Broad categories of fluid flow and canonical geometries

Category	Canonical geometry
Internal Flows	Circular pipe

exact laminar
D_h turbulent — noncircular ducts

h_m — transitions : contraction or diffuser
fittings

h_p, h_t — pumps / turbines

$$\frac{p_1}{\gamma} + z_1 + \alpha_1 \frac{V_1^2}{2g} + h_p = \frac{p_2}{\gamma} + z_2 + \alpha_2 \frac{V_2^2}{2g} + h_t + h_L + \Sigma h_m$$

External Flows

BL theory — Slender bodies — flat plate BL

C_D — bluff body — Sphere : Stokes flow

appendages

$$\frac{F}{\mu U R} = 6\pi$$

$$C_D = 24/Re$$

$2/3\ C_f$ & $1/3\ C_p$

Free shear flows

mixing layers	2D
jets	2D and axi
wakes	2D and axi

Circular pipe

laminar flow $Re = \frac{U_{ave} D}{\nu} < 2000$

$u(r) = u_{max}\left(1 - r^2/R^2\right)$ $\quad u_{max} = \frac{R^2}{4\mu}\left(-\frac{d\hat{p}}{dx}\right)$ $\quad \hat{p} = p + \gamma z$

$\tau_w = \frac{8\mu U_{ave}}{D}$ $\quad \Delta h = h_1 - h_2 = h_L = f\,\frac{L}{D}\frac{V^2}{2g}$

$f = 64/Re$ $\quad C_f = 16/Re$ $\quad Po = C_f Re = 16$

Re only determines transition; since, creeping flow and no inertia force and mass absent

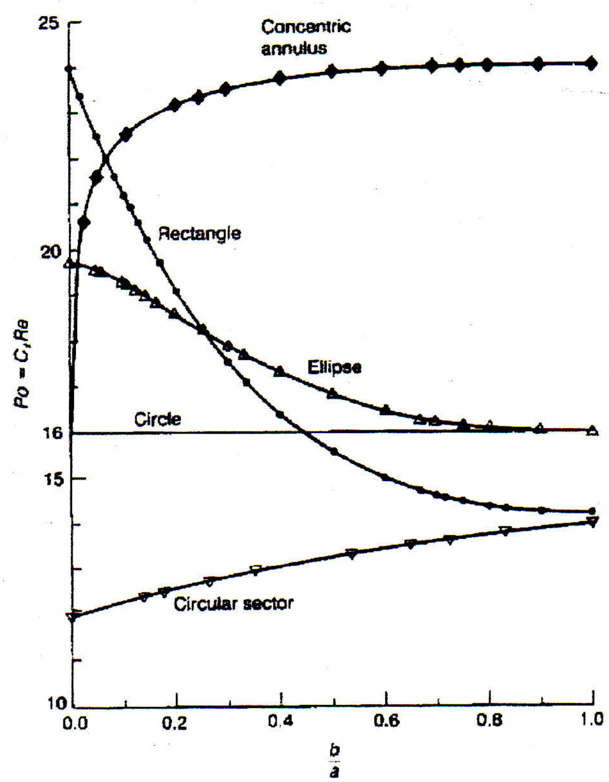


FIGURE 3-13
Comparison of Poiseuille numbers for various duct cross sections when Reynolds number is scaled by the hydraulic diameter. [*Numerical data taken from Shah and London (1978).*]

turbulent flow

Kolmogorov scales:

$$\varepsilon = \frac{u_0^2}{\tau_0} = \frac{u_0^3}{l_0}$$ $u_0^2 = k$ = kinetic energy

$l_0 = L_\delta$ = width of flow

is size of largest eddy

dissipation $= \frac{d}{dt}(KE)$ where $\frac{d(KE)}{dt}$ is from largest scales

ε occurs at smallest scales (energy cascade) at which turbulence is isotropic and has universal form

