Chapter 7: Boundary Layer Theory

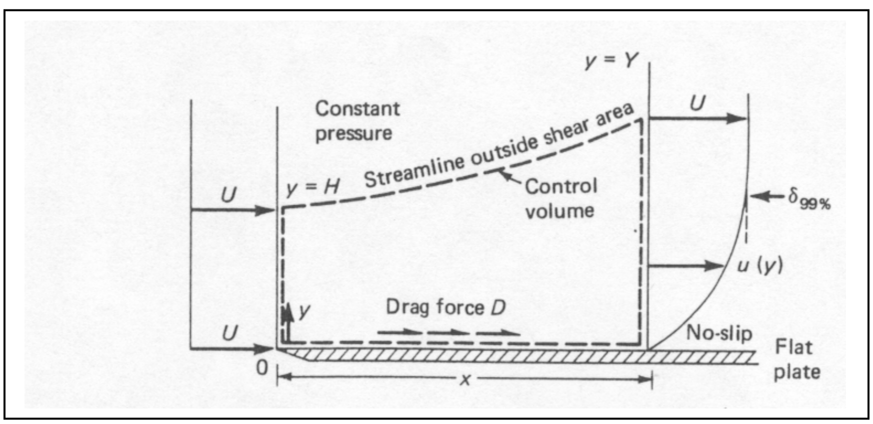
7.1. Introduction:

Boundary layer flows: External flows around streamlined bodies at high Re have viscous (shear and no-slip) effects confined close to the body surfaces and its wake, but are nearly inviscid far from the body.

Applications of BL theory: *aerodynamics* (airplanes, rockets, projectiles), *hydrodynamics* (ships, submarines, torpedoes), *transportation* (automobiles, trucks, cycles), *wind engineering* (buildings, bridges, water towers), and *ocean engineering* (buoys, breakwaters, cables).

7.2 Flat-Plate Momentum Integral Analysis & Laminar approximate solution

Consider flow of a viscous fluid at high Re past a flat plate, i.e., flat plate fixed in a uniform stream of velocity $U\hat{i}$.



Boundary-layer thickness arbitrarily defined by y = $\delta_{99\%}$ (where, $\delta_{99\%}$ is the value of y at u = 0.99U). Streamlines outside $\delta_{99\%}$ will deflect an amount δ^* (**the displacement thickness**). Thus the streamlines move outward from $y = H$ at $x = 0$ to $y = Y = \delta = H + \delta^*$ at $x = x_1$.

<u>Conservation of mass</u>:

$$\int_{CS} \rho \underline{V} \bullet \underline{n} dA = \mathbf{0} = -\int_0^H \rho U dy + \int_0^{H+\delta^*} \rho u dy$$

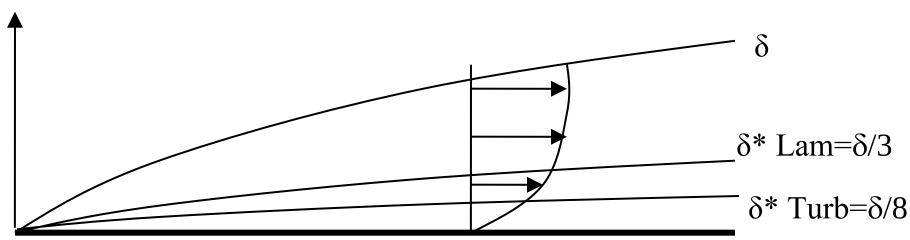
Assuming incompressible flow (constant density), this relation simplifies to

$$UH = \int_0^Y u dy = \int_0^Y (U + u - U) dy = UY + \int_0^Y (u - U) dy$$

Note: $Y = H + \delta^*$, we get the definition of displacement thickness:

$$\delta^* = \int_0^Y \left(1 - \frac{u}{U}\right) dy$$

δ^* (a function of x only) is an important measure of effect of BL on external flow. To see this more clearly, consider an alternate derivation based on an equivalent discharge/flow rate argument:



$$\underbrace{\int_{\delta^*}^{\delta} U dy}_{\text{Inviscid flow about } \delta^* \text{ body}} = \int_0^{\delta} u dy$$

Flowrate between δ^* and δ of inviscid flow=actual flowrate, i.e., inviscid flow rate about displacement body = equivalent viscous flow rate about actual body

$$\int_0^{\delta} U dy - \int_0^{\delta^*} U dy = \int_0^{\delta} u dy \Rightarrow \delta^* = \int_0^{\delta} \left(1 - \frac{u}{U}\right) dy$$

w/o BL - displacement effect=actual discharge

For 3D flow, in addition it must also be explicitly required that δ^* is a stream surface of the inviscid flow continued from outside of the BL.

<u>Conservation of x-momentum</u>:

$$\sum F_x = -D = \int_{CS} \rho u \underline{V} \bullet \underline{n} dA = -\int_0^H \rho U \left(U dy\right) + \int_0^Y \rho u \left(u dy\right)$$

$Drag = D = \rho U^2 H - \int_0^Y \rho u^2 dy$ = Fluid force on plate = - Plate force on CV (fluid)

Again assuming constant density and using continuity: $H = \int_0^Y \frac{u}{U} dy$

$$D = \rho U^2 \int_0^Y u / U dy - \int_0^Y u^2 dy = \int_0^x \tau_w dx$$

$$\frac{D}{\rho U^2} = \theta = \int_0^Y \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

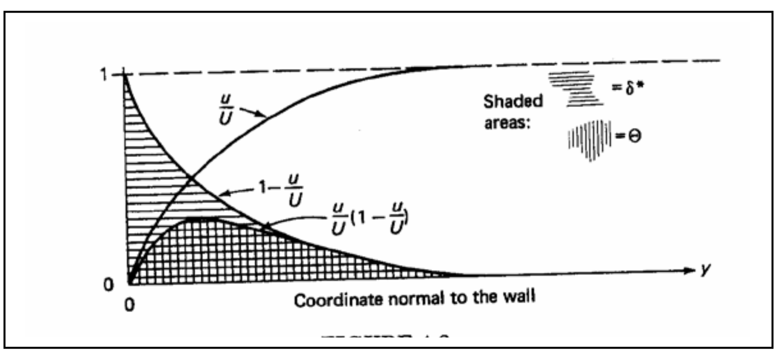
where, θ is the **momentum thickness (a function of x only)**, an important measure of the drag.

$$C_D = \frac{2D}{\rho U^2 x} = \frac{2\theta}{x} = \frac{1}{x}\int_0^x C_f dx \quad \text{Per unit span}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \Rightarrow C_f = \frac{d}{dx}\left(x C_D\right) = 2\frac{d\theta}{dx}$$

$$\frac{d\theta}{dx} = \frac{C_f}{2} \qquad \tau_w = \rho U^2 \frac{d\theta}{dx}$$

Special case 2D momentum integral equation for $p_x = 0$



Simple velocity profile approximations:

$$u = U(2y/\delta - y^2/\delta^2)$$

u(0) = 0 no slip
u(δ) = U matching with outer flow
$u_y(\delta)=0$

Use velocity profile to get $C_f(\delta)$ and θ(δ) and then integrate momentum integral equation to get $\delta(Re_x)$

δ* = δ/3
θ = 2δ/15
H= δ*/θ= 5/2

$$\tau_w = 2\mu U/\delta$$

$$\Rightarrow C_f = \frac{2\mu U/\delta}{1/2\rho U^2} = 2\frac{d\theta}{dx} = 2\frac{d}{dx}(2\delta/15);$$

$$\therefore \delta d\delta = \frac{15\mu dx}{\rho U}$$

$$\delta^2 = \frac{30\mu dx}{\rho U}$$

$$\delta/x = 5.5/\mathrm{Re}_x^{1/2}$$

$$Re_x = Ux/\nu$$

$$\delta^*/x = 1.83/\mathrm{Re}_x^{1/2}$$

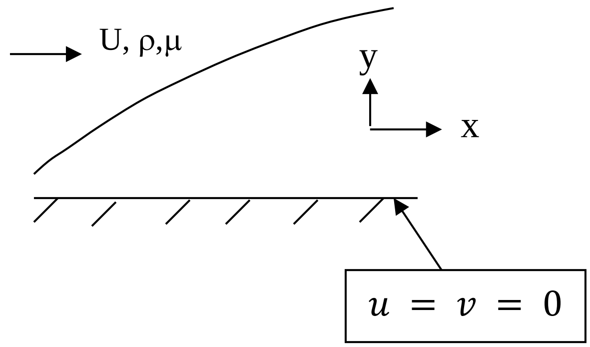
$$\theta/x = 0.73/\mathrm{Re}_x^{1/2}$$

$$C_D = 1.46/\mathrm{Re}_L^{1/2} = 2C_f(L)$$

10% error, cf. Blasius

$$\frac{\partial u}{\partial y} = U\left(\frac{2}{\delta} - \frac{2y^2}{\delta^2}\right)\Big]_{y=0} = \frac{2U}{\delta}\nu$$

7.3. Boundary layer approximations, equations and comments



2D NS, ρ=constant, neglect g

$$u_x + v_y = 0$$
$$u_t + uu_x + vu_y = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(u_{xx} + u_{yy})$$
$$v_t + uv_x + vv_y = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(v_{xx} + v_{yy})$$

Introduce non-dimensional variables that includes scales such that all variables are of O(1):

$$x^* = x / L$$
$$y^* = \frac{y}{L}\sqrt{\mathrm{Re}}$$
$$t^* = tU / L$$
$$u^* = u / U$$
$$v^* = \frac{v}{U}\sqrt{Re}$$
$$p^* = \frac{p - p_0}{\rho U^2}$$
$$Re_x = UL/\nu$$

The NS equations become (drop *)

$$u_x + v_y = 0$$
$$u_t + uu_x + vu_y = -p_x + \underline{\frac{1}{Re}u_{xx}} + u_{yy}$$
$$\frac{1}{Re}\left(v_t + uv_x + vv_y\right) = -p_y + \underline{\frac{1}{Re^2}v_{xx}} + \underline{\frac{1}{Re}v_{yy}}$$

For large Re (BL assumptions) the underlined terms drop out and the BL equations are obtained.

Therefore, y-momentum equation reduces to

$$p_y = 0$$
$$i.e.\ p = p(x,t)$$
$$\Rightarrow p_x = -\rho(U_t + UU_x)$$ From Euler/Bernoulli equation for external flow

<u>2D BL equations:</u>

$$u_x + v_y = 0$$
$$u_t + uu_x + vu_y = (U_t + UU_x) + \nu u_{yy}$$

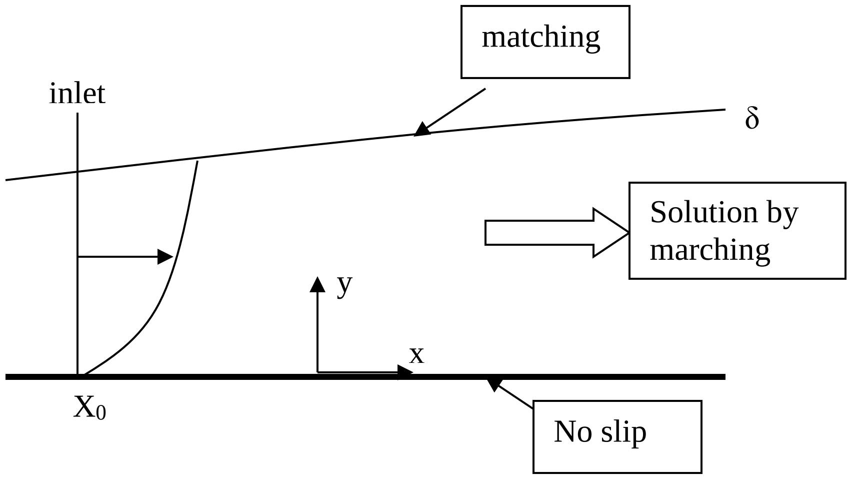
Note:

(1) U(x,t), p(x,t) impressed on BL by the external flow.

(2) $\frac{\partial^2}{\partial x^2} = 0$: i.e. longitudinal (or stream-wise) diffusion is neglected.

(3) Due to (2), the equations are parabolic in x. Physically, this means all downstream influences are lost other than that contained in external flow. A marching solution is possible.

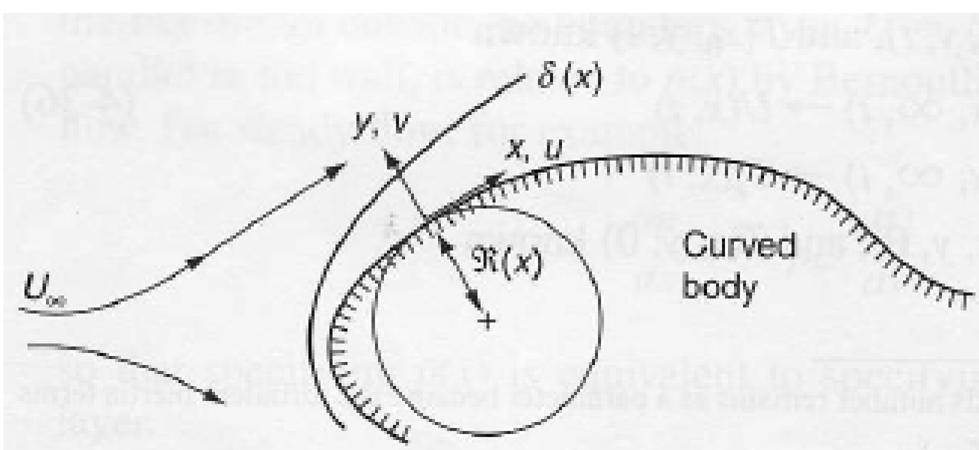
(4) Boundary conditions



No slip: $u(x,0,t) = v(x,0,t) = 0$
Initial condition: $u(x,y,0)$ known
Inlet condition: $u(x_0,y,t)$ given at x_0
Matching with outer flow: $u(x,\infty,t) = U(x,t)$

(5) When applying the boundary layer equations one must keep in mind the restrictions imposed on them due to the basic BL assumptions
→ not applicable for thick BL or separated flows (although they can be used to estimate occurrence of separation).

(6) Curvilinear coordinates



Although BL equations have been written in Cartesian Coordinates, they apply to curved surfaces provided δ << R and x, y are curvilinear coordinates measured along and normal to the surface, respectively. In such a system we would find under the BL assumptions

$$p_y = \frac{\rho u^2}{R}$$

Assume *u* is a linear function of *y*: $u = Uy/\delta$

$$\frac{dp}{dy} = \frac{\rho U^2 y^2}{R\delta^2}$$

$$p(\delta) - p(0) \propto \frac{\rho U^2 \delta}{3R}$$

Or

$\dfrac{\Delta p}{\rho U^2} \propto \dfrac{\delta}{3R}$; therefore, we require $\delta << R$

(7) Practical use of the BL theory

For a given body geometry:

(a) Inviscid theory gives p(x) → integration gives L and D = 0

(b) BL theory gives → $\delta^*(x)$, $\tau_w(x)$, $\theta(x)$, etc. and predicts separation if any

(c) If separation present then no further information → must use inviscid models, BL equation in inverse mode, or NS equation.

