The exam is closed book and closed notes.

Consider steady, two-dimensional, incompressible viscous flow of a Newtonian fluid between parallel plates a distant $h$ apart, as shown in the Figure below. The upper plate moves at constant velocity $U$ while the lower plate is at rest. The pressure varies linearly in the $x$ direction: $p = p(x) = Cx$. If the plates are very wide and very long and the flow is fully developed, neglect gravity and find the velocity distribution between the plates.

**Incompressible Continuity Equation:**

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

**Incompressible Navier-Stokes Equations in Cartesian Coordinates:**

\[
\begin{align*}
\rho g_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) &= \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \\
\rho g_y - \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) &= \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \\
\rho g_z - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) &= \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)
\end{align*}
\]
Solution:

KNOWN: $p(x)$

(1) FIND: $u(y)$

ASSUMPTIONS: incompressible, steady, Newtonian flow, no gravity effects

ANALYSIS:

The plate is very wide and very long and the flow is fully developed, therefore the flow is essentially axial: $v = w = 0$.

Continuity:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (0.5)
\]

\[
\frac{\partial u}{\partial x} + 0 + 0 = 0 \quad (0.5)
\]

\[
\Rightarrow u = u(y) \text{ only} \quad (0.5)
\]

$x$-momentum:

\[
\rho g_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \rho \left( \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \quad (1)
\]

\[
0 - \frac{\partial p}{\partial x} + \mu \left( 0 + \frac{\partial^2 u}{\partial y^2} + 0 \right) = \rho \left( 0 + 0 + 0 + 0 \right) \quad (1)
\]

\[
\frac{\partial p}{\partial x} = C \quad (0.5)
\]

\[
\Rightarrow \mu \frac{\partial^2 u}{\partial y^2} = C
\]

Integrate twice:

\[
u = \frac{1}{\mu} C \frac{y^2}{2} + C_1 y + C_2 \quad (1)
\]
Boundary conditions:

\[ \text{at } y = 0: \ u(y) = 0 \quad \rightarrow \quad C_2 = 0 \quad (0.5) \]

\[ \text{at } y = h: \ u(y) = U \quad \rightarrow \quad C_1 = \frac{U}{h} - \frac{Ch}{2\mu} \quad (1) \]

Replace and find:

\[ u = \frac{1}{\mu} \cdot \frac{C y^2}{2} + \left( \frac{U}{h} - \frac{Ch}{2\mu} \right) y \quad (0.5) \]