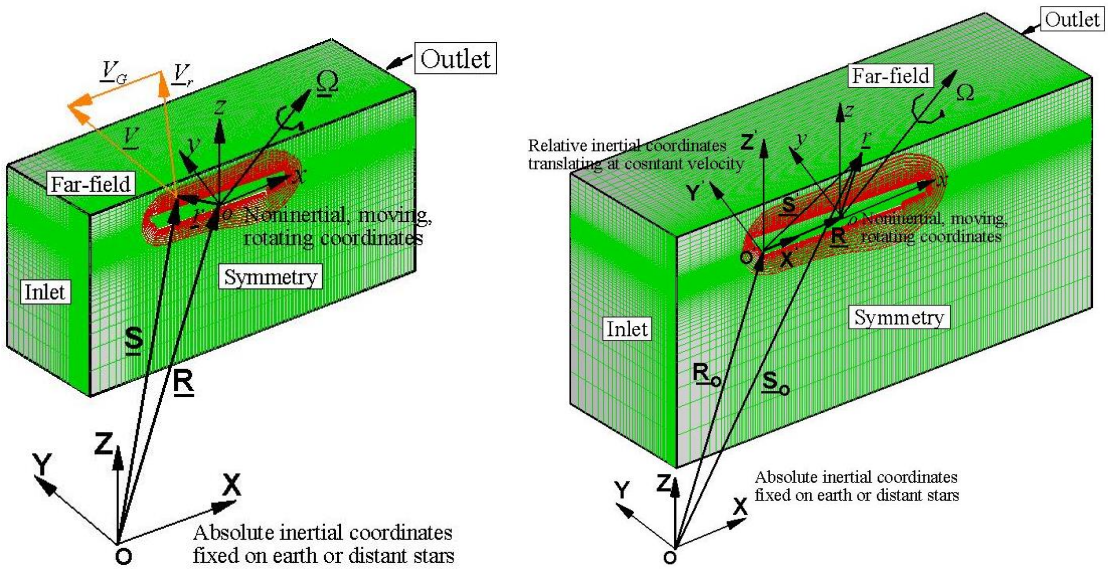

Figure 1 Definition of different coordinates for ship Hydrodynamics simulations with absolute inertial earth-fixed coordinates (X,Y,Z), non-inertial ship-fixed coordinates (x,y,z), and relative inertial coordinates (X',Y',Z'): (a) solve in (X,Y,Z), (b) solve in (X',Y',Z').

1. Velocity correlations

1.1 Correlation between (X,Y,Z) and (x,y,z)

If $\underline{r}$ represents the position vector of a fluid particle in (x,y,z), the position vector of that fluid particle in (X,Y,Z) is:

$$\underline{S} = \underline{R} + \underline{r} \tag{5}$$

The absolute velocity in (X,Y,Z) and relative velocity in (x,y,z) are defined by Eqns. (6) and (7), respectively:

$$\underline{V} = d\underline{S}/dt \tag{6}$$

$$\underline{V}_r = \tilde{d}\underline{r}/dt \tag{7}$$

Differentiate Eqn. (5) is differentiated respect to time and with use of the correlation of d/dt in (X,Y,Z) and $\tilde{d}/dt$ in (x,y,z) [Greenwood, 1988]:

$$\frac{d}{dt} = \frac{\tilde{d}}{dt} + \underline{\Omega}\times \tag{8}$$

The velocities are related by:

$$\underline{V} = \underline{V}_r + \underline{V}_{CS} \tag{9}$$

1.2 Correlation between (X,Y,Z) and (X',Y',Z')

Derivation similar to 1.1 is applied in (X',Y',Z'). The absolute velocity in (X,Y,Z) and relative velocity in (X',Y',Z') are defined by equations (10) and (11), respectively:

$$\underline{V} = d\underline{S}_O/dt \tag{10}$$

$$\underline{V}_r' = d\underline{S}'/dt \tag{11}$$

Use Equation (8) and differentiate $\underline{S}' = \underline{R}' + \underline{r}$ respect to time:

$$\underline{V}_r' = \underline{V}_r + \underline{V}_{CS}' \tag{12}$$

Differentiate $\underline{S}_O = \underline{R}_O + \underline{S}'$ respect to time:

$$\underline{V} = d\left(\underline{R}_O + \underline{S}'\right)/dt = \dot{\underline{R}}_O + \underline{V}_r' \tag{13}$$