(d) If separation is absent, integration of $\tau_w(x)$ → frictional resistance and body + δ^*, inviscid theory gives → p(x) for drag + δ^*, can go back to (2) for more accurate BL calculation including viscous – inviscid interaction

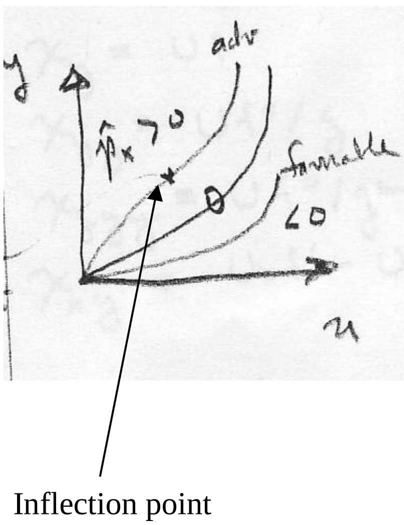
(8) Separation and shear stress

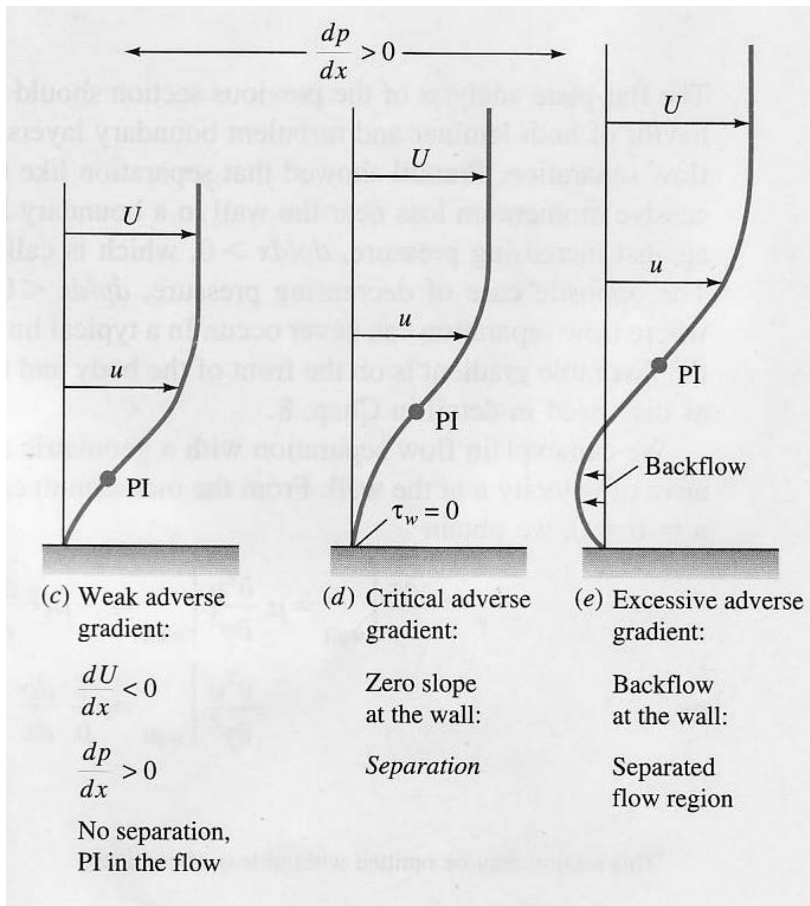
At the wall, $u = v = 0 \rightarrow u_{yy} = \frac{1}{\mu} p_x$

1st derivative u gives $\tau_w \rightarrow \tau_w = \mu u_y\Big|_w$

$\tau_w = 0$ separation

2nd derivative u depends on p_x





7.4. Laminar Boundary Layer ($Re_{trans} = 5 \times 10^5 - 3 \times 10^6$) - Similarity solutions (2D, steady, incompressible): method of reducing PDE to ODE by appropriate similarity transformation; also, as a result of transformation at least one coordinate lacks origin such that the solution collapses to same form at all length or time scales

$$u_x + v_y = 0$$
$$uu_x + vu_y = UU_x + \nu u_{yy}$$

BCs: $u(x,0) = v(x,0) = 0$

$u(x,\infty) = U(x)$

+ inlet condition

For Similarity $\dfrac{u(x,y)}{U(x)} = F\left(\dfrac{y}{g(x)}\right)$ expect $g(x)$ related to $\delta(x)$

Or in terms of stream function ψ : $u = \psi_y \quad v = -\psi_x$

For similarity $\psi = U(x)g(x)f(\eta) \qquad \eta = y/g(x)$

$u = \psi_y = Uf' \quad v = -\psi_x = -(U_x gf + Ug_x f - Ug_x \eta f')$

BC:

$$u(x,0) = 0 \Rightarrow U(x)f'(0) = 0 \Rightarrow f'(0) = 0$$
$$v(x,0) = 0 \Rightarrow U_x(x)g(x)f(0) + U(x)g_x(x)f(0)$$
$$-U(x)g_x(x) \times 0 \times f'(0) = 0$$
$$\Rightarrow \left(U_x(x)g(x) + U(x)g_x(x)\right)f(0) = 0$$
$$\Rightarrow f(0) = 0$$
$$u(x,\infty) = U(x) \Rightarrow U(x)f'(\infty) = U(x) \Rightarrow f'(\infty) = 1$$

Write boundary layer equations in terms of ψ

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} = UU_x + \nu \psi_{yyy}$$

Substitute

$$\psi_{yy} = Uf''/g$$

$$\psi_{yyy} = Uf'''/g^2$$

$$\psi_{xy} = U_x f' - Uf''\eta g_x / g$$

Assemble them together:

$$\left(Uf'\right)\left(U_x f' - \cancel{Uf''\eta\frac{g_x}{g}}\right) - \left(U_x gf + Ug_x f - \cancel{Ug_x\eta f'}\right)\left(Uf''/g\right)$$

$$= UU_x + \nu\left(U\ f'''/g^2\right)$$

$$UU_x f'^2 - UU_x ff'' - U^2\, g_x/g\ ff'' = UU_x + \nu\frac{U}{g^2}\, f'''$$

$$UU_x f'^2 - \frac{U}{g}(Ug)_x\, ff'' = UU_x + \nu\frac{U}{g^2}\, f'''$$

$$f''' + \underbrace{\boxed{\frac{g}{\nu}(Ug)_x}}_{C_1}\, ff'' + \underbrace{\boxed{\frac{g^2}{\nu}U_x}}_{C_2}\left(1 - f'^2\right) = 0$$

Where for similarity C_1 and C_2 are constant or function η only

- i.e. for a chosen pair of C_1 and C_2 → $U(x)$, $g(x)$ can be found (Potential flow is NOT known a priori)
- Then solution of $f''' + C_1 ff'' + C_2\left(1 - f'^2\right) = 0$ gives $f(\eta)$ → $u(x, y)$, $\tau_w = \mu\left.\frac{\partial u}{\partial y}\right|_w = \frac{\mu U f''(0)}{g}$, δ, δ*,θ, H, C_f, C_D

The Blasius Solution for Flat-Plate Flow

U=constant→ $U_x = 0 \rightarrow C_2 = 0$

Then $C_1 = \dfrac{U}{\nu} g g_x$

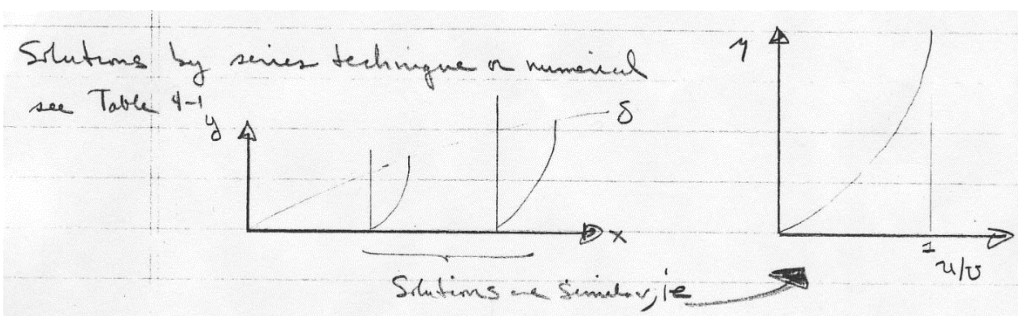
$$\frac{d}{dx}\left(g^2\right) = \frac{2C_1\nu}{U} \Longrightarrow g(x) = \left[2C_1\nu x/U\right]^{1/2}$$

Let $C_1 = 1$, then $g(x) = \sqrt{\dfrac{2\nu x}{U}} \Longrightarrow \eta = y\sqrt{\dfrac{U}{2\nu x}}$

$$\Longrightarrow \boxed{\begin{gathered} f''' + 1/2\, f f'' = 0 \\ f(0) = f'(0) = 0,\ f'(\infty) = 1 \end{gathered}}$$

Blasius equations for Flat Plate Boundary Layer

Solutions by series technique or numerical



$\dfrac{u}{U} = 0.99$ when $\eta = 3.5$ → $\dfrac{\delta}{x} = \dfrac{5}{\sqrt{\mathrm{Re}_x}}$ $\mathrm{Re}_x = \dfrac{Ux}{\nu}$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \int_0^\infty \left(1 - f'\right) d\eta \sqrt{\frac{2\nu x}{U}} \rightarrow \boxed{\frac{\delta^*}{x} = \frac{1.7208}{\sqrt{\mathrm{Re}_x}}}$$

$$\theta = \int_0^\infty \left(1 - \frac{u}{U}\right)\frac{u}{U} dy = \int_0^\infty \left(1 - f'\right) f' \sqrt{\frac{2\nu x}{U}} d\eta \rightarrow \boxed{\frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}}$$

So, $\boxed{\dfrac{\delta^*}{\theta} = H = 2.59}$

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_w = \frac{\mu U f''(0)}{\sqrt{2\nu x/U}} \quad \rightarrow \quad \boxed{C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}} = \frac{\theta}{x}}$$

$$\boxed{C_D = \frac{D}{\frac{1}{2}\rho U^2 L} = \int_0^L C_f \frac{dx}{L} = \frac{1.328}{\sqrt{\mathrm{Re}_L}}}; \qquad \mathrm{Re}_L = \frac{UL}{\nu};$$

$$\boxed{\frac{v}{U} = \frac{\eta f' - f}{\sqrt{2Re_x}} \ll 1} \qquad \text{for} \quad \mathrm{Re}_x >> 1$$

TABLE 4-1
Numerical solution of the Blasius flat-plate relation, Eq. (4-45)

η	$f(\eta)$	$f'(\eta)$	$f''(\eta)$
0.0	0.0	0.0	0.46960
0.1	0.00235	0.04696	0.46956
0.2	0.00939	0.09391	0.46931
0.3	0.02113	0.14081	0.46861
0.4	0.03755	0.18761	0.46725
0.5	0.05864	0.23423	0.46503
0.6	0.08439	0.28058	0.46173
0.7	0.11474	0.32653	0.45718
0.8	0.14967	0.37196	0.45119
0.9	0.18911	0.41672	0.44363
1.0	0.23299	0.46063	0.43438
1.1	0.28121	0.50354	0.42337
1.2	0.33366	0.54525	0.41057
1.3	0.39021	0.58559	0.39598
1.4	0.45072	0.62439	0.37969
1.5	0.51503	0.66147	0.36180
1.6	0.58296	0.69670	0.34249
1.7	0.65430	0.72993	0.32195
1.8	0.72887	0.76106	0.30045
1.9	0.80644	0.79000	0.27825
2.0	0.88680	0.81669	0.25567
2.2	1.05495	0.86330	0.21058
2.4	1.23153	0.90107	0.16756
2.6	1.41482	0.93060	0.12861
2.8	1.60328	0.95288	0.09511
3.0	1.79557	0.96905	0.06771
3.2	1.99058	0.98037	0.04637
3.4	2.18747	0.98797	0.03054
3.6	2.38559	0.99289	0.01933
3.8	2.58450	0.99594	0.01176
4.0	2.78388	0.99777	0.00687
4.2	2.98355	0.99882	0.00386
4.4	3.18338	0.99940	0.00208
4.6	3.38329	0.99970	0.00108
4.8	3.58325	0.99986	0.00054

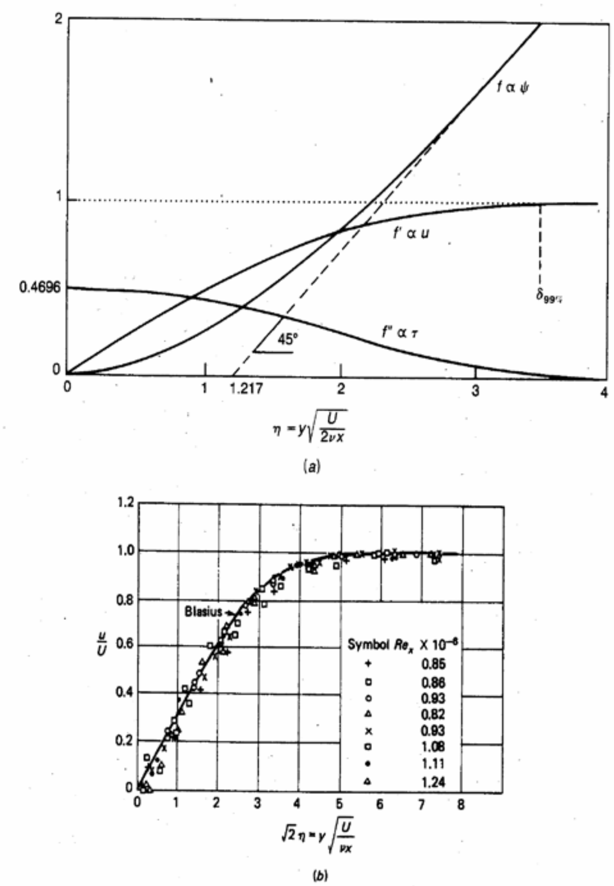


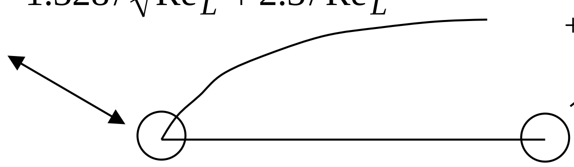
FIGURE 4-6
The Blasius solution for the flat-plate boundary layer: (*a*) numerical solution of Eq. (4-45); (*b*) comparison of $f' = u/U$ with experiments by Liepmann (1943).