$\eta = (\nu^3/\varepsilon)^{1/4}$ length $\eta / l_0 = Re^{-3/4}$

$u_\eta = (\varepsilon \nu)^{1/4}$ velocity $u_\eta / u_0 = Re^{-1/4}$

$\tau_\eta = (\nu/\varepsilon)^{1/2}$ time $\tau_\eta / \tau_0 = Re^{-1/2}$

micro scale << large scale and range of scales increase Re power law

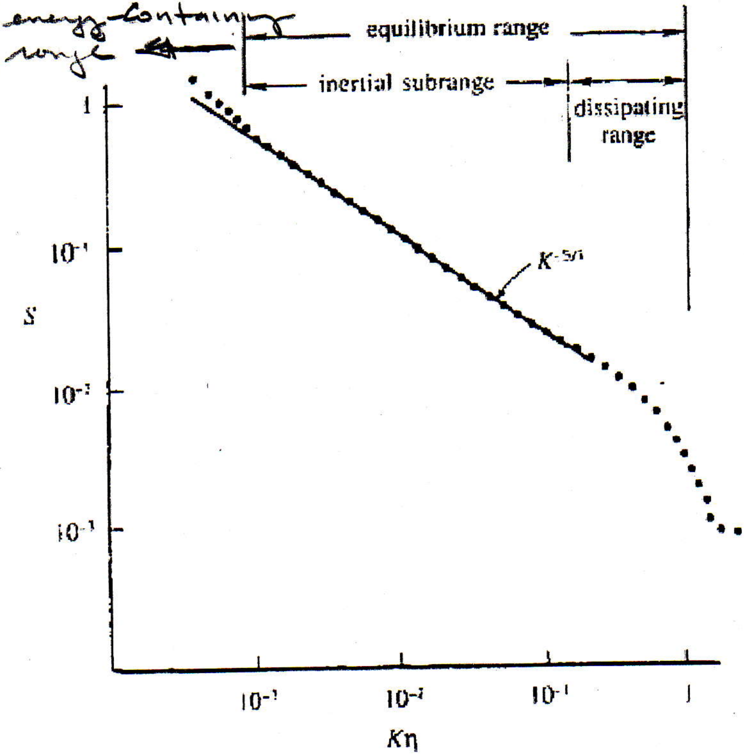


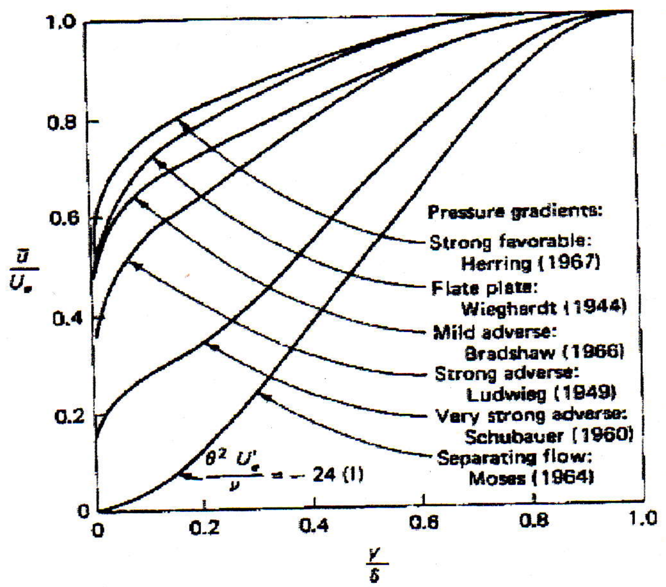
Fig. 12.12 A typical wavenumber spectrum observed in the ocean, plotted on a log–log scale. The unit of S is arbitrary, and the dots represent hypothetical data.

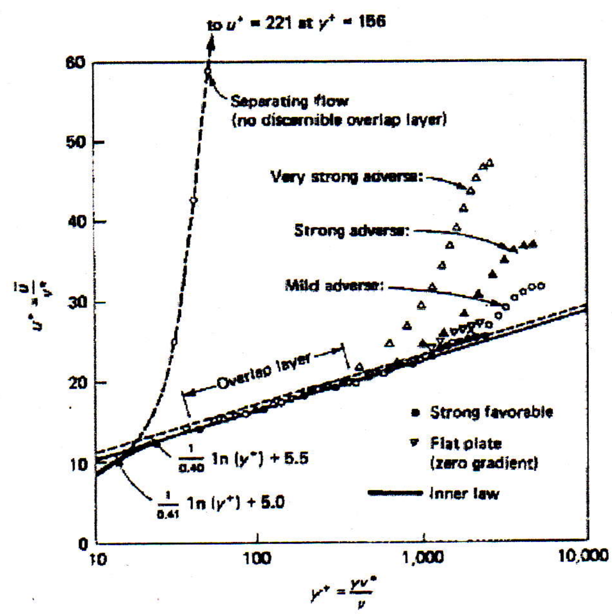
triple layer mean velo. profile:

$u^+ = y^+$ $\qquad u^+ = u/u^*$ $\quad y^+ = \frac{y u^*}{\nu}$ $\quad u^* = \sqrt{\tau_w/\rho}$

$u^+ = \frac{1}{\kappa}\ln y^+ + B$

$\frac{U-u}{u^*} = -\frac{1}{\kappa}\ln y/\delta + A$





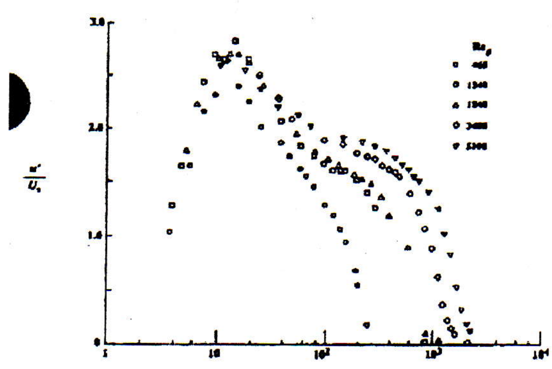


Fig 18. Variation of the distribution of turbulence intensity in wall variables with Reynolds number. Boundary layer data from Purtell *et al* (1981).

$f = f(Re, \varepsilon/d)$ based on $u_{ave} = \dfrac{Q}{A} = \dfrac{1}{\pi R^2}\displaystyle\int_0^R U\, 2\pi r\, dr$

and $U(r)$ from log-law

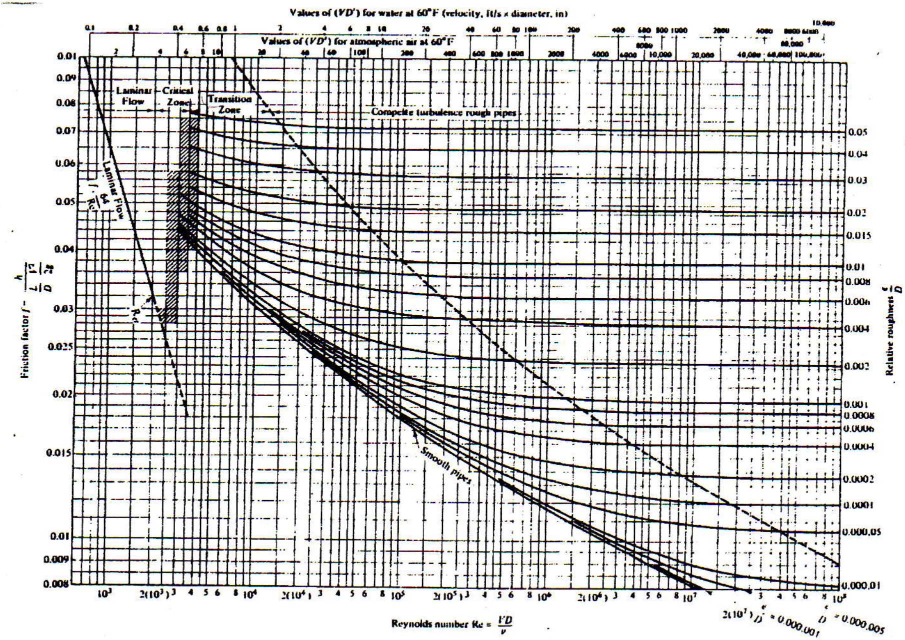


Fig. 6.13 **The Moody chart for pipe friction with smooth and rough walls.** (*From Ref. 8, by permission of the ASME.*)

$$h_f = -\Delta h = -\left(\frac{\Delta p}{\gamma} + \Delta z\right) = f\frac{L}{D}\frac{V^2}{2g}$$

