Consider steady, two-dimensional, incompressible viscous flow of a Newtonian fluid between parallel plates a distance $h$ apart, as shown in the Figure below. The upper plate moves at constant velocity $U_t$ while the lower plate moves at constant velocity $U_b$. The pressure varies linearly in the $x$ direction ($p(x) = Cx$). If the plates are very wide and very long and the flow is parallel ($v = w = 0$), neglect gravity and find the velocity distribution between the plates using continuity and momentum equations.

**Incompressible Continuity Equation:**
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

**Incompressible Navier-Stokes Equations in Cartesian Coordinates:**
\[
\begin{align*}
\rho g_x &= \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \\
\rho g_y &= \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) = \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) \\
\rho g_z &= \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) = \rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)
\end{align*}
\]
Solution:  

Continuity:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

\[ \frac{\partial u}{\partial x} + 0 + 0 = 0 \quad (3) \]

\[ \frac{\partial u}{\partial x} = 0: \text{ the flow is fully developed } \Rightarrow u = u(y) \quad (5) \quad (+1) \]

\( x \)-momentum:

\[ \rho g_x - \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \]

\[ 0 - \frac{\partial p}{\partial x} + \mu \left( 0 + \frac{\partial^2 u}{\partial y^2} + 0 \right) = \rho \left( 0 + 0 + 0 + 0 \right) \quad (4) \]

\[ \frac{\partial p}{\partial x} = C \]

\[ \Rightarrow \mu \frac{\partial^2 u}{\partial y^2} = C \quad (+3) \]

Integrate twice:

\[ u = \frac{1}{\mu} C \frac{y^2}{2} + C_1 y + C_2 \quad (+1) \]

Boundary conditions:

at \( y = 0 \): \( u(y) = U_b \) \quad \Rightarrow \quad C_2 = U_b

at \( y = h \): \( u(y) = U_t \) \quad \Rightarrow \quad C_1 = \frac{U_t - U_b}{h} \frac{Ch}{2\mu}

Replace and find:

\[ u = \frac{1}{\mu} C \frac{y^2}{2} + \left( \frac{U_t - U_b}{h} - \frac{Ch}{2\mu} \right) y + U_b \quad (+2) \]