	C_D	Re_L
Oseen	3-226 (3rd edition,vicous flows)	<1
Blasius		100<Re<Re_{tr}~3 ×10^6

LE Higher order correction

$$C_D = 1.328/\sqrt{\mathrm{Re}_L} + 2.3/\mathrm{Re}_L$$

Similar breakdown occurs at Trailing edge. From triple – deck theory the correction is $+2.661/\mathrm{Re}_L^{7/8}$



Rex small therefore local breakdown of BL approximation

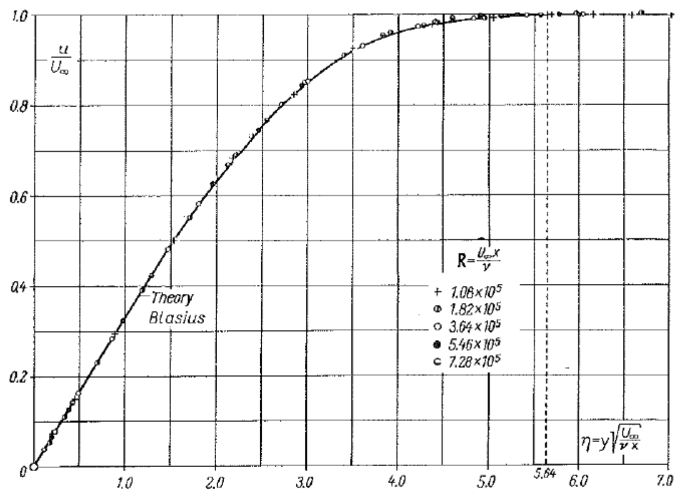


Fig. 7.9. Velocity distribution in the laminar boundary layer on a flat plate at zero incidence, as measured by Nikuradse [20]

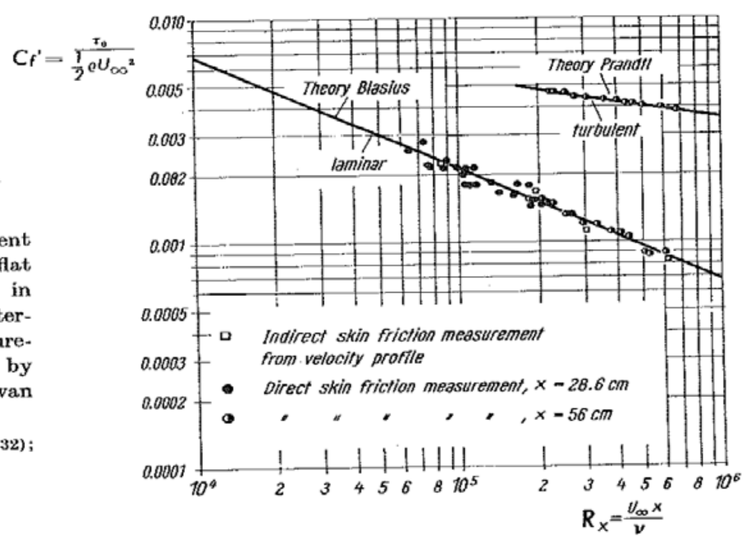


Fig. 7.10. Local coefficient of skin friction on a flat plate at zero incidence in incompressible flow, determined from direct measurement of shearing stress by Liepmann and Dhawan [6, 18]

Theory: laminar from eqn. (7.32); turbulent from eqn. (21.12)

Falkner-Skan Wedge Flows

$$f''' + C_1 ff'' + C_2\left(1 - f'^2\right) = 0$$
$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$
$$C_1 = \frac{g}{\nu}(Ug)_x \qquad C_2 = \frac{g^2}{\nu}U_x$$

$$\left.\begin{aligned} f &= f(\eta) \\ \eta &= y/g(x) \\ u/U &= f'(\eta) \end{aligned}\right\}$$ Similarity form of BL eq.

(Blasius Solution: C_2=0, C_1=1)

Consider
$$\begin{aligned}\left(Ug^2\right)_x &= 2Ugg_x + g^2U_x \\ &= 2Ugg_x + 2g^2U_x - g^2U_x \\ &= 2g(Ug)_x - g^2U_x \\ &= 2\nu C_1 - \nu C_2\end{aligned}$$

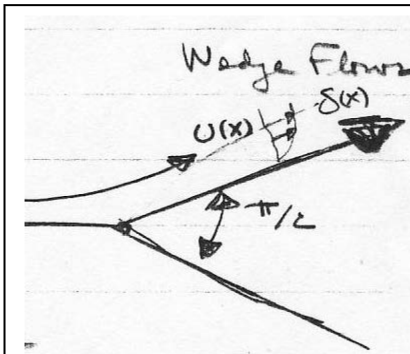
Hence → $\left(Ug^2\right)_x = \nu\left(2C_1 - C_2\right), \qquad C_2 = \frac{g^2}{\nu}U_x$

Choose C_1=1 and C_2 arbitrary=C,

Integrate $\quad Ug^2 = \nu(2-C)x$

Combine $\quad \dfrac{U_x}{U} = \dfrac{C}{2-C}\dfrac{1}{x}$

$$C = g^2U_x/\nu$$



$$\ln U = \frac{C}{2-C}\ln x + k$$

Then
$$\begin{cases} U(x) = kx^{C/(2-C)} \\ g(x) = \sqrt{\dfrac{\nu(2-C)}{k}}\,x^{\frac{1-C}{2-C}} \end{cases}$$

Change constants

$$U(x) = kx^m$$

$$\eta = \frac{y}{g} = y\sqrt{\frac{m+1}{2}\frac{U}{\nu x}}$$

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0, \quad \beta = \frac{2m}{m+1}, \quad m = \frac{\beta}{2-\beta}$$

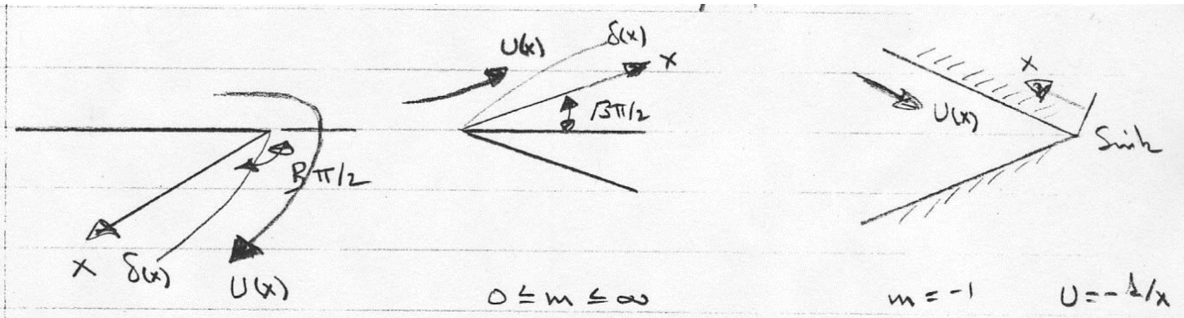
$$f(0) = f'(0) = 0 \qquad f'(\infty) = 1$$

Solutions for $-0.19884 \leq \beta \leq 1.0$

Separation ($\tau_w = 0$)

Solutions show many commonly observed characteristics of BL flow:

- The parameter β is a measure of the pressure gradient, dp/dx. For $\beta > 0$, $dp/dx < 0$ and the pressure gradient is favorable. For $\beta < 0$, the $dp/dx > 0$ and the pressure gradient is adverse.
- Negative β solutions drop away from Blasius profiles as separation approached
- Positive β solutions squeeze closer to wall due to flow acceleration
- Accelerated flow: $\tau_{\max}$ near wall
- Decelerated flow: $\tau_{\max}$ moves toward $\delta/2$



$-\frac{1}{2} \le m \le 0$

$-2 \le \beta \le 0$

Expansion Corner

$\beta = -.19884$ (Separation)

$0 \le m \le \infty$

$0 \le \beta \le 2$

Wedge Flow

$m=0$, $\beta=0$ } Flat Plate

$m=1$, $\beta=1$ } Stagnation Point

$m = -1$ $\quad U = -k/x$

$\beta = \infty$ $\quad \eta = \frac{y}{x}\sqrt{\frac{U}{\nu}}$

Sink : Simple Sol involving tanh
Sec. 4.37

BL Sol of Jeffery-Hamel flow

Also, $\beta = 4$, $m = -2$: doublet flow near a plane wall

$\beta = 5$, $m = -5/3$: doublet near a 90° corner

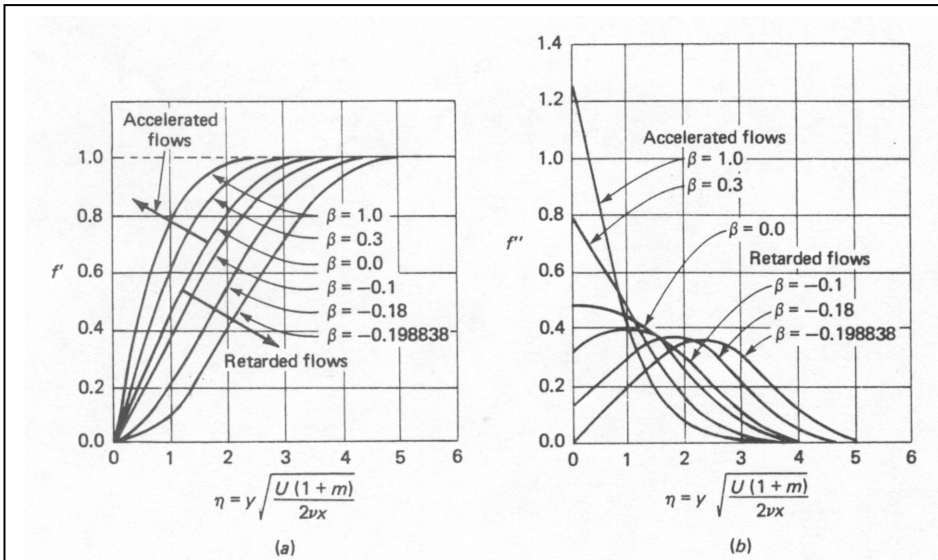


FIGURE 4-11
(*a*) Velocity profiles and (*b*) shear-stress profiles for the Falkner-Skan equation.

7.5. Momentum Integral Equation

Historically similarity and other AFD methods used for idealized flows and momentum integral methods for practical applications, including pressure gradients.

Momentum integral equation, which is valid for both laminar and turbulent flow:

$$\int_{y=0}^{\infty}\left(\text{BL form of momentum equation} + (u-U)\text{continuity}\right)dy$$

$$\frac{\tau_w}{\rho U^2} = \underbrace{\frac{1}{2}C_f = \frac{d\theta}{dx}}_{\text{For flat plate equation}\rightarrow \frac{dU}{dx}=0} + (2+H)\frac{\theta}{U}\frac{dU}{dx}$$

$$\theta = \int_0^{\delta}\frac{u}{U}\left(1-\frac{u}{U}\right)dy;$$

$$H = \frac{\delta^*}{\theta};$$

$$\delta^* = \int_0^{\delta}\left(1-\frac{u}{U}\right)dy$$

Momentum: $uu_x + vu_y = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$

The pressure gradient is evaluated form the outer potential flow using Bernoulli equation

$$p + \frac{1}{2}\rho U^2 = \text{constant}$$

$$p_x + \frac{1}{2}\rho 2UU_x = 0$$

$$-p_x = \rho UU_x$$

$$(u-U)\underbrace{\left(u_x+v_y\right)}_{Continuity}=uu_x+uv_y-Uu_x-Uv_y$$

$$\underbrace{uu_x+vu_y-UU_x-\frac{1}{\rho}\tau_y}_{0}+\underbrace{uu_x+uv_y-Uu_x+Uv_y}_{0}=0$$

$$-\frac{1}{\rho}\tau_y=-2uu_x-vu_y+UU_x-uv_y+Uu_x+Uv_y$$

$$=\frac{\partial}{\partial x}(uU-u^2)+(U-u)U_x+\frac{\partial}{\partial y}(vU+vu)$$

$$\int_0^\infty -\frac{1}{\rho}\tau_y dy=-(\cancelto{0}{\tau_\infty}-\tau_w)/\rho=\frac{\partial}{\partial x}\int_0^\infty u(U-u)dy+U_x\int_0^\infty (U-u)dy+\cancelto{0}{(vU-vu)\Big|_0^\infty}$$

$$\frac{\tau_w}{\rho}=\frac{\partial}{\partial x}\left[U^2\int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy+U_x\int_0^\infty (U-u)dy\right]=$$

$$U^2\theta_x+2UU_x\theta+U_x\delta^*$$

$$\frac{C_f}{2}=\frac{d\theta}{dx}+\left(2\theta+\delta^*\right)\frac{1}{U}\frac{dU}{dx}$$

$$\frac{C_f}{2}=\frac{d\theta}{dx}+(2+H)\frac{\theta}{U}\frac{dU}{dx},\ H=\frac{\delta^*}{\theta}$$

$$\frac{\tau_w}{\rho U^2}=\frac{1}{2}C_f=\theta_x+(2+H)\frac{\theta}{U}U_x$$

Historically two approaches for solving the momentum integral equation for specified potential flow U(x):

1. Guessed Profiles
2. Empirical Correlations

Best approach is to use empirical correlations to get integral parameters (δ, δ^*, θ, H, C_f, C_D) after which use these to get velocity profile u/U

Thwaites Method

Multiply momentum integral equation by $\dfrac{U\theta}{\nu}$

$$\frac{\tau_w\theta}{\mu U}=\frac{U\theta}{\nu}\frac{d\theta}{dx}+\frac{\theta^2}{\nu}\frac{dU}{dx}\left(2+H\right)$$

The equation is dimensionless and, LHS and H can be correlated with pressure gradient parameter $\lambda=\dfrac{\theta^2}{\nu}\dfrac{dU}{dx}$ as shear and shape-factor correlations

$$\boxed{\begin{aligned}&\frac{\tau_w\theta}{\mu U}=S\left(\lambda\right)=(\lambda+0.09)^{0.62}\\&H=\delta^*/\theta=H\left(\lambda\right)=\sum_{i=0}^{5}a_i(0.25-\lambda)^i\end{aligned}}$$

a_i = (2, 4.14, -83.5, 854, -3337, 4576)

Note

$$\frac{U\theta}{\nu}\frac{d\theta}{dx}=\frac{1}{2}U\frac{d}{dx}\left(\frac{\theta^2}{\vartheta}\right)$$

Substitute above into momentum integral equation

$$S(\lambda)=\frac{1}{2}U\frac{d}{dx}\left(\frac{\theta^2}{\nu}\right)+\lambda\left(2+H\right)$$

$$U\frac{d\left(\lambda/U_x\right)}{dx}=2\left[S-\lambda\left(2+H\right)\lambda\right]=F\left(\lambda\right)$$

$F\left(\lambda\right)=0.45-6\lambda$ based on AFD and EFD

Define $z = \frac{\theta^2}{\nu}$ so that $\lambda = z\frac{dU}{dx}$

$$U\frac{dz}{dx} = 0.45 - 6\lambda = 0.45 - 6z\frac{dU}{dx}$$

$$U\frac{dz}{dx} + 6z\frac{dU}{dx} = 0.45$$

i.e. $\frac{1}{U^5}\frac{d}{dx}\left(zU^6\right) = 0.45$

$$zU^6 = 0.45\int_0^x U^5 dx + C$$

$$\Rightarrow \boxed{\theta^2 = \theta_0^2 + \frac{0.45\nu}{U^6}\int_0^x U^5 dx}$$