$\Delta h = h_2 - h_1$

$$h_f = .316\left(\frac{\mu}{\rho VD}\right)^{1/4}\frac{L}{D}\frac{V^2}{2g}$$

$$h_f \propto V^{1.75}$$

(recall $h_f \propto V$ for laminar flow)

Other useful relationships

Power law fit to velocity profile:

$$\frac{u}{u_{max}} = \left(\frac{y}{r_o}\right)^m \qquad y = r_o - r$$

$$m = m(Re)$$

$$\frac{u_{max}}{u^*} = \frac{1}{\kappa}\ln\frac{r_o u^*}{r} + B$$

$$\frac{V}{u_{max}} = \left(1 + 1.33 f^{1/2}\right)^{-1}$$

TABLE 10.1 EXPONENTS FOR POWER-LAW EQUATION AND RATIO OF MEAN TO MAXIMUM VELOCITY

Re →	**4×10^3**	**2.3×10^4**	**1.1×10^5**	**1.1×10^6**	**3.2×10^6**
$m \rightarrow$	$\frac{1}{6.0}$	$\frac{1}{6.6}$	$\frac{1}{7.0}$	$\frac{1}{8.8}$	$\frac{1}{10.0}$
$\overline{V}/V_{max} \rightarrow$	0.791	0.807	0.817	0.850	0.865

SOURCE: Schlichting (36). Used with permission of the McGraw-Hill Companies.

flat plate BL

laminar

$u_x + v_y = 0$

$u u_x + v u_y = \nu u_{yy}$

$u(x,0) = v(x,0) = 0$

$u(x,\delta) = U$

$$\frac{u}{U} = 0.99 \text{ when } \eta = 3.5 \rightarrow \boxed{\frac{\delta}{x} = \frac{5}{\sqrt{\mathrm{Re}_x}}} \quad \mathrm{Re}_x = \frac{Ux}{\nu}$$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \int_0^\infty \left(1 - f'\right) d\eta \sqrt{\frac{2\nu x}{U}} \rightarrow \boxed{\frac{\delta^*}{x} = \frac{1.7208}{\sqrt{\mathrm{Re}_x}}}$$

$$\theta = \int_0^\infty \left(1 - \frac{u}{U}\right) \frac{u}{U} dy = \int_0^\infty \left(1 - f'\right) f' \sqrt{\frac{2\nu x}{U}} d\eta \rightarrow \frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

So, $\boxed{\dfrac{\delta^*}{\theta} = H = 2.59}$

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_w = \frac{\mu U f''(0)}{\sqrt{2\nu x/U}} \quad \rightarrow \quad \boxed{C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} = \frac{\theta}{x}}$$

$$\boxed{C_D = \frac{D}{\frac{1}{2}\rho U^2 L} = \int_0^L C_f \frac{dx}{L} = \frac{1.328}{\sqrt{\mathrm{Re}_L}}} \qquad \mathrm{Re}_L = \frac{UL}{\nu}$$

$$\frac{v}{U} = \frac{\eta f' - f}{\sqrt{2\,\mathrm{Re}_x}} << 1 \quad \text{for} \quad \mathrm{Re}_x >> 1$$

TABLE 4-1
Numerical solution of the Blasius flat-plate relation, Eq. (4-45)

η	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
0.0	0.0	0.0	0.46960
0.1	0.00235	0.04696	0.46956
0.2	0.00939	0.09391	0.46931
0.3	0.02113	0.14081	0.46861
0.4	0.03755	0.18761	0.46725
0.5	0.05864	0.23423	0.46503
0.6	0.08439	0.28058	0.46173
0.7	0.11474	0.32653	0.45718
0.8	0.14967	0.37196	0.45119
0.9	0.18911	0.41672	0.44363
1.0	0.23299	0.46063	0.43438
1.1	0.28121	0.50354	0.42337
1.2	0.33366	0.54525	0.41057
1.3	0.39021	0.58559	0.39598
1.4	0.45072	0.62439	0.37969
1.5	0.51503	0.66147	0.36180
1.6	0.58296	0.69670	0.34249
1.7	0.65430	0.72993	0.32195
1.8	0.72887	0.76106	0.30045
1.9	0.80644	0.79000	0.27825
2.0	0.88680	0.81669	0.25567
2.2	1.05495	0.86330	0.21058
2.4	1.23153	0.90107	0.16756
2.6	1.41482	0.93060	0.12861
2.8	1.60328	0.95288	0.09511
3.0	1.79557	0.96905	0.06771
3.2	1.99058	0.98037	0.04637
3.4	2.18747	0.98797	0.03054
3.6	2.38559	0.99289	0.01933
3.8	2.58450	0.99594	0.01176
4.0	2.78388	0.99777	0.00687
4.2	2.98355	0.99882	0.00386
4.4	3.18338	0.99940	0.00208
4.6	3.38329	0.99970	0.00108
4.8	3.58325	0.99986	0.00054