2. Reynolds transport theorem (RTT)

RTT for flow with an arbitrary CV moving at $\underline{V}_R$ is applied:

$$\frac{dB_{SYS}}{dt} = \frac{d}{dt}\int_{CV} \beta\rho\, d\forall + \int_{CS} \beta\rho \underline{V}_R \cdot \underline{n}\, dA \tag{14}$$

For a non-deforming CV, RTT for incompressible flow:

$$\frac{dB_{syst}}{dt} = \rho \int_{CV} \frac{\partial \beta}{\partial t} d\forall + \rho \int_{CS} \beta \underline{V}_R \cdot \underline{n} dA \tag{15}$$

To convert it to the differential form of RTT, the flux term is transformed to a volume integral using the Gauss divergence theorem:

$$\begin{aligned} \frac{dB_{syst}}{dt} &= \rho \int_{CV} \frac{\partial \beta}{\partial t} d\forall + \rho \int_{CV} \nabla \cdot \left(\beta \underline{V}_R \right) d\forall \\ &= \rho \int_{CV} \left\{ \frac{\partial \beta}{\partial t} + \nabla \cdot \left(\beta \underline{V}_R \right) \right\} d\forall \end{aligned} \tag{16}$$

Taking limit for elemental CV for which integrand is independent of the volume $\forall$ and divided by $\forall$:

$$\frac{dB_{syst}}{dt} = \rho \left\{ \frac{\partial \beta}{\partial t} + \nabla \cdot \left(\beta \underline{V}_R \right) \right\} \tag{17}$$

3. Application of RTT for (X, Y, Z):

3.1 Integral form for a CV

Conservation of mass with $B_{syst} = M$, $\beta = 1$ for Eqn.(15) and note steady flow with $\underline{V}_R = \underline{V} - \underline{V}_{CS}$,

$$\int_{CS} \left(\underline{V} - \underline{V}_{CS} \right) \cdot \underline{n} dA = 0 \tag{18}$$

Conservation of momentum with $B_{syst} = M\underline{V}$ and $\beta = \underline{V}$ for Eqn. (15):

$$\frac{d(M\underline{V})}{dt} = \sum \underline{F} = \rho \int_{CV} \frac{\partial \underline{V}}{\partial t} d\forall + \rho \int_{CS} \underline{V} (\underline{V} - \underline{V}_{CS}) \cdot \underline{n} dA \tag{19}$$

For steady flow:

$$\sum \underline{F} = \rho \int_{CS} \underline{V}(\underline{V} - \underline{V}_{CS}) \cdot \underline{n} dA \tag{20}$$

3.2 Differential form

For **mass conservation**, substitute $B_{syst} = M$, $\beta = 1$ and $\underline{V}_R = \underline{V} - \underline{V}_G$ (when the CV is limited to a point, $\underline{V}_G$ is the local grid velocity so $\underline{V}_{CS} = \underline{V}_G$) to Eqn. (17) and note $\nabla \cdot \underline{V}_G = 0$:

$$\nabla \cdot \underline{V} = 0 \tag{21}$$

For **momentum equation**, substitute $B_{syst} = M\underline{V}$ and $\beta = \underline{V}$ to Eqn. (17):

$$\frac{d}{dt}\left(M\underline{V}\right)_{syst} = \rho \left\{ \frac{\partial \underline{V}}{\partial t} + \nabla \cdot \left[\underline{V}\left(\underline{V} - \underline{V}_G\right) \right] \right\} = \sum \underline{f} \tag{22}$$