$\theta_0(x=0) = 0$ and U(x) known from potential flow solution

Complete solution:

$$\lambda = \lambda(\theta) = \frac{\theta^2}{\nu}\frac{dU}{dx}$$

$$\frac{\tau_w \theta}{\mu U} = S(\lambda)$$

$$\delta^* = \theta H(\lambda)$$

Accuracy: mild p_x $\pm 5\%$ and strong adverse p_x (τ_w near 0) $\pm 15\%$

i. Pohlhausen Velocity Profile:

$$\frac{u}{U} = f(\eta) = a\eta + b\eta^2 + c\eta^3 + d\eta^4 \text{ with } \eta = \frac{y}{\delta}$$

a, b, c, d determined from boundary conditions

1) $y = 0 \rightarrow u = 0,\ u_{yy} = -\frac{U}{\nu}U_x$

2) $y = \delta \rightarrow u = U,\ u_y = 0,\ u_{yy} = 0$

No slip is automatically satisfied.

separation

$$F(\eta) = 2\eta - 2\eta^3 + \eta^4$$

$$G(\eta) = \frac{\eta}{6}(1-\eta)^3$$

$$\rightarrow \frac{u}{U} = F(\eta) + \Lambda G(\eta),\ -12 \le \Lambda \le 12$$

(experiment: $\Lambda_{separation} = -5$)

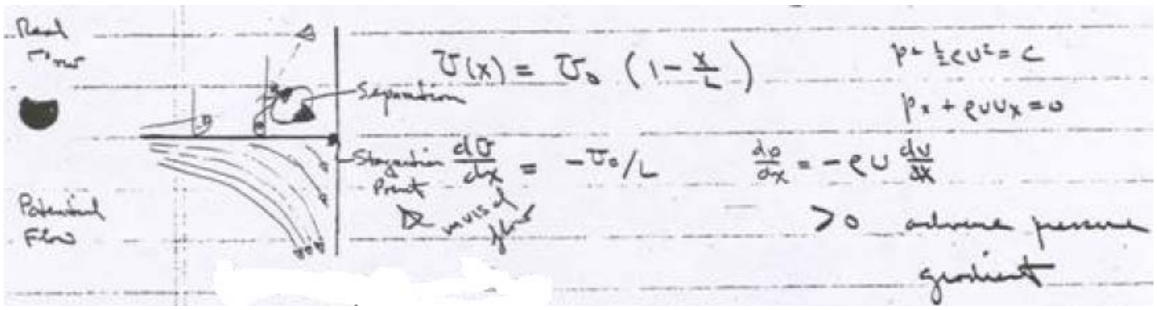
$$\Lambda = \frac{\delta^2}{\nu}\frac{dU}{dx} = -p_x\frac{\delta^2}{\mu U}$$

pressure gradient parameter related to

$$\lambda = \lambda(\Lambda) = \left(\frac{37}{315} - \frac{\Lambda}{945} + \frac{\Lambda^2}{9072}\right)\Lambda$$

Profiles are fairly realistic, except near separation. In guessed profile methods u/U directly used to solve momentum integral equation numerically, but accuracy not as good as empirical correlation methods; therefore, use Thwaites method to get λ, etc., and then use λ to get Λ and plot u/U.

ii. Howarth linearly decelerating flow (example of exact solution of steady state 2D boundary layer)



Howarth proposed a linearly decelerating external velocity distribution $U(x) = U_0\left(1-\frac{x}{L}\right)$ as a theoretical model for laminar boundary layer study.

Use Thwaites's method to compute:

a) X_{sep}

b) $C_f\left(\frac{x}{L}=0.1\right)$

Note $U_x = -U_0/L$

Solution

$$\theta^2 = \frac{0.45\nu}{U_0^6\left(1-\frac{x}{L}\right)^6}\int_0^x U_0^5\left(1-\frac{x}{L}\right)^5 dx = 0.075\frac{\nu L}{U_0}\left[\left(1-\frac{x}{L}\right)^{-6}-1\right]$$

can be evaluated for given L, Re_L

(Note: $\begin{matrix}\theta = 0 \rightarrow x = 0, \\ \theta = \infty \rightarrow x = L\end{matrix}$)

$$\lambda = \frac{\theta^2}{\nu}\frac{dU}{dx} = -0.075\left[\left(1-\frac{x}{L}\right)^{-6}-1\right]$$

$$\lambda_{sep} = -0.09 \Rightarrow \frac{X_{sep}}{L} = 0.123$$

3% higher than exact solution =0.1199

$$C_f\left(\frac{x}{L} = 0.1\right)$$ → i.e. just before separation

$$\lambda = -0.0661$$

$$S(\lambda) = 0.099 = \frac{1}{2} C_f \operatorname{Re}_\theta$$

$$C_f = \frac{2(0.099)}{\operatorname{Re}_\theta}$$

Compute Re_θ in terms if Re_L

$$\theta^2 = 0.075 \frac{\nu L}{U_0}\left[(1-0.1)^{-6} - 1\right] = 0.0661 \frac{\vartheta L}{U_0}$$

$$\frac{\theta^2}{L^2} = 0.0661 \frac{\nu L}{U_0} = \frac{0.0661}{\operatorname{Re}_L}$$

$$\frac{\theta}{L} = \frac{0.257}{\operatorname{Re}_L^{1/2}}$$

$$\operatorname{Re}_\theta = \frac{\theta}{L}\operatorname{Re}_L = 0.257 \operatorname{Re}_L^{1/2}$$

$$C_f = \frac{2(0.099)}{0.257}\operatorname{Re}_L^{-1/2} = 0.77 \operatorname{Re}_L^{-1/2}$$

To complete solution must specify Re_L

Consider the complex potential

$$F(z)=\frac{a}{2}z^2=\frac{a}{2}r^2e^{2i\theta}$$

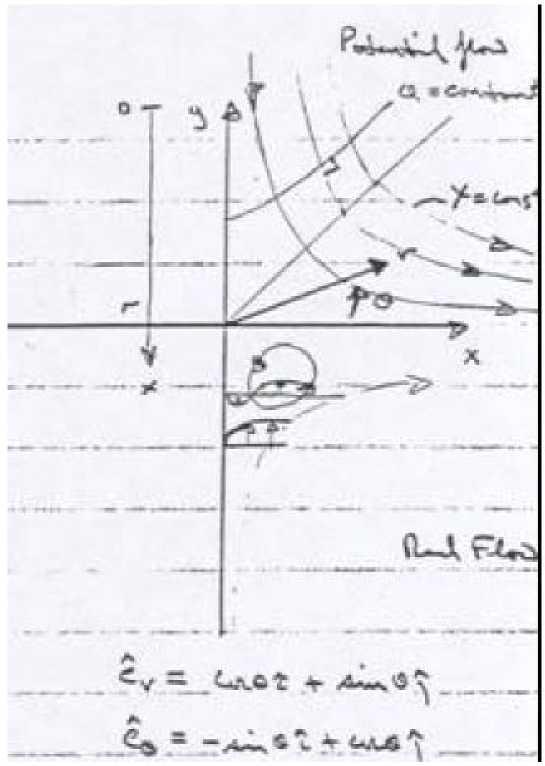
$$\varphi=\mathrm{Re}[F(z)]=\frac{a}{2}r^2\cos 2\theta$$

$$\psi=\mathrm{Im}[F(z)]=\frac{a}{2}r^2\sin 2\theta$$

Orthogonal rectangular hyperbolas

φ: asymptotes y = ± x

ψ: asymptotes x=0, y=0



$$\begin{cases} \underline{V}=\nabla\varphi=\varphi_r\hat{e}_r+\frac{1}{r}\varphi_\theta\hat{e}_\theta \\ v_r=ar\cos 2\theta \\ v_\theta=-ar\sin 2\theta \end{cases}$$

$\frac{\pi}{2}\le\theta\le 0$ (flow direction as shown)

$$\underline{V}=v_r(\cos\theta\hat{i}+\sin\theta\hat{j})+v_\theta(-\sin\theta\hat{i}+\cos\theta\hat{j})=$$
$$(v_r\cos\theta-v_\theta\sin\theta)\hat{i}+(v_r\sin\theta+v_\theta\cos\theta)\hat{j}$$

Potential flow slips along surface: (consider $\theta=90^\circ$)

1) determine *a* such that $v_r=U_0$ at r=L, $\theta=90^\circ$

$v_r=aL\cos(2\times 90)=U_0\Rightarrow aL=-U_0$, i.e. $a=-\frac{U_0}{L}$

2) let $U(x)=v_r$ at x=L-r:

$\Rightarrow v_r=a(L-x)\cos(2\times 90)=U(x)$

$$Or:\ U(x)=-a(L-x)=\frac{U_0}{L}(L-x)=U_0(1-\frac{x}{L})$$

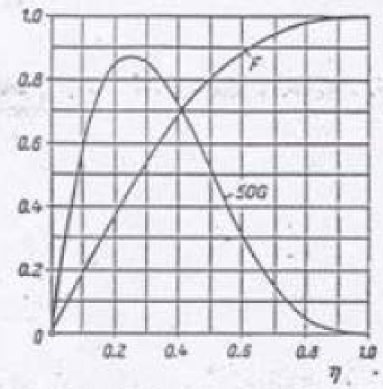


Fig. 10.3. The functions $F(\eta)$ and $G(\eta)$ for the velocity distribution in the boundary layer from eqns. (10.22) and (10.23)

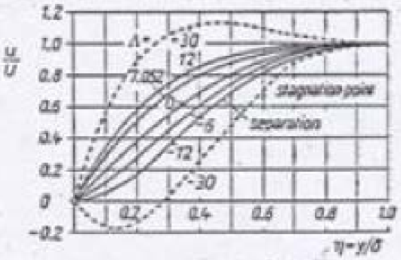


Fig. 10.4. The one-parameter family of velocity profiles from eqn. (10.22)

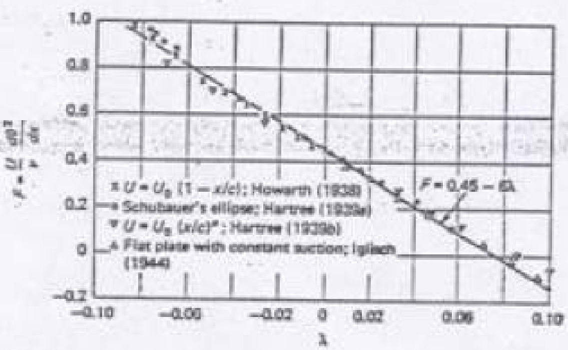


FIGURE 4-23
Empirical correlation of the boundary-layer function $F(\lambda)$ in Eq. (4-189). [*After Thwaites (1949).*]

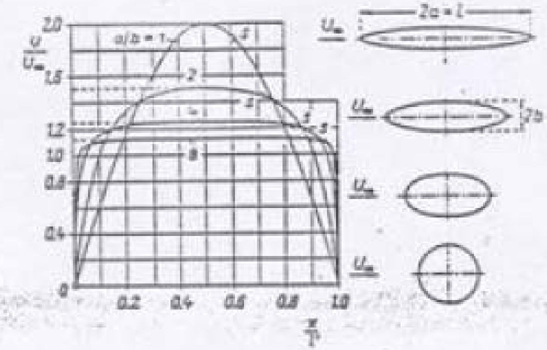


Fig. 10.9. Potential velocity distribution function on elliptical cylinders of slenderness $a/b = 1, 2, 4, 8$, the direction of the stream being parallel to the major axis

S = position of point of separation

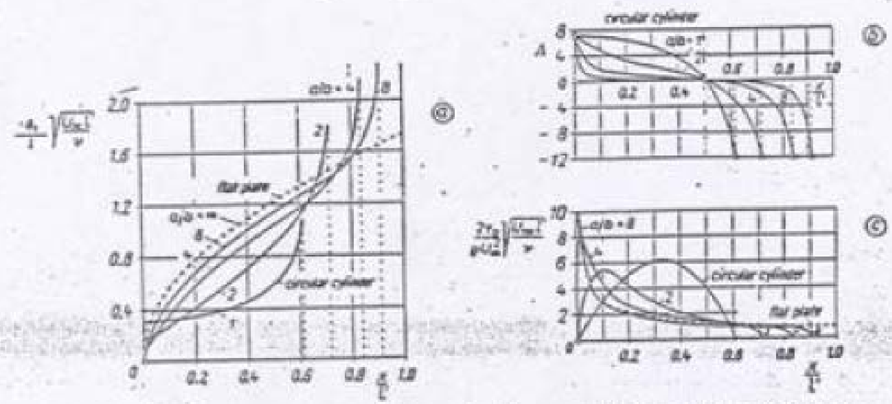


Fig. 10.10. Results of the calculation of boundary layers on elliptical cylinders of slenderness $a/b = 1, 2, 4, 8$, Fig. 10.9. a) displacement thickness of the boundary layer, b) shape factor c) shearing stress at the wall, $2\,l'$ = circumference of the ellipse; $a/b = 1$ circular cylinder; $a/b = \infty$ flat plate

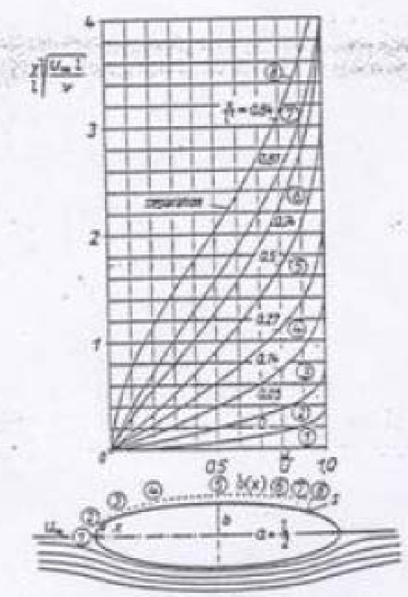


Fig. 10.11. Velocity profiles in the laminar boundary layer on an elliptical cylinder. Ratio of axes $a/b = 4$

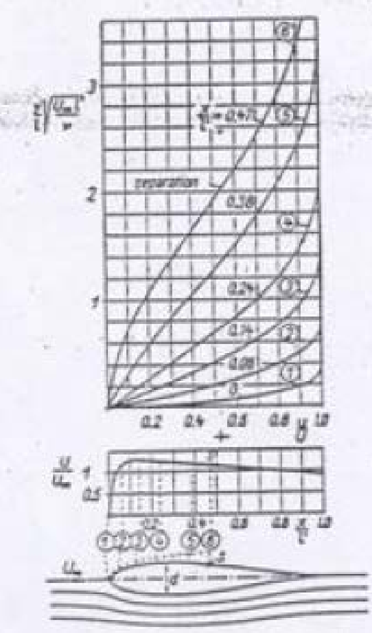


Fig. 10.12. Velocity profiles in the laminar boundary layer and potential velocity function for a Zhukovskii aerofoil J 015 of thickness ratio $d/l = 0{\cdot}15$ at an angle of incidence $\alpha = 0$

Refer back to Table 4-6 (Lec 26) with regard to relative performance of integral methods

7.6. Turbulent Boundary Layer

1. *Introduction: Transition to Turbulence*

Chapter 6 described the transition process as a succession of Tollmien-Schlichting waves, development of Λ - structures, vortex decay and formation of turbulent spots as preliminary stages to fully turbulent boundary-layer flow.