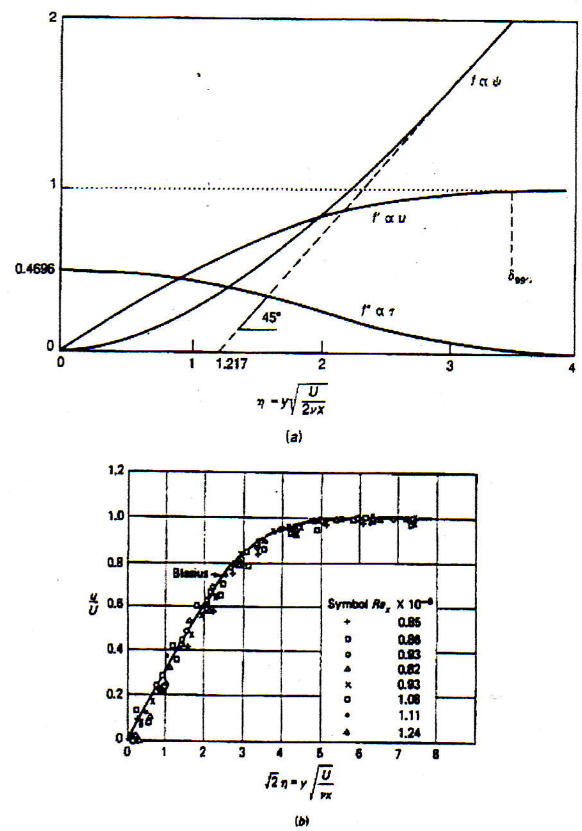


FIGURE 4-6
The Blasius solution for the flat-plate boundary layer: (a) numerical solution of Eq. (4-45); (b) comparison of $f' = u/U$ with experiments by Liepmann (1943).

Approximate solution Turbulent Boundary-Layer

$Re_t \sim 3 \text{ X } 10^6$ for a flat plate boundary layer

$Re_{crit} \sim 500{,}000$

$$\frac{c_f}{2} = \frac{d\theta}{dx}$$

as was done for the approximate laminar flat plate boundary-layer analysis, solve by expressing $c_f = c_f(\delta)$ and $\theta = \theta(\delta)$ and integrate, i.e.

assume log-law valid across entire turbulent boundary-layer

$$\frac{u}{u^*} = \frac{1}{\kappa}\ln\frac{yu^*}{\nu} + B$$

neglect laminar sub layer and velocity defect region

at $y = \delta$, $u = U$

$$\frac{U}{u^*} = \frac{1}{\kappa}\ln\frac{\delta u^*}{\nu} + B$$

$$Re_\delta\left(\frac{c_f}{2}\right)^{1/2}$$

or $$\left(\frac{2}{c_f}\right)^{1/2} = 2.44\ln\left[Re_\delta\left(\frac{c_f}{2}\right)^{1/2}\right] + 5$$

$$c_f \cong .02\,Re_\delta^{-1/6} \quad \text{power-law fit}$$

$c_f(\delta)$

Next, evaluate

$$\frac{d\theta}{dx} = \frac{d}{dx}\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy$$

can use log-law or more simply a power law fit

$$\left.\begin{aligned} &\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} \\ &\theta = \frac{7}{72}\delta = \theta(\delta) \end{aligned}\right\}$$

Note: can not be used to obtain $c_f(\delta)$ since $\tau_w \to \infty$

$$\Rightarrow \qquad \tau_w = c_f \frac{1}{2}\rho U^2 = \rho U^2 \frac{d\theta}{dx} = \frac{7}{72}\rho U^2 \frac{d\delta}{dx}$$

$$Re_\delta^{-1/6} = 9.72\frac{d\delta}{dx}$$

or $\quad \dfrac{\delta}{x} = .16\,Re_x^{-1/7}$

$\delta \propto x^{6/7}$ almost linear

i.e., much faster growth rate than laminar boundary layer

$$c_f = \frac{.027}{Re_x^{1/7}}$$

$$C_f = \frac{.031}{Re_L^{1/7}} = \frac{7}{6}C_f(L)$$

Alternate forms given in text depending on experimental information and power-law fit used, etc. (i.e., dependent on Re range.)

Some additional relations given in texts for larger Re are as follows:

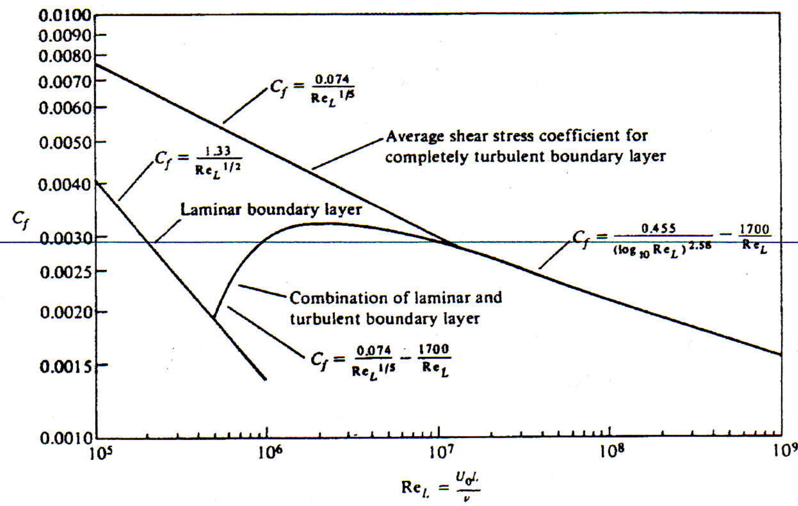
Total shear-stress coefficient

$$C_f = \frac{.455}{(\log_{10} Re_L)^{2.58}} \frac{-1700}{Re_L} \qquad Re > 10^7$$

$$\frac{\delta}{L} = c_f(.98 \log Re_L - .732)$$

Local shear-stress coefficient

$$c_f = (2 \log Re_x - .65)^{-2.3}$$



Finally, a composite formula that takes into account both the initial laminar boundary-layer (with translation at $Re_{CR} = 500{,}000$) and subsequent turbulent boundary layer is $C_f = \frac{.074}{Re_L^{1/5}} - \frac{1700}{Re_L}$ $\qquad 10^5 \le Re \le 10^7$

Free Shear Flows

Viscous flows which develop & spread in an infinite ambient fluid ie without walls or other boundaries

(1) mixing layer ie free shear layer between parallel streams of different U

(2) jets

(3) wakes

For large Re with dominant flow direction x, BL assumptions are valid (& assuming far from origin of mixing, jet, or wake)

$v \ll u$

$u_x \ll u_y$ & $u_{xx} \ll u_{yy}$

$p_y \sim 0$

Additionally

$p_x \sim 0$ since no boundaries &
far from origin

2D shear flow equations in Cartesian coordinates

$$u_x + v_y = 0$$

$$u u_x + v u_y = \nu u_{yy}$$

In all cases similarity solutions are possible. Note that not applicable in near field developing region where BL & $p_x \sim 0$ assumptions not valid.