The 2nd term on the left-hand-side can be simplified using the divergence operator expansion with the use of the continuity equation,

$$\nabla \cdot \left[\underline{V}\left(\underline{V} - \underline{V}_G\right) \right] = \underline{V} \underbrace{\nabla \cdot \left(\underline{V} - \underline{V}_G\right)}_{=0 \quad continuity} + \left(\underline{V} - \underline{V}_G\right) \cdot \nabla \underline{V} \tag{23}$$

After the body and surface forces are expressed per unit volume, the NS equation in (X,Y,Z) is:

$$\rho \left[\underbrace{\frac{\partial \underline{V}}{\partial t}}_{1} + \underbrace{\left(\underline{V} - \underline{V}_G\right) \cdot \nabla \underline{V}}_{2} \right] = -\nabla\left(p + \gamma Z\right) + \underbrace{\mu \nabla^2 \underline{V}}_{3} \tag{24}$$

4. Application of RTT for $\left(X', Y', Z'\right)$:

4.1 Integral form for a CV

For **mass conservation**, $B_{syst} = M$ and $\beta = 1$ for Eqn. (15) and note $\underline{V}_R = \underline{V}'_r$. For steady flow:

$$\int_{CS} \underline{V}'_r \cdot \underline{n} dA = 0 \tag{25}$$

Conservation of momentum with $B_{syst} = M\left(\underline{V}'_r + \underline{V}'_{CS}\right)$, $\beta = \underline{V}'_r + \underline{V}'_{CS}$ and $\underline{V}_R = \underline{V}'_r$

$$\frac{d[M\left(\underline{V}'_r + \underline{V}'_{CS}\right)]}{dt} =$$

$$\sum \underline{F} = \rho \int_{CV} \frac{\partial\left(\underline{V}'_r + \underline{V}'_{CS}\right)}{\partial t} d\forall + \rho \int_{CS} \left(\underline{V}'_r + \underline{V}'_{CS}\right) \underline{V}'_r \cdot \underline{n} dA \tag{26}$$

For steady flow with the use of continuity:

$$\sum \underline{F} = \rho \int_{CS} \left(\underline{V}'_r + \underline{V}'_{CS}\right) \underline{V}'_r \cdot \underline{n} dA$$

$$= \rho \int_{CS} \underline{V}'_r \underline{V}'_r \cdot \underline{n} dA + \cancelto{0}{\rho \underline{V}'_{CS} \int_{CS} \underline{V}'_r \cdot \underline{n} dA} \tag{27}$$

$$\sum \underline{F} = \rho \int_{CS} \underline{V}'_r \underline{V}'_r \cdot \underline{n} dA \tag{28}$$

4.2 Differential form

For mass conservation, replace $\underline{V}$ in Eqn. (21) using Eqn. (13):

$$\nabla \cdot \underline{V}'_r = 0 \tag{29}$$

For conservation of momentum, $\underline{V}_G$ and $\underline{V}$ in equation (24) are substituted by Eqns. (4) and (13). Note the time derivatives in (X',Y',Z') and (X,Y,Z) are the same as they are both inertial coordinates, term [1] in equation (24) becomes:

$$\frac{\partial \underline{V}}{\partial t} = \underset{0}{\ddot{\underline{R}}_O} + \frac{\partial \underline{V}_r'}{\partial t} = \frac{\partial \underline{V}_r'}{\partial t} \tag{30}$$

For term [2] in Eqn. (24),

$$\left(\underline{V} - \underline{V}_G\right) \cdot \nabla \underline{V} = \left(\underline{V}_r' + \dot{\underline{R}}_O - \underline{V}_G' - \dot{\underline{R}}_O\right) \cdot \nabla \left(\underline{V}_r' + \dot{\underline{R}}_O\right) = \left(\underline{V}_r' - \underline{V}_G'\right) \cdot \nabla \underline{V}_r' \tag{31}$$

For term [3] in Eqn. (24),

$$\nabla^2 \underline{V} = \nabla^2 \left(\underline{V}_r' + \dot{\underline{R}}_O\right) = \nabla^2 \underline{V}_r' \tag{32}$$