The phenomena observed during the transition process are similar for the flat plate boundary layer and for the plane channel flow, as shown in the following figure based on measurements by M. Nishioka et al. (1975). Periodic initial perturbations were generated in the BL using an oscillating cord.

For typical commercial surfaces transition occurs at $\mathrm{Re}_{x,tr} \approx 5\times10^5$. However, one can delay the transition to $\mathrm{Re}_{x,tr} \approx 3\times10^6$ with care in polishing the wall.

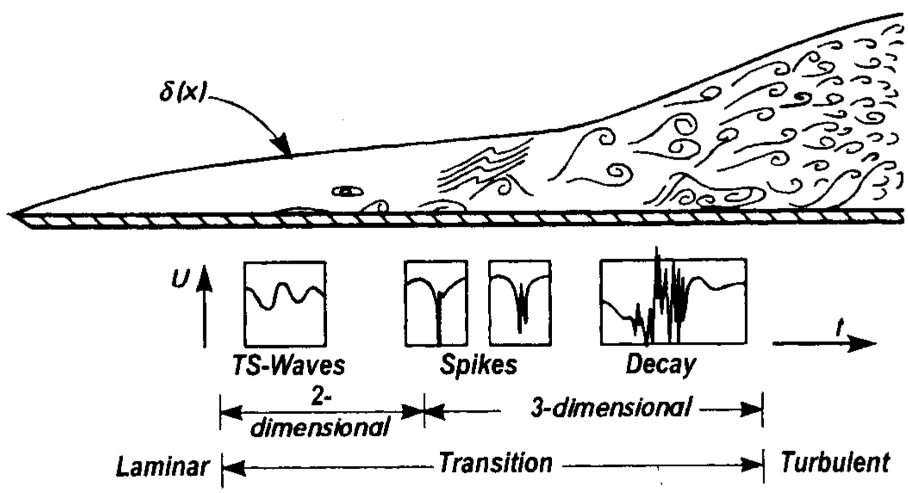


Fig. 15.38. Signals found at different regions in the transition at a plate at zero incidence, after M. Nishioka et al. (1975, 1990)

2. *Reynolds Average of 2D boundary layer equations*

$$u = \overline{u} + u'; \quad v = \overline{v} + v'; \quad w = \overline{w} + w'; \quad p = \overline{p} + p';$$

Substituting *u*, *v* and *w* into continuity equation and taking the time average we obtain,

$$\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}+\frac{\partial \overline{w}}{\partial z}=0 \qquad \frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0$$

Similarly for the momentum equations and using continuity (neglecting g),

$$\rho\frac{D\overline{V}}{Dt}=-\nabla\overline{p}+\nabla\cdot\tau_{ij}$$

Where

$$\tau_{ij}=\underbrace{\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}_{\text{Laminar}}-\underbrace{\rho\overline{u_i'u_j'}}_{\text{Turbulent}}$$

Assume

a. $\delta(x) << x$ which means $\overline{v} << \overline{u}$, $\dfrac{\partial}{\partial x} << \dfrac{\partial}{\partial y}$

b. mean flow structure is two-dimensional: $\overline{w}=0$, $\dfrac{\partial}{\partial z}=0$

Note the mean lateral turbulence is actually not zero, $\overline{w'^2}\neq 0$, but its z derivative is assumed to vanish.

Then, we get the following BL equations for incompressible steady flow:

$$\boxed{\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}=0} \qquad \text{Continuity}$$

$$\boxed{\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}\approx U_e\frac{dU_e}{dx}+\frac{1}{\rho}\frac{\partial \tau}{\partial y}} \qquad \text{x-momentum}$$

$$\boxed{\frac{\partial p}{\partial y}\approx -\rho\frac{\partial \overline{v'^2}}{\partial y}} \qquad \text{y-momentum}$$

Where U_e is the free-stream velocity and:

$$\tau = \mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'}$$

Note:

- The equations are solved for the time averages $\bar{u}$ and $\bar{v}$
- The shear stress now consists of two parts: 1. first part is due to the molecular exchange and is computed from the time-averaged field as in the laminar case; 2. The second part appears additionally and is due to turbulent motions.
- The additional term is new unknown for which a relation with the average field of the velocity must be constructed via a turbulence model.

Integrate y- momentum equation across the boundary layer

$$p \approx p_e(x) - \rho \overline{v'^2}$$

So, unlike laminar BL, there is a slight variation of pressure across the turbulent BL due to velocity fluctuations normal to the wall, which is no more than 4% of the stream-wise velocity and thus can be neglected. The Bernoulli relation is assumed to hold in the inviscid free-stream:

$$dp_e / dx \approx -\rho U_e dU_e / dx$$

Assume the free stream conditions, $U_e(x)$ is known, the boundary conditions:

No slip: $\bar{u}(x,0) = \bar{v}(x,0) = 0$

Free stream matching: $\bar{u}(x,\delta) = U_e(x)$

3. ***Momentum Integral Equations valid for BL solutions***

The momentum integral equation has the identical form as the laminar-flow relation:

$$\frac{d\theta}{dx} + (2+H)\frac{\theta}{U_e}\frac{dU_e}{dx} = \frac{\tau_w}{\rho U_e^{\,2}} = \frac{C_f}{2}$$

For laminar flow:

(C_f, H, θ) are correlated in terms of simple parameter $\lambda = \frac{\theta^2}{\upsilon}\frac{dU_e}{dx}$

For Turbulent flow:

(C_f, H, θ) cannot be correlated in terms of a single parameter. Additional parameters and relationships are required that model the influence of the turbulent fluctuations. There are many possibilities all of which require a certain amount of empirical data. As an example we will review the $\pi-\beta$ method.

4. ***Flat plate boundary layer (zero pressure gradient)***

a. Smooth flat plate

$\mathrm{Re}_t = 5\times10^5 \sim 3\times10^6$ for a flat plate boundary layer

$\mathrm{Re}_{crit} \sim 100{,}000$

$$\frac{c_f}{2} = \frac{d\theta}{dx}$$

as was done for the approximate laminar flat plate boundary-layer analysis, solve by expressing $c_f = c_f(\delta)$ and $\theta = \theta(\delta)$ and integrate, i.e. assume log-law valid across entire turbulent boundary-layer

$$\frac{u}{u^*} = \frac{1}{\kappa}\ln\frac{yu^*}{\nu} + B$$

neglect laminar sub layer and velocity defect region

at y = δ, u = U

$$\frac{U}{u^*} = \frac{1}{\kappa}\ln\frac{\delta u^*}{\nu} + B$$

$$\frac{\delta u^*}{\nu} = \mathrm{Re}_\delta\left(\frac{c_f}{2}\right)^{1/2}$$

$$\left.\begin{aligned} &\text{or}\quad \left(\frac{2}{c_f}\right)^{1/2} = 2.44\ln\left[\mathrm{Re}_\delta\left(\frac{c_f}{2}\right)^{1/2}\right] + 5 \\ &c_f \cong .02\,\mathrm{Re}_\delta^{-1/6} \text{ power-law fit} \end{aligned}\right\} c_f(\delta)$$

Next, evaluate

$$\frac{d\theta}{dx} = \frac{d}{dx}\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right)dy$$

can use log-law or more simply a power law fit

$$\left.\begin{aligned} &\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} \\ &\theta = \frac{7}{72}\delta = \theta(\delta) \end{aligned}\right\}$$

Note: cannot be used to obtain $c_f(\delta)$ since $\tau_w \to \infty$

$$\Rightarrow \qquad \tau_w = c_f\frac{1}{2}\rho U^2 = \rho U^2\frac{d\theta}{dx} = \frac{7}{72}\rho U^2\frac{d\delta}{dx}$$

$$\mathrm{Re}_\delta^{-1/6} = 9.72\frac{d\delta}{dx}$$

or $\dfrac{\delta}{x} = 0.16\,\mathrm{Re}_x^{-1/7}$

$\delta \propto x^{6/7}$ almost linear

i.e., much faster growth rate than laminar boundary layer

$$c_f = \frac{0.027}{\mathrm{Re}_x^{1/7}}$$

$$\tau_{w,turb} = \frac{0.0135\mu^{1/7}\rho^{6/7}U^{13/7}}{x^{1/7}}$$

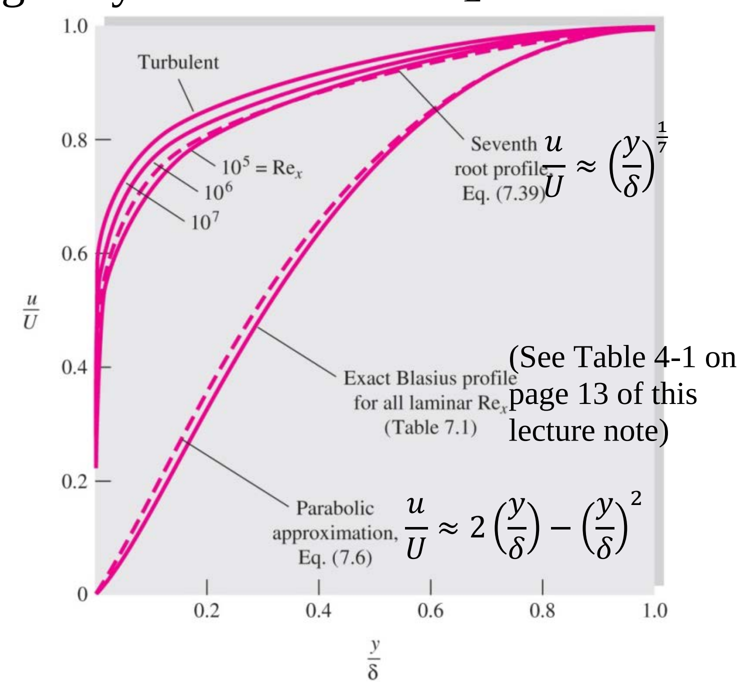
$\tau_{w,turb}$ decreases slowly with x, increases with ρ and U^2 and insensitive to μ

$$C_D = C_f = \frac{0.031}{Re_L^{1/7}} = \frac{7}{6}c_f(L)$$

$$\delta^* = \frac{1}{8}\delta$$

$$H = \frac{\delta^*}{\theta} = 1.3$$

These formulas are for a fully turbulent flow over a smooth flat plate from the leading edge; in general, give better results for sufficiently large Reynolds number $\mathrm{Re}_L > 10^7$.



Comparison of dimensionless laminar and turbulent flat-plate velocity profiles (Ref: White, F. M., Fluid Mechanics, 7th Ed., McGraw-Hill)

Alternate forms by using the same velocity profile $u/U = (y/\delta)^{1/7}$ assumption but using an experimentally determined shear stress formula $\tau_w = 0.0225\rho U^2(\nu/U\delta)^{1/4}$ are:

$$\frac{\delta}{x} = 0.37\,\mathrm{Re}_x^{-1/5} \qquad c_f = \frac{0.058}{\mathrm{Re}_x^{1/5}} \qquad C_f = \frac{0.074}{\mathrm{Re}_L^{1/5}}$$

shear stress: $$\tau_w = \frac{0.029\rho U^2}{\mathrm{Re}_x^{1/5}}$$

These formulas are valid only in the range of the experimental data, which covers $\mathrm{Re_L} = 5 \times 10^5 \sim 10^7$ for smooth flat plates.

Other empirical formulas are by using the logarithmic velocity-profile instead of the 1/7-power law:

$$\frac{\delta}{L} = c_f(0.98\log Re_L - 0.732)$$

$$c_f = (2\log Re_x - 0.65)^{-2.3}$$

$$C_f = \frac{0.455}{(\log_{10} Re_L)^{2.58}}$$

These formulas are also called as the *Prandtl-Schlichting skin-friction formula* and valid in the whole range of $\mathrm{Re_L} \le 10^9$.

For these experimental/empirical formulas, the boundary layer is usually “tripped” by some roughness or leading edge disturbance, to make the boundary layer turbulent from the leading edge.

No definitive values for turbulent conditions since depend on empirical data and turbulence modeling.

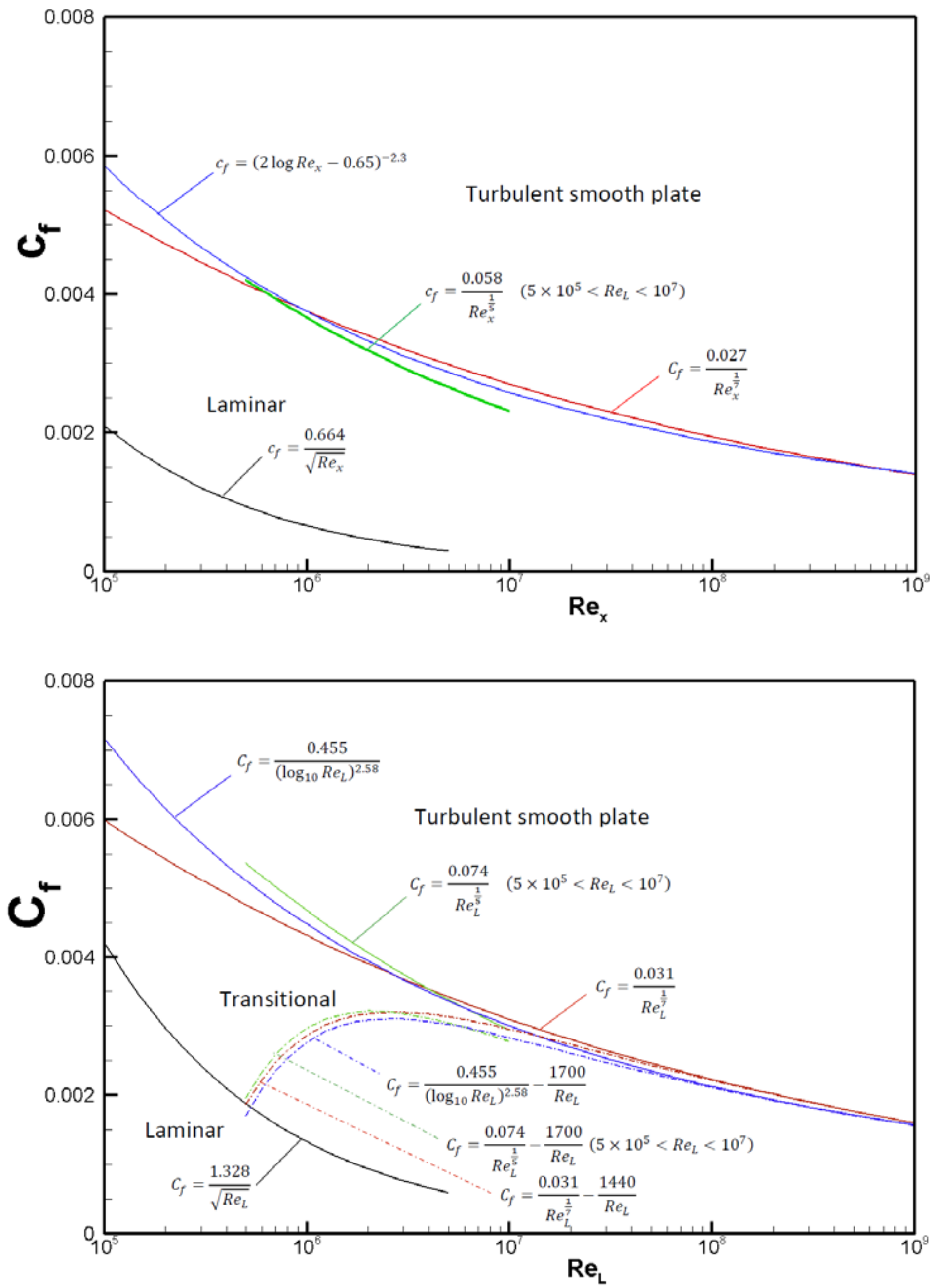
Finally, composite formulas that take into account both the initial laminar boundary layer and subsequent turbulent boundary layer, i.e. in the transition region ($5 \times 10^5 < Re_L < 8 \times 10^7$) where the laminar drag at the leading edge is an appreciable fraction of the total drag:

$$C_f = \frac{0.031}{Re_L^{\frac{1}{7}}} - \frac{1440}{Re_L}$$

$$C_f = \frac{0.074}{Re_L^{\frac{1}{5}}} - \frac{1700}{Re_L}$$

$$C_f = \frac{0.455}{(\log_{10} Re_L)^{2.58}} - \frac{1700}{Re_L}$$

with transitions at $Re_t = 5 \times 10^5$ for all cases.