2D Jet

Note: $Re_{cut} = 4$

$Re_{trans} \sim 30$

Assume: $Re = \frac{u_0 h}{\nu_\infty} >> 1$

$M = \rho \int_{-\infty} u^2 dy = \text{constant} = \rho u_0^2 h$

$u_x + v_y = 0$

$u u_x + v u_y = \nu u_{yy}$

$u_y = v = 0 \qquad y = 0$

$u \to 0 \qquad y \to \infty$ ($v \neq 0$ due to entrainment)

Entrainment = jet entrains surrounding fluid due to viscous forces

Similarity Solution:

$\chi = a x^m f(\eta) \qquad \eta = \frac{y}{b x^n}$

m, n exponents

n = rate of jet spreading

a, b make χ, η dimensionless

$u = \chi_y$

$v = -\chi_x$

$f''(0) = f(0) = 0$

$f'(\infty) = 0$

$$\chi_y \chi_{xy} - \chi_x \chi_{yy} = \nu \chi_{yyy}$$

$$\frac{ab}{\nu} x^{m+n-1} \left[(m-n) f'^2 - m f f'' \right] = f'''$$

Since both LHS & RHS $f(\eta)$: $m + n = 1$

$$M = \rho a^2 b^{-1} x^{2m-n} \int_{-\infty}^{\infty} f'^2 \, d\eta = \text{constant} \; : \; 2m-n=0$$

$$m = 1/3 \quad n = 2/3 \quad \text{ie} \quad \eta = \frac{y}{b x^{2/3}} \propto x^{-2/3}$$

jet width increases as $x^{2/3}$

$$f''' + \frac{ab}{3\nu}\left(f'^2 + f f''\right) = 0$$

$$f = \tanh \eta$$

$$u = \frac{a}{b} x^{-1/3} \operatorname{sech}^2 \eta \quad \text{or} \quad \frac{u}{u_{max}} = \operatorname{sech}^2 \eta$$

x, η dimensionless & f, u without constants

y

$$a = \left(\frac{9\nu M}{2\rho}\right)^{1/3} \quad b = \left(\frac{48\nu^2\rho}{M}\right)^{1/3}$$

$$u_{max} = \left(\frac{3M^2}{32\rho^2\nu}\right)^{1/3} x^{-1/3} \quad \text{decays as } x^{-1/3}$$

$$Q = \int_{-\infty}^{\infty} u \, dy = \left(\frac{36 M \nu}{\rho}\right)^{1/3} x^{1/3} \quad \text{increases as } x^{1/3}$$

due to entrainment

$$\frac{dM}{dx} = 0 \qquad \frac{dQ}{dx} > 0 \qquad \underbrace{\frac{d}{dx}\int_{-\infty}^{\infty} u^3 \, dy < 0}_{}$$

$x=0$ singularity
practical application use
$x - x_0$ where x_0 = virtual origin

KE flux decreases due
to viscous dissipation

Theory not valid near slot, only for large $Re = \frac{\dot{m}}{\mu} \propto x^{1/3}$

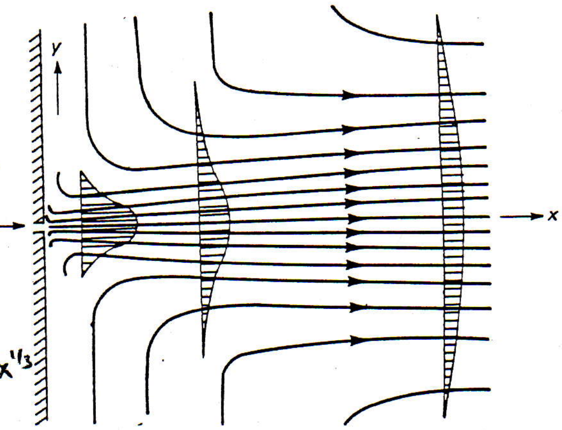


FIGURE 4-18
Definition sketch for the two-dimensional laminar free jet. [*After Schlichting (1933a).*]

Summary 2D Laminar Jet

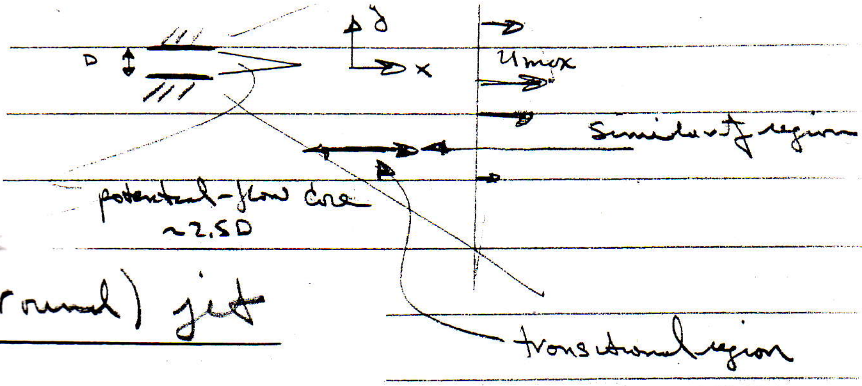
$$u(x,y) = u_{max} \operatorname{sech}^2\left[.2752\left(\frac{M\rho}{\mu^2 x^2}\right)^{1/3} y\right]$$

$$u_{max} = .4543\left(\frac{M^2}{\rho\mu x}\right)^{1/3} \qquad \propto x^{-\frac{1}{3}}$$

$$\text{Width} = 2y\Big|_{\frac{u}{u_{max}}=.01} = 21.8\left(\frac{x^2\mu^2}{M\rho}\right)^{1/3} \qquad \propto x^{2/3}$$