Equations (30) to (32) are substituted back to equation (24) and note gradient, divergence, and Laplacian operators are frame invariant. The governing equations in terms of $\underline{V}_r'$ are obtained:

$$\rho\left[\frac{\partial \underline{V}_r'}{\partial t} + \left(\underline{V}_r' - \underline{V}_G'\right) \cdot \nabla \underline{V}_r'\right] = -\nabla\left(p + \gamma Z'\right) + \mu \nabla^2 \underline{V}_r' \tag{33}$$

5. Application of RTT for (*x*,*y*,*z*):

5.1 Integral form for a CV

For **mass conservation**, $B_{syst} = M$, $\beta = 1$ for Eqn.(15) and note steady flow,

$$\int_{CS} \underline{V}_R \cdot \underline{n} dA = 0 \tag{34}$$

Conservation of momentum with $B_{syst} = M\underline{V} = M\left(\underline{V}_r + \underline{V}_{CS}\right)$, $\beta = \underline{V}$

$$\frac{d\left(M\underline{V}\right)}{dt} =$$

$$\sum \underline{F} = \rho \int_{CV} \frac{\partial \underline{V}}{\partial t} d\forall + \rho \int_{CS} \left(\underline{V}_r + \underline{V}_{CS}\right)\underline{V}_R \cdot \underline{n} dA \tag{35}$$

$$= \rho \int_{CV} \frac{\partial \underline{V}}{\partial t} d\forall + \rho \int_{CS} \underline{V}_r \underline{V}_R \cdot \underline{n} dA + \rho \underline{V}_{CS} \cancelto{0}{\int_{CS} \underline{V}_R \cdot \underline{n} dA}$$

$$\frac{\partial \underline{V}}{\partial t} = \ddot{\underline{R}} + \frac{\partial \underline{V}_r}{\partial t} + \frac{\partial\left(\underline{\Omega} \times \underline{r}\right)}{\partial t}$$

$$= \frac{\tilde{\partial} \underline{V}_r}{\partial t} + \underbrace{\ddot{\underline{R}} + 2\underline{\Omega} \times \underline{V}_r + \dot{\underline{\Omega}} \times \underline{r} + \underline{\Omega} \times \left(\underline{\Omega} \times \underline{r}\right)}_{\underline{a}_{rel}} \tag{36}$$

where

$$\dot{\underline{\Omega}} = \frac{\partial \underline{\Omega}}{\partial t} = \frac{\tilde{\partial} \underline{\Omega}}{\partial t} + \underbrace{\underline{\Omega} \times \underline{\Omega}}_{=0} = \frac{\tilde{\partial} \underline{\Omega}}{\partial t} \tag{37}$$

So,

$$\sum \underline{F} - \int_{CV} \underline{a}_{rel} dm = \rho \int_{CV} \frac{\tilde{\partial} \underline{V}_r}{\partial t} d\forall + \rho \int_{CS} \underline{V}_r \underline{V}_R \cdot \underline{n} dA \tag{38}$$

Note when the CV is moving together with (x,y,z), the $\underline{V}_R$ Fin the above equations will be replaced by $\underline{V}_r$

5.2 Differential form

For mass conservation, replace $\underline{V}$ in Eqn. (21) using Eqn. (9):

$$\nabla \cdot \underline{V}_r = 0 \tag{39}$$

The differential form for momentum equation in (x,y,z) can be derived from Eqn. (24), note

$$\underline{V}_G = \dot{\underline{R}} + \underline{\Omega} \times \underline{r} \tag{40}$$

$$\underline{V} = \underline{V}_r + \underline{V}_G \tag{41}$$

For term [1], from Eqn.(36),

$$\frac{\partial \underline{V}}{\partial t} = \frac{\tilde{\partial} \underline{V}_r}{\partial t} + \underline{a}_{rel} \tag{42}$$