Local friction coefficient c_f (top) and friction drag coefficient C_f(bottom) for a flat plate parallel to the upstream flow.

b. Influence of roughness

The influence of roughness can be analyzed in an exactly analogous manner as done for pipe flow i.e.

$$u^+ = \frac{1}{\kappa}\ln\frac{yu^*}{\upsilon} + B + \Delta B(\varepsilon^+) \qquad ①$$

$$\Delta B(\varepsilon^+) = -\frac{1}{\kappa}\ln(1+0.3\varepsilon^+)$$

i.e. rough wall velocity profile shifts downward by a constant amount $\Delta B(\varepsilon^+)$ which, increases with $\varepsilon^+ = \varepsilon u^* / \vartheta$

A complete rough-wall analysis can be done using the composite log-law in a similar manner as done for a smooth wall i.e. determine $C_f(\delta)$ and $\theta(\delta)$ from ① and equate using momentum integral equation

$$C_f(\delta) = 2\frac{d}{dx}\theta(\delta)$$

Then eliminate δ to get $C_f(x, \varepsilon / x)$

However, analysis is complicated: solution is Fig. 7.6. For fully rough-flow a curve fit to the C_f and C_D equations is given by,

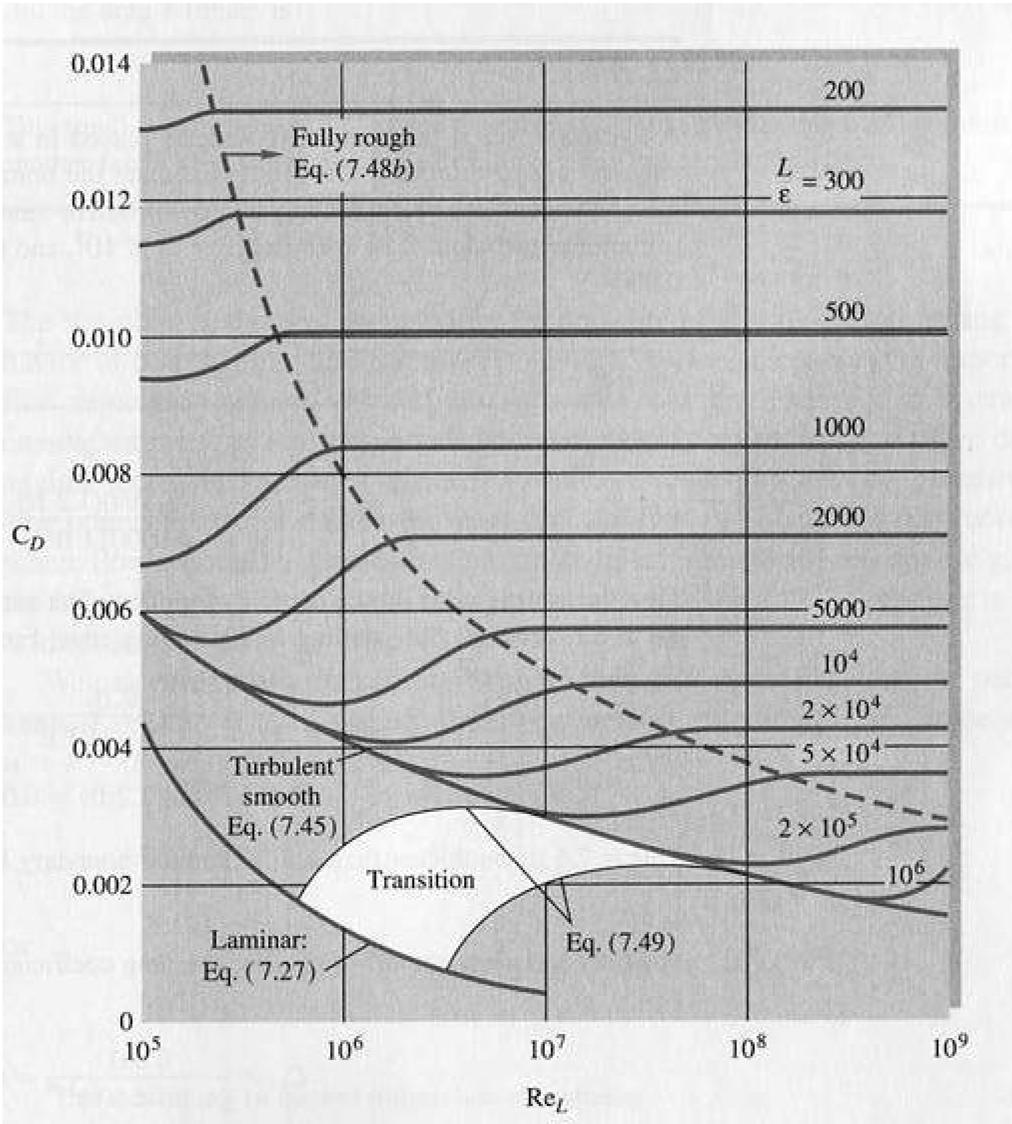


Fig. 7.6 Drag coefficient of laminar and turbulent boundary layers on smooth and rough flat plates.

$$\left.\begin{aligned} C_f &= (2.87 + 1.58\log\frac{x}{\varepsilon})^{-2.5} \\ C_D &= (1.89 + 1.62\log\frac{L}{\varepsilon})^{-2.5} \end{aligned}\right\} \text{ Fully rough flow}$$

Again, shown on Fig. 7.6. along with transition region curves developed by Schlichting which depend on $Re_t = \begin{cases} 5\times10^5 \\ 3\times10^6 \end{cases}$

5. ***Boundary layer with pressure gradient***

$$u_x + v_y = 0$$

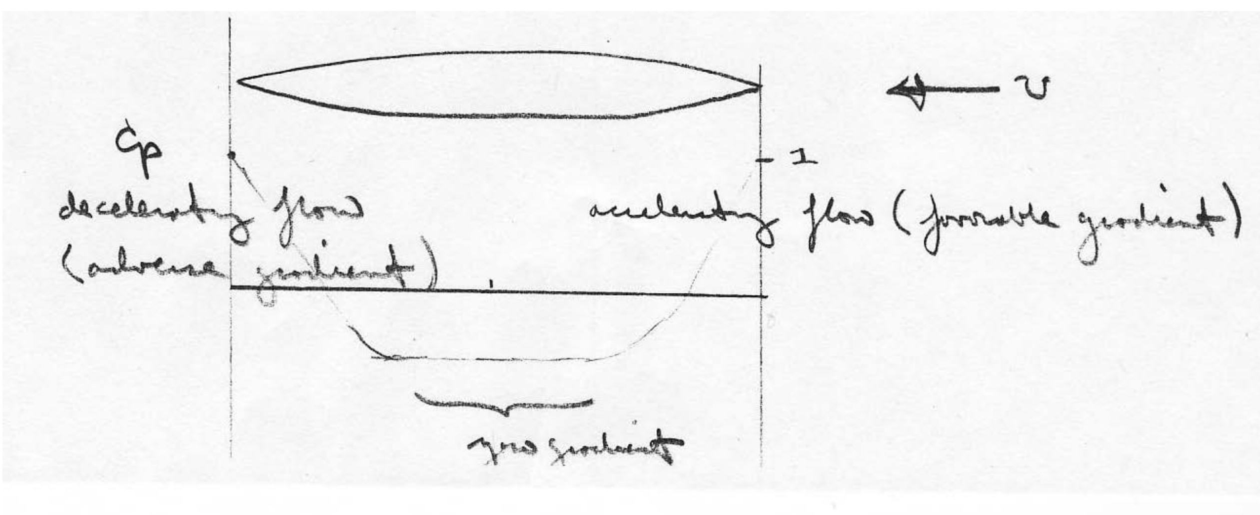
$$uu_x + vu_y = \boxed{-\frac{\partial}{\partial x}(p/\rho)} + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

$$\tau = \mu \frac{\partial u}{\partial y} - \rho \overline{u'v'}$$

The pressure gradient term has a large influence on the solution. In particular, adverse pressure gradient (i.e. increasing pressure) can cause flow separation. Recall that the y momentum equation subject to the boundary layer assumptions reduced to

$p_y = 0$ i.e. $p = p_e$ = constant across BL.

That is, pressure (which drives BL equations) is given by external inviscid flow solution which in many cases is also irrotational. Consider a typical inviscid flow solution (chapter 8)



Even without solving the BL equations we can deduce information about the shape of the velocity profiles just by evaluating the BL equations at the wall (y = 0)

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p_e}{\partial x}$$

$$where \quad \frac{\partial p_e}{\partial x} = -\rho U_e \frac{dU_e}{dx}$$

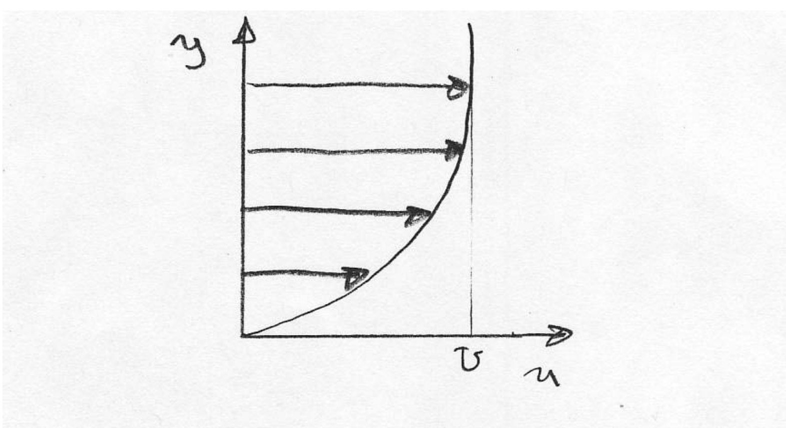
which, shows that the curvature of the velocity profile at the wall is related to the pressure gradient.

Effect of Pressure Gradient on Velocity Profiles

Point of inflection: a point where a graph changes between concave upward and concave downward.

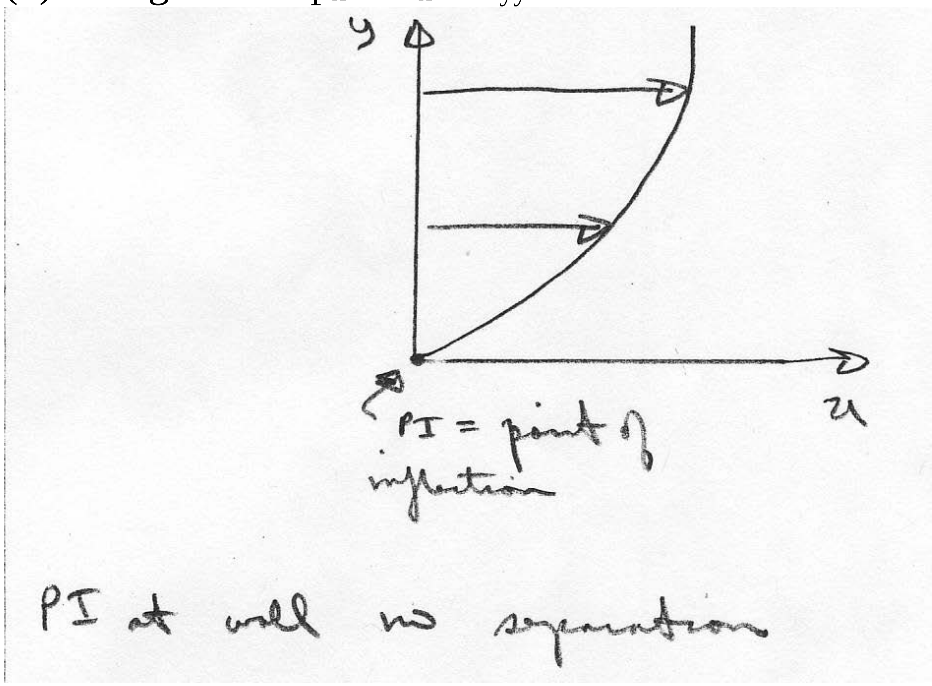
The point of inflection is basically the location where second derivative of u is zero, i.e. $\dfrac{\partial^2 u}{\partial y^2} = 0$

(a) favorable gradient: $p_x<0$, $U_x>0$, $u_{yy}<0$

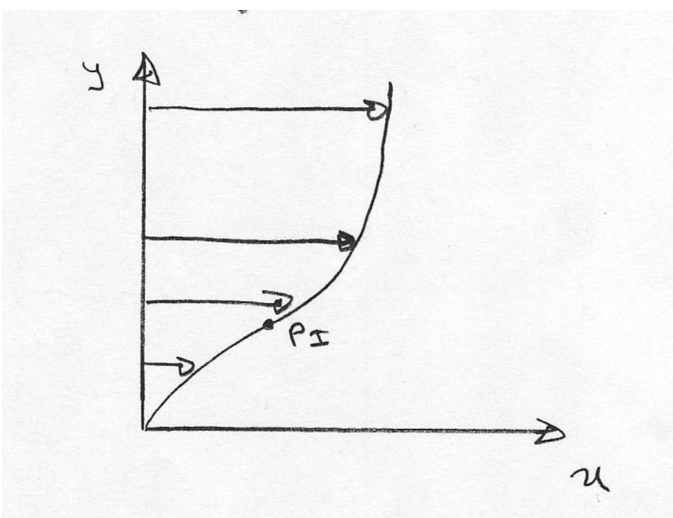


No point of inflection i.e. curvature is negative all across the BL and BL is very resistant to separation. Note $u_{yy}(\delta)<0$ in order for u to merge smoothly with U.