$$v_\infty = -.55\left(\frac{M\mu}{\rho x^2}\right)^{1/3}$$

$$Q = 3.302\left(\frac{M\mu x}{\rho^2}\right)^{1/3} \qquad \propto x^{1/3}$$



Axisymmetric (round) jet

$$Re = \frac{u_0 r_0}{\nu} \gg 1$$

$$M = 2\pi\rho \int_0^\infty u^2 r\,dr = \text{constant}$$

$$u_x + \frac{1}{r}(rv)_r = 0$$

$$u u_x + v u_r = \frac{\nu}{r}(r u_r)_r \qquad \frac{\partial}{\partial r} \gg \frac{\partial}{\partial x}$$

$$v = 0,\ u_y = 0 \qquad r = 0$$

$$u = 0 \qquad r = \infty$$

Similarity solution:

$$\psi = x^m F(\eta) \qquad \eta = \frac{r}{x^n}$$

m, n determined as with 2D jet out M and momentum equation independent x

$$u \propto x^{m-2n} \quad u_x \propto x^{m-2n-1} \quad u_r \propto x^{m-3n} \quad \frac{1}{r}(r u_r)_r \propto x^{m-4n}$$

$$2m - 4n + 2n = 0 \qquad 2m - 4n - 1 = m - 4n$$

$$m = n = 1$$

$$\psi = \nu x F(\eta) \qquad \eta = r/x \qquad \text{increase } x$$

$$u = \frac{\nu}{x}\frac{F'}{\eta} \qquad v = \frac{\nu}{x}\left(F' - \frac{F}{\eta}\right)$$

$$\frac{FF'}{\eta^2} - \frac{F'^2}{\eta} - \frac{FF''}{\eta} = \frac{d}{d\eta}\left(F'' - \frac{F'}{\eta}\right)$$

integration $\qquad FF' = F' - \eta F''$

$$F = \frac{\xi^2}{1+\frac{1}{4}\xi^2} \qquad \xi = c\eta$$

$$c = \left(\frac{3J}{16\pi\rho\nu^2}\right)^{1/2} \text{ from } M = \text{constant}$$

$$u = \frac{3M}{8\pi\mu x}\left(1 + \frac{c^2\eta^2}{4}\right)^{-2} \qquad u_{max} \propto x^{-1}$$

$\dot m \neq \dot m(M)$
slow speed spreads rapidly; high speed spreads slowly.

$$\dot m = \rho Q = 8\pi\mu x \qquad \propto x$$

such that former entrains more fluid & $\dot m_{fast} = \dot m_{slow}$ for same μ, x

Laminar Jet Asymptotic Laws:

		u_{max}	W_{max}	$\dot m$	δ
Same solution as for turbulent flow (solutions are indistinguishable except $\nu = \nu_t$)	AXI	x^{-1}		x	x
	AXI with W (rotationally symmetric)	x^{-1}	x^{-2}	x	x
	2D	$x^{-1/3}$		$x^{1/3}$	$x^{2/3}$
	2D turbulent	$x^{-1/2}$			x

2D Far Wake Nonlifting Body

Assume : $Re = \frac{u_0 L}{\nu} \gg 1$

u_0 = uniform stream
L = length of body

$$F = \text{drag} = \rho u_0^2 \underbrace{\int_{-\infty}^{\infty} \frac{u_w}{u_0}\left(1 - \frac{u_w}{u_0}\right) dy}_{\Theta \text{ = wake momentum thickness}} = \text{Constant}$$

$$F = \rho u_0^2 \Theta$$

from x-momentum using continuity.

$u = u_0 - u_w$, u_w = wake velocity
= velocity defect profile

$$C_D = \frac{F}{\frac{1}{2}\rho u_0^2 A}$$

$A = L \times S$, S = span length
or often projected area used for A

$$u_x + v_y = 0$$

$$(u_0 - u) u_x + v u_y = \nu u_{yy}$$

$$u_y = v = 0 \quad y = 0$$

$$u(x, \pm\infty) = 0$$

$u_w = u_0 - u$
$v_w = v$
$u \ll v$; $\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$
write NS for u_w, v_w
apply BL assumptions for u, v

Similarity Solution :

$$\chi = a x^m f(\eta) \qquad \eta = \frac{y}{b x^n}$$

$$u = \chi_y \qquad v = -\chi_x$$

$$u_0 u_x \propto u_0 \delta^2 x^{2n-1}\left[(m-n)f' - nf''\right]$$

$$-uu_x + vu_y \propto a\delta^2 x^{m+n-1}\left[(n-m)f'^2 + 2n\eta f'f'' - mff''\right]$$

$$\nu u_{yy} \propto \nu f'''$$

$$F = \rho a x^m \int_{-\infty}^{\infty} \left(u_0 - \frac{a}{\delta} x^{m-n} f'\right) f' \, d\eta$$

However, it is not possible to choose m, n such that BL, F independent x, ie, an additional assumption is required to achieve similarity. If $n = 1/2$, $m = 0$, only terms remain with $x^{-1/2}$; therefore must also assume $x \to \infty$, which is why called "far wake" solution.

$$f''' + \frac{u_0^2 \delta}{2\nu}\left(\eta f'' + f'\right) = 0$$

$$F = \rho a u_0 \int_{-\infty}^{\infty} f' \, d\eta = \rho u_0 Q$$

$$f' = \exp(-\eta^2) \quad \text{if} \quad \delta = \left(\frac{4\nu}{u_0}\right)^{1/2}$$

$$\Rightarrow a = F/\rho u_0 \sqrt{\pi}$$

$$\eta = y\sqrt{u_0/4\nu x}$$ width increases as $x^{1/2}$

$$\chi = \frac{f}{2\rho u_0}\,\mathrm{erf}(\eta)$$

$$u = \left[\frac{f^2}{4\pi\rho^2 u_0 \nu x}\right]^{1/2} \exp(-\eta^2)$$ defect decreases $x^{-1/2}$