For term [2],

$$\left(\underline{V} - \underline{V}_G\right) \cdot \nabla \underline{V} = \underline{V}_r \cdot \nabla \left(\underline{V}_r + \underline{V}_G\right) = \underline{V}_r \cdot \nabla \underline{V}_r \tag{43}$$

For term [3],

$$\nabla^2 \underline{V} = \nabla^2 \underline{V}_r + \underbrace{\nabla^2 \underline{V}_G}_{0} = \nabla^2 \underline{V}_r \tag{44}$$

The above four equations are substituted back to the differential form of the momentum equation in (X,Y,Z) and note gradient, divergence, and Laplacian operators are frame invariant. The governing equations in terms of $\underline{V}_r$ are obtained:

$$\rho \left[\frac{\tilde{\partial} \underline{V}_r}{\partial t} + \underline{V}_r \cdot \nabla \underline{V}_r \right] = \underbrace{-\rho \underline{a}_{rel}}_{body\ force} - \nabla \left(p + \gamma z \right) + \mu \nabla^2 \underline{V}_r \tag{45}$$

where

$$\underline{a}_{rel} = \dot{\underline{R}} + 2\underline{\Omega} \times \underline{V}_r + \underline{\Omega} \times \left(\underline{\Omega} \times \underline{r} \right) + \dot{\underline{\Omega}} \times \underline{r} \tag{46}$$

6. Comments on application of different approaches

6.1 Solving fluid mechanics problems analytically using integral forms

As shown in the lecture notes, it is easier to solve the continuity and momentum equation in the relative inertial coordinates.

6.2 Solving differential equations using CFD [Xing et al., 2008]

The governing differential equations (GDEs) for continuity and momentum in (*X*,*Y*,*Z*) and (*X´*,*Y´*,*Z´*) are transformed from the physical domain in Cartesian coordinates (X, Y, Z, t) to the computational domain in non-orthogonal curvilinear coordinates (ξ, η, ζ, τ) using the chain rule without involving grid velocity for the time derivative transformation:

$$\frac{1}{J}\frac{\partial}{\partial \xi^j}\left(b_i^j U_i\right) = 0 \tag{47}$$

$$\frac{\partial U_i}{\partial \tau} + \frac{1}{J} b_j^k \left(U_j - U_{Gj}\right)\frac{\partial U_i}{\partial \xi^k} = -\frac{1}{J} b_i^k \frac{\partial p}{\partial \xi^k} + \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\frac{b_i^j b_i^k}{J\,\mathrm{Re}_{eff}}\frac{\partial U_i}{\partial \xi^k}\right) + \frac{b_j^k}{J}\frac{\partial \nu_t}{\partial \xi^k}\frac{b_i^l}{J}\frac{\partial U_j}{\partial \xi^l} + S_i \tag{48}$$

Eqs. (47) and (48) are identical to Warsi *et al.*, who transformed the standard NS equations in (X, Y, Z, t) to a fixed computational domain (ξ, η, ζ, τ) with the inclusion of the grid velocity $\partial X_i/\partial \tau$ in the time derivative transformation. In the present derivation of Eqs. (24) and (33) the solution domain and control volume are coincident with each other, which conceptually better shows the relationship between the moving but non-deforming control volume and solution domain and additionally provides the continuous $\underline{V}_G/\underline{V}'_G$ form of the NS equations. Compared with Eq. (33), Eq. (24) allows non-constant $\underline{V}_C$. When $\underline{V}_C = 0$, Eq. (33) reduces to Eq. (24). Transformation of the continuous $\underline{V}_G$ form of the NS equations in (*X*,*Y*,*Z*) into (*x*,*y*,*z*) clearly shows the difference of GDEs using the absolute inertial earth-fixed or non-inertial ship-fixed coordinate systems. Compared with Eq. (45), application of Eq. (24) simplifies the specification of boundary conditions, saves computational cost by reducing the solution domain size, and can be easily applied to

simulate multi-objects such as ship-ship interactions. In general, implementation of Eq. (24) to simulate captive, semi-captive, or full 6DOF ship motions is straightforward.

References

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