(b) zero gradient: $p_x = U_x = u_{yy} = 0$

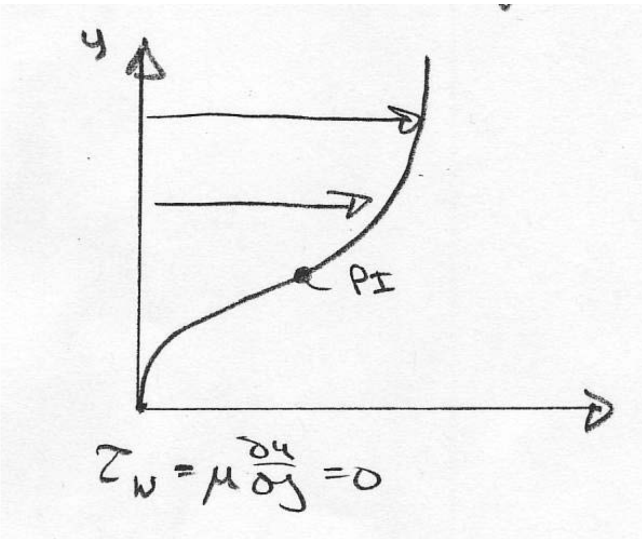


(c) weak adverse gradient: $p_x>0$, $U_x<0$, $u_{yy}>0$



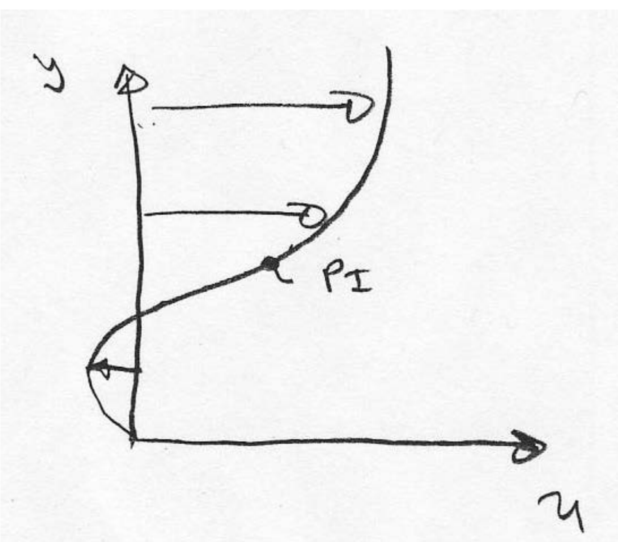
PI in flow, still no separation

(d) critical adverse gradient: $p_x>0$, $U_x<0$, $u_{yy}>0$, $u_y = 0$



PI in flow, incipient separation

(e) excessive adverse gradient: $p_x>0$, $U_x<0$, $u_{yy}>0$, $\underbrace{u_y < 0}_{\tau_w < 0}$



PI in flow, backflow near wall i.e. separated flow region

i.e. main flow breaks away or separates from the wall: large increase in drag and loss of performance:

$H_{separation} = 3.5$ laminar

$= 2.4$ turbulent

6. *π-β Method*

As mentioned earlier, the momentum integral equation for turbulent flow has the identical form as the laminar-flow relation:

$$\frac{d\theta}{dx}=\frac{C_f}{2}-(2+H)\frac{\theta}{U_e}\frac{dU_e}{dx} \qquad \text{(I)}$$

With U(x) assumed known, there are three unknown C_f, H, θ for turbulent flow. Thus, at least two additional relations are needed to find unknowns. There are many possibilities for additional relations all of which require a certain amount of empirical data. As an example we will review the π−β method.

Cole's law of the wake:
By adding the wake to the log-law, the velocity profile for both overlap and outer layers can be written as:

$$u^+=\frac{1}{\kappa}\ln y^+ + B+\frac{2\Pi}{\kappa}f(\eta)$$

where

$$\eta = y/\delta$$

$$f(\eta)=\sin^2\left(\frac{\pi}{2}\eta\right)=3\eta^2-2\eta^3$$

$$\Pi=\kappa A/2$$

The quantity Π is called Coles' wake parameter.

By integrating wall-wake law across the boundary layer:

$$\lambda=\frac{\sqrt{2}}{c_f}=a(\Pi)\frac{H}{H-1}$$

$$a(\Pi)=\frac{2+3.179\Pi+1.5\Pi^2}{\kappa(1+\Pi)}$$

$$\mathrm{Re}_\theta=\frac{U\theta}{\upsilon}=\frac{1+\Pi}{\kappa H}\exp(\kappa\lambda-\kappa B-2\Pi)$$

If we eliminate Π between these formulas, we obtain a unique relation among $C_f = 2/\lambda^2$, H and θ:

$$\begin{cases} C_f = 2/\lambda^2 = 2/[a(\Pi)\dfrac{H}{H-1}]^2 \\ a(\Pi) = \dfrac{2+3.179\Pi+1.5\Pi^2}{\kappa(1+\Pi)} \\ \mathrm{Re}_\theta = \dfrac{U\theta}{\upsilon} = \dfrac{1+\Pi}{\kappa H}\exp(\kappa\lambda-\kappa B-2\Pi) \end{cases} \qquad \text{(II)}$$

<u>Clauser's equilibrium parameter β:</u>

For outer layer,

$$U_e - \overline{u} = f(\tau_w, \rho, y, \delta, \frac{dp}{dx})$$

Using dimensional analysis:

$$\frac{U_e-\overline{u}}{(\tau_w/\rho)^{1/2}} = g(\frac{y}{\delta}, \frac{\delta}{\tau_w}\frac{dp}{dx})$$

Clauser (1954) replaced δ by displacement thickness δ^*:

$$\frac{U_e-\overline{u}}{(\tau_w/\rho)^{1/2}} = g(\frac{y}{\delta}, \beta)$$

$$\beta = \frac{\delta^*}{\tau_w}\frac{dp}{dx} = -\lambda^2 H\frac{\theta}{U_e}\frac{dU_e}{dx}$$

β is called Clauser's equilibrium parameter.

Das (1987) showed that EFD data points fit into the following polynomial correlation:

$$\beta = -0.4 + 0.76\Pi + 0.42\Pi^2$$

Therefore:

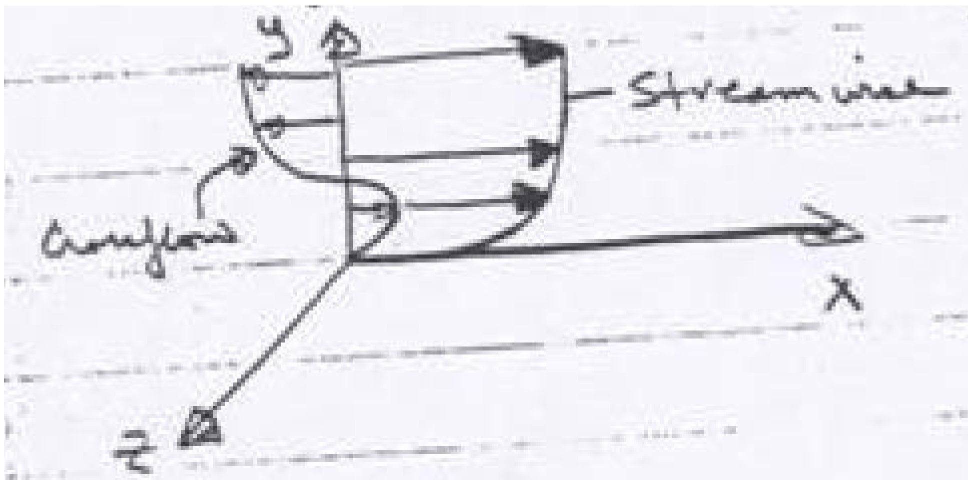
$$-\lambda^2 H\frac{\theta}{U_e}\frac{dU_e}{dx} = -0.4+0.76\Pi+0.42\Pi^2 \qquad \text{(III)}$$

If we eliminate Π using that $\mathrm{Re}_\theta = \frac{U\theta}{\upsilon} = \frac{1+\Pi}{\kappa H}\exp(\kappa\lambda - \kappa B - 2\Pi)$, we obtain another relation among $C_f = 2/\lambda^2$, H and θ.

Equations (I), (II), and (III) can be solved simultaneously using say a Runge-Kutta method to find C_f, H, θ. Equations are solved with initial condition for $\theta(x_0)$ and integrated to $x=x_0+\Delta x$ iteratively. Estimated θ gives Re_θ and Π, β gives H. Lastly C_f is evaluated using Re_θ and H. Iterations required until all relations satisfied and then proceed to next Δx.

7. 3-D Integral methods

Momentum integral methods perform well (i.e. compare well with experimental data) for a large class of both laminar and turbulent 2D flows. However, for ***3D flows they do not***, primarily due to the inability of correlating the cross flow velocity components.



The cross flow is driven by $\frac{\partial p}{\partial z}$, which is imposed on BL from the outer potential flow U(x,z).

3-D boundary layer equations

$$uu_x + vu_y + wu_z = -\frac{\partial}{\partial x}(p/\rho) + \vartheta u_{yy} - \frac{\partial}{\partial y}(\overline{u'v'});$$

$$uw_x + vw_y + ww_z = -\frac{\partial}{\partial z}(p/\rho) + \vartheta w_{yy} - \frac{\partial}{\partial y}(\overline{v'w'});$$

$$u_x + v_y + w_z = 0;$$

$$+ \quad \text{closure equations}$$

Differential methods have been developed for this reason as well as for extensions to more complex and non-thin boundary layer flows.

7.7 Separation

What causes separation?

The increasing downstream pressure slows down the wall flow and can make it go backward-flow separation.

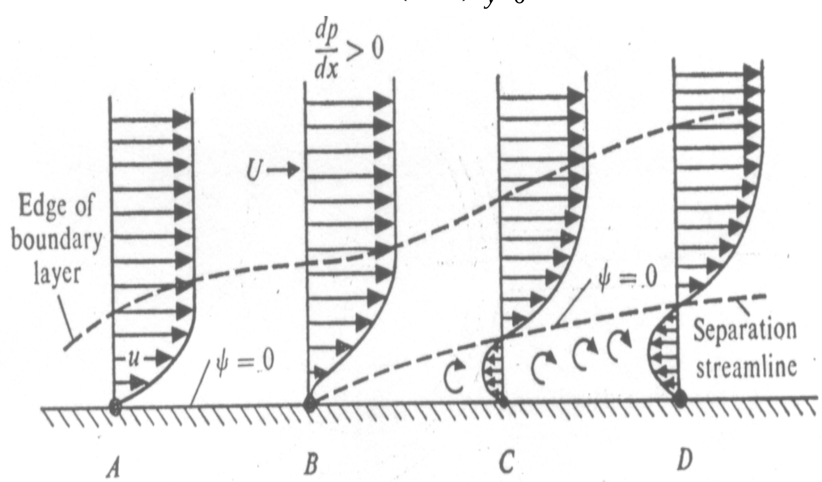
$dp/dx > 0$ adverse pressure gradient, flow separation may occur.

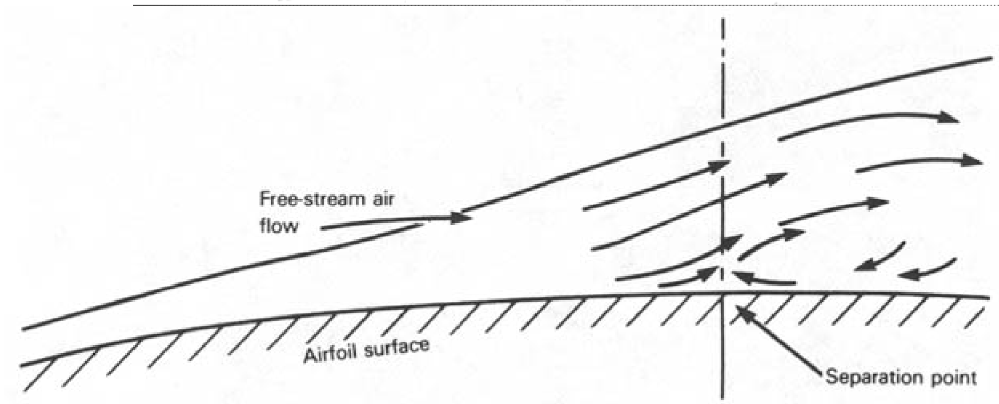
$dp/dx < 0$ favorable gradient, flow separation can never occur

Previous analysis of BL was valid before separation.

Separation Condition

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} = 0$$





Note: 1. Due to backflow close to the wall, a strong thickening of the BL takes place and BL mass is transported away into the outer flow

2. At the point of separation, the streamlines leave the wall at a certain angle.

Separation of Boundary Layer

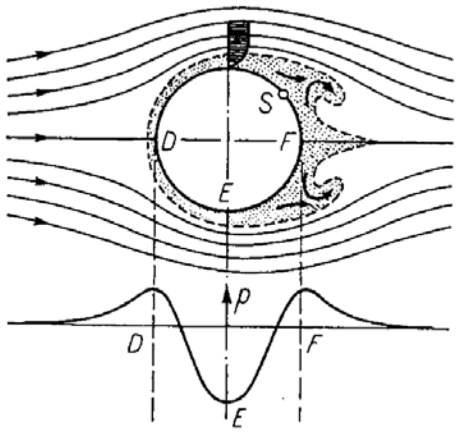


Fig. 2.6. Separation of the boundary layer and vortex formation at a circular cylinder (schematic). S = separation point

Notes:

1. D to E, pressure drop, pressure is transformed into kinetic energy.
2. From E to F, kinetic energy is transformed into pressure.
3. A fluid particle directly at the wall in the boundary layer is also acted upon by the same pressure distribution as in the outer flow (inviscid).
4. Due to the strong friction forces in the BL, a BL particle loses so much of its kinetic energy that is cannot manage to get over the "pressure gradient" from E to F.
5. The following figure shows the time sequence of this process:
   a. reversed motion begun at the trailing edge
   b. boundary layer has been thickened, and start of the reversed motion has moved forward considerably.
   c. and d. a large vortex formed from the backflow and then soon separates from the body.

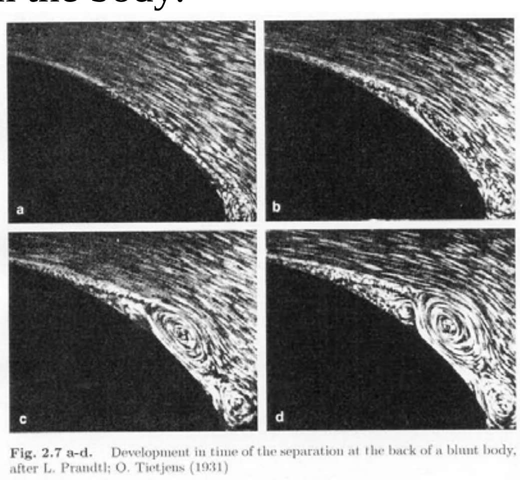


Fig. 2.7 a-d. Development in time of the separation at the back of a blunt body, after L. Prandtl; O. Tietjens (1931)

Examples of BL Separations (two-dimensional)

Features: The entire boundary layer flow breaks away at the point of zero wall shear stress and, having no way to diverge left or right, has to go up and over the resulting separation bubble or wake.