In wake $\frac{F}{\rho u_0^2} = \Theta = \text{constant}$

In far wake $Q = u_0\Theta = u_0\delta^*$

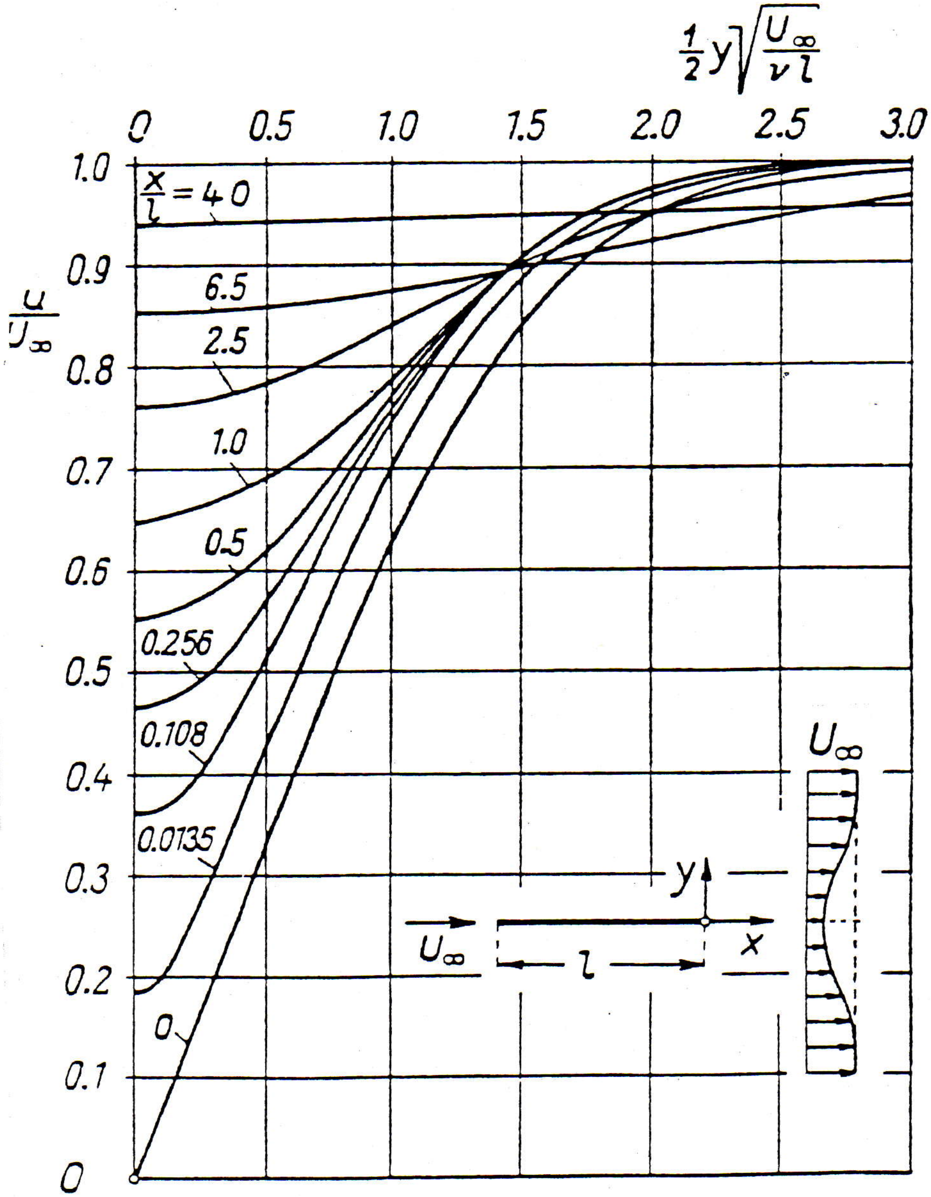
ie $\Theta = \delta^* = \text{constant}$ in far wake of lifting body

Same solution if make Oseen approximation for convective acceleration term

$$u_w \frac{\partial u_w}{\partial x} = u_0 \frac{\partial u}{\partial x} \qquad Re < 1$$

Therefore, solution valid all $Re > 0$.

All bodies produce parabolic wake.



Summary 2D Laminar Wake

$$u = u_0 - u_w = C_D \left(\frac{Re_L}{16\pi}\right)^{1/2} \left(\frac{L}{x}\right)^{1/2} \exp\left(-\frac{U_0 y^2}{4 x \nu}\right)$$

flat plate: $\dfrac{u(x,0)}{u_0} = \dfrac{.664}{\sqrt{\pi}} \left(\dfrac{L}{x}\right)^{1/2} \qquad x > 3L$

Axisymmetric Wake

$$u = u_0 - u_w$$

$$u_0 u_x = \frac{\nu}{r}(r u_r)_r$$

$$u_r = 0 \quad r = 0$$
$$u = 0 \quad r = \infty$$

$$F = 2\pi \rho u_0 \int_0^\infty u r \, dr = \text{constant} = C_D \tfrac{1}{2} \rho u_0^2 L^2$$

$$u = C u_0 \frac{f(\eta)}{x} \qquad \eta = \frac{1}{2} r \sqrt{\frac{u_0}{\nu x}} \qquad \propto x^{1/2}$$

$$(\eta f')' + 2\eta^2 f' + 4\eta f = 0$$

$$f'(0) = 0 \qquad f(\infty) = 0$$

$$f(\eta) = \exp(-\eta^2)$$

Same shape as 2D with r replace y

$$u = \frac{u_0 C_D}{8\pi} Re_L \frac{L}{x} \exp\left(-\frac{u_0 r^2}{4 x \nu}\right) \qquad u_{max} \propto x^{-1}$$

Wake Asymptotic Laws:

	$(u_1)_{max}$	δ	
2D	$x^{-1/2}$	$x^{1/2}$	same turbulent
AXI	x^{-1}	$x^{1/2}$	but different for axi
	$x^{-2/3}$	$x^{1/3}$	

2D Mixing Layer

Recit = 0

Assume : BL assumptions, $p_x = 0$ &
that two uniform streams
start mixing at $x=0$ with
$y=0$ dividing streamline

$$u_{\alpha_x} + v_{\alpha_y} = 0$$

$$u_\alpha u_{\alpha_x} + v_\alpha v_{\alpha_y} = \nu_\alpha u_{\alpha_{yy}}$$

$$v_\alpha = 0 \quad , \quad u_1 = u_2 \quad , \quad \mu_1 u_{1y} = \mu_2 u_{2y} \qquad y = 0$$

$$u_1 = U_1 \quad , \quad u_2 = U_2 \qquad y = \pm\infty$$

Similarity Solution:

$$\psi_\alpha = (\nu_\alpha U_1 x)^{1/2} f_\alpha(\eta_\alpha) \qquad \eta_\alpha = \left(\frac{U_1}{\nu_\alpha x}\right)^{1/2} y$$

Blasius type similarity ψ, η with
U_1 used as reference velocity, which allows
$U_2 = 0$

$$u_\alpha = \frac{u_\alpha}{U_1} = f'_\alpha$$

$$v_\alpha = \frac{1}{2}\left(\frac{U_1 \nu_\alpha}{x}\right)^{1/2}(\eta_\alpha f'_\alpha - f_\alpha)$$

$$2f'''_\alpha + f_\alpha f''_\alpha = 0$$

$$f_1(0)=f_2(0)=0,\quad f_1'(0)=f_2'(0),\quad f_1''(0)=\underbrace{\left(\frac{\rho_2\mu_2}{\rho_1\mu_1}\right)^{1/2}}_{k^{1/2}}f_2''(0)$$

$$f_1(\infty)=1 \quad f_2(-\infty)=U_2/U_1$$

$k^{1/2} = 245$ air/water
$= 1$ identical fluids

As with Blasius, solution is found by numerical integration.

k↑ lower layer moves slower
eg for air-water wind driven flow

$k=1$, $U_c \Rightarrow$ not antisymmetric

$f_2(-\infty)/\sqrt{2} = -.619$ analogy flat plate with slowing

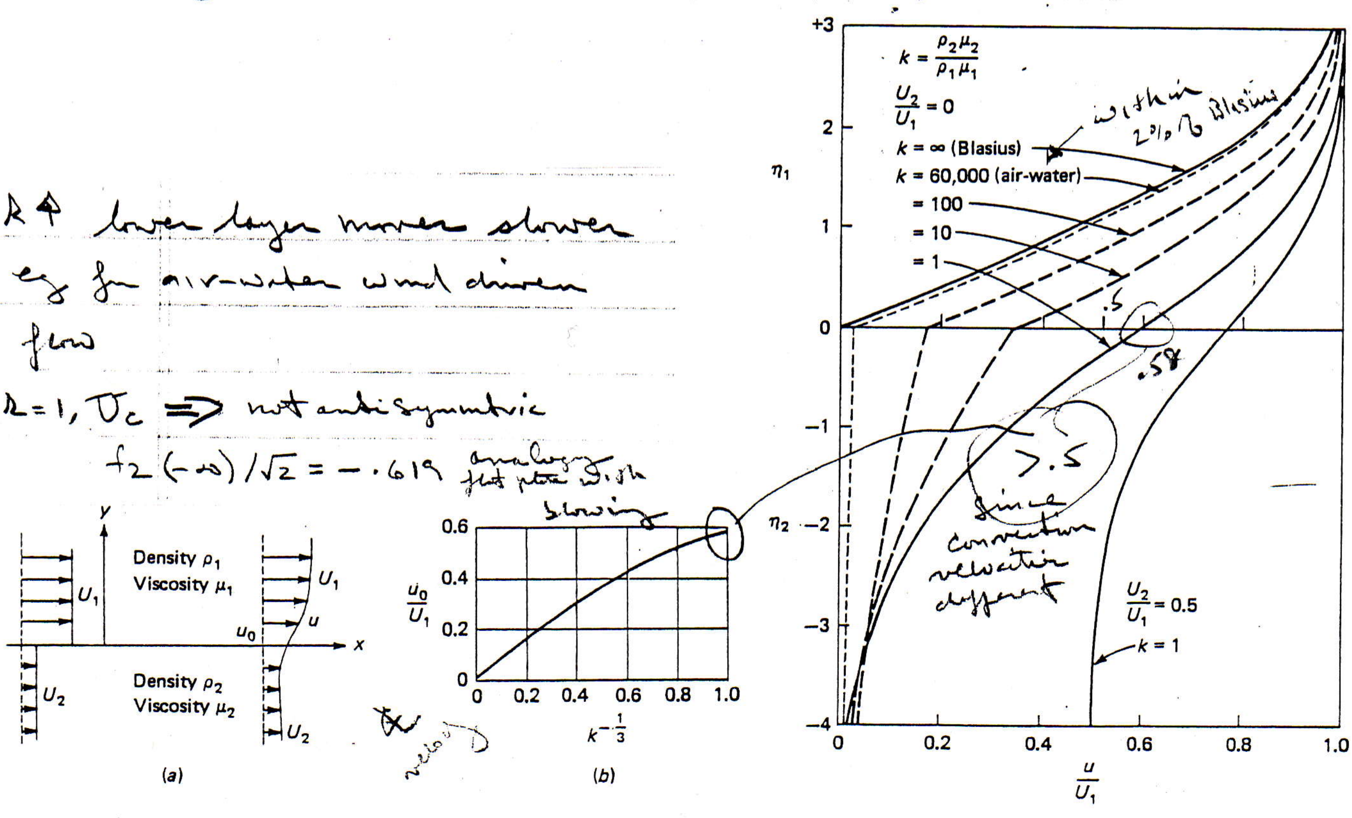


FIGURE 4-17
Velocity distribution between two parallel streams of different properties: (*a*) geometry; (*b*) velocities at the interface ($U_2 = 0$). [*After Lock (1951).*] (*By permission of The Clarendon Press, Oxford.*)