1. Plane wall(s)

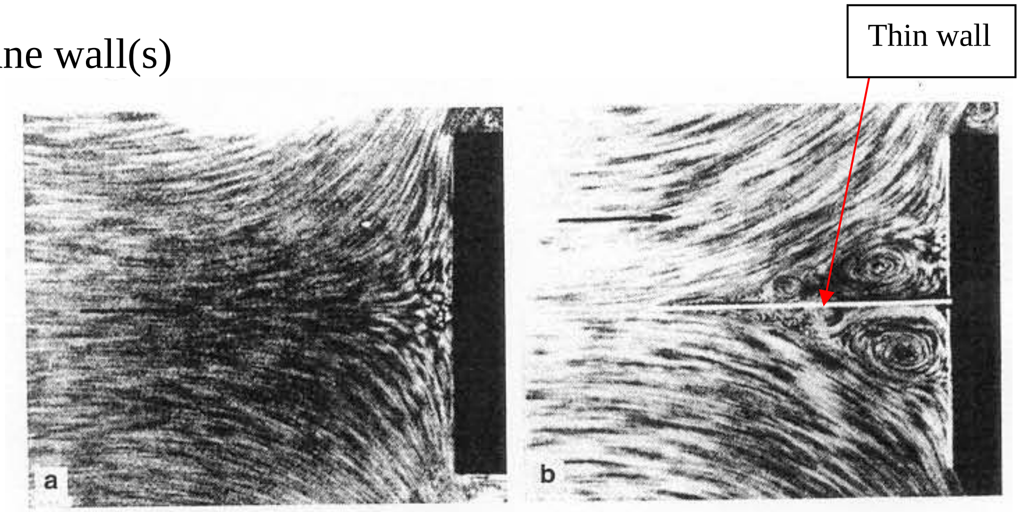
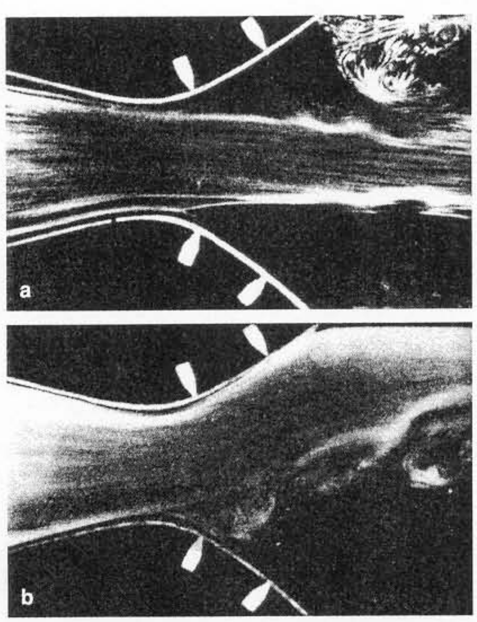


Fig. 2.10. Stagnation point flow, after H. Föttinger (1939), (**a**) free stagnation–point flow without separation, (**b**) retarded stagnation–point flow, with separation

(a). Plane stagnation-point flow: no separation on the streamlines of symmetry (no wall friction present), and no separation at the wall (favorable pressure gradient)

(b).Flat wall with right angle to the wall: flow separate, why?

2. Diffuser flow:



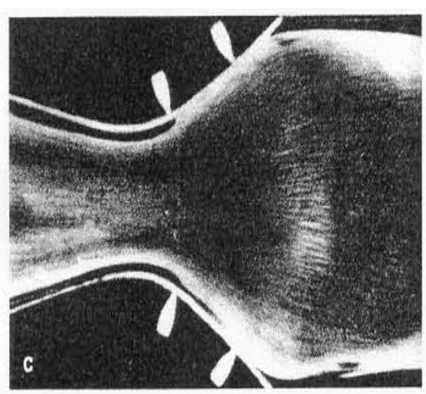
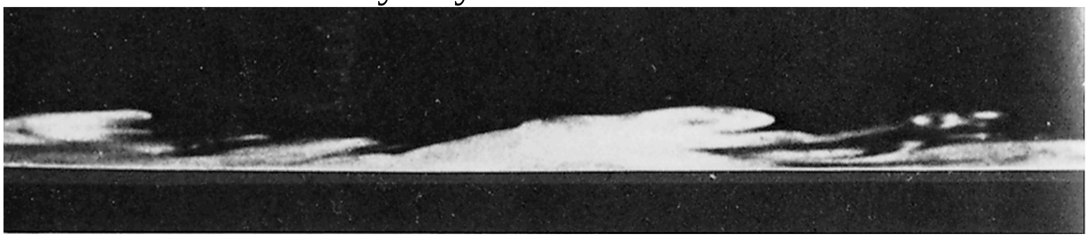
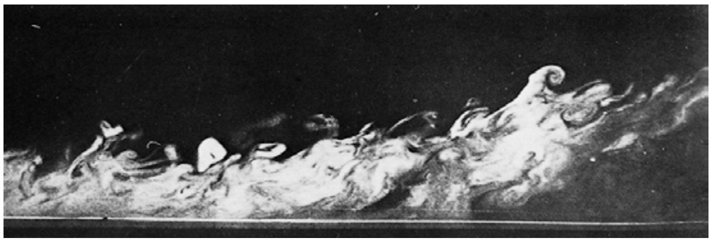


Fig. 2.9. Flow in a widening channel (diffuser) (**a**) separation at both diffuser walls, (**b**) suction of the boundary layer at the upper diffuser wall, (**c**) suction at both diffuser walls (after L. Prandtl; O. Tietjens (1931))

3. Turbulent Boundary Layer



(a)



(b)

Influence of a strong pressure gradient on a turbulent flow
(a) a strong negative pressure gradient may re-laminarize a flow
(b) a strong positive pressure gradient causes a strong boundary layer top thicken. (Photograph by R.E. Falco)

<u>Examples of BL Separations (three-dimensional)</u>

Features: unlike 2D separations, 3D separations allow many more options.

There are four different special points in separation:

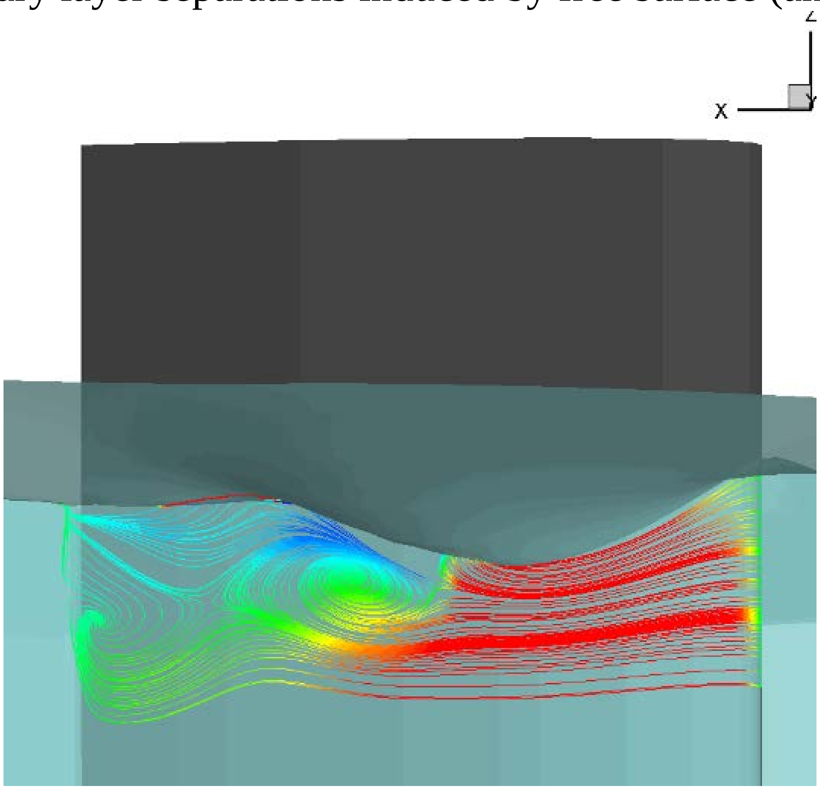
(1). *A nodal Point*, where an infinite number of surface streamlines merged tangentially to the separation line

(2). *A saddle point*, where only two surface streamlines intersect and all others divert to either side

(3). *A focus, or spiral node,* which forms near a saddle point and around which an infinite number of surface streamlines swirl

(4). *A three-dimensional singular point*, not on the wall, generally serving as the center for a horseshoe vortex.

1. Boundary layer separations induced by free surface (animation)



CFDSHIP-IOWA

2. Separation regions in corner flow

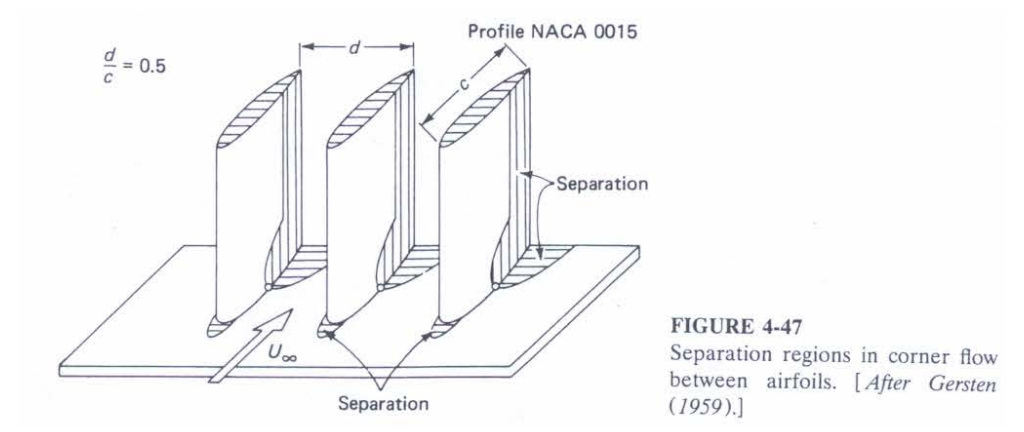


FIGURE 4-47
Separation regions in corner flow between airfoils. [*After Gersten* (*1959*).]

3. 3D separations on a round-nosed body at angle of attack

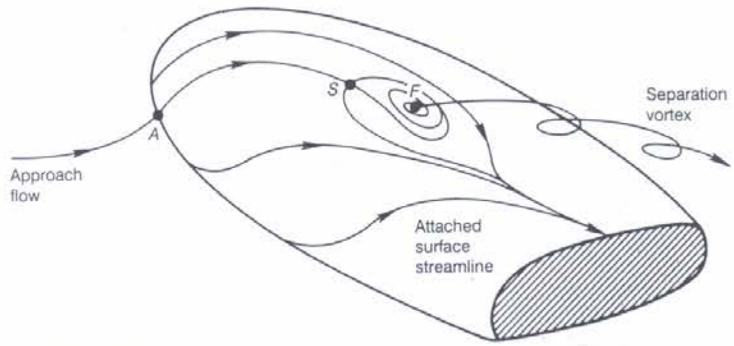


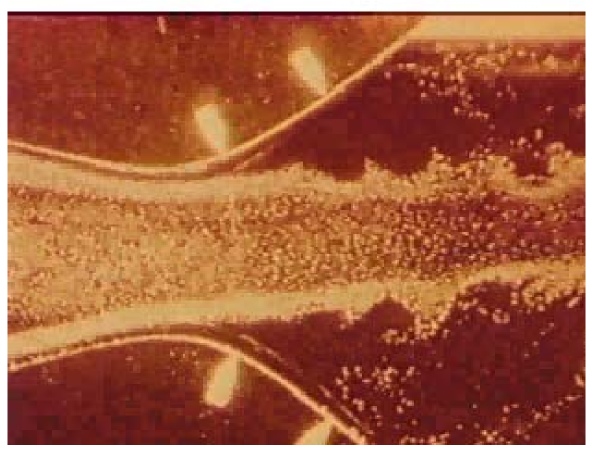
FIGURE 4-49. Three-dimensional separation on a round-nosed body at angle of attack, first described by Legendre (1965). Point A is a nodal attachment point, point S is a saddle point, and point F is a focus of separation.

<u>Video Library (animations from "Multi-media Fluid Mechanics", Homsy, G. M., etc.)</u>

Conditions Producing Separation

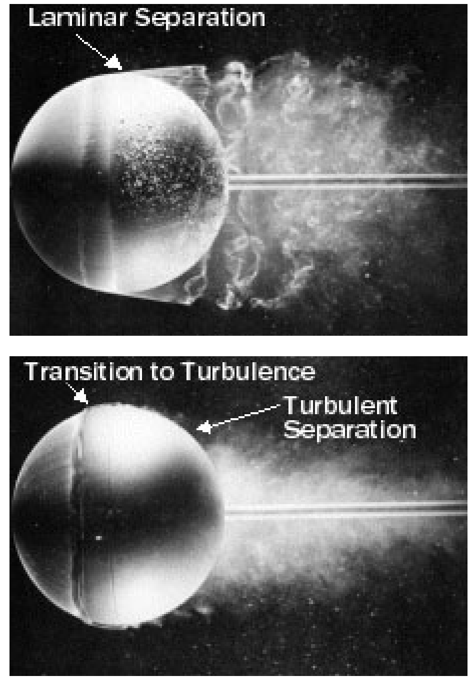
Separations on airfoil (different attack angles)

Leading edge separation

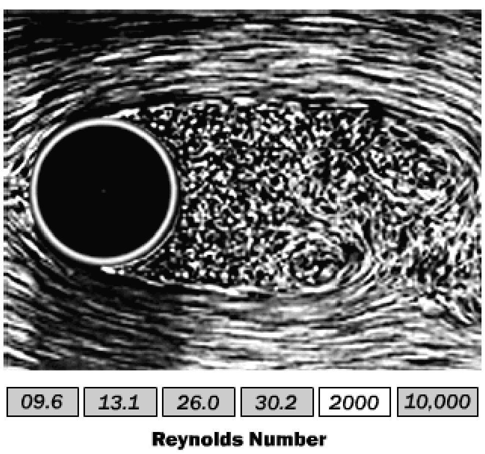


Separations in diffuser

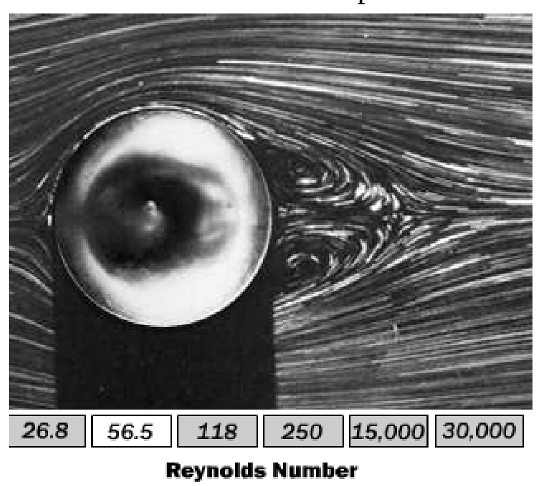
Effect of body shape on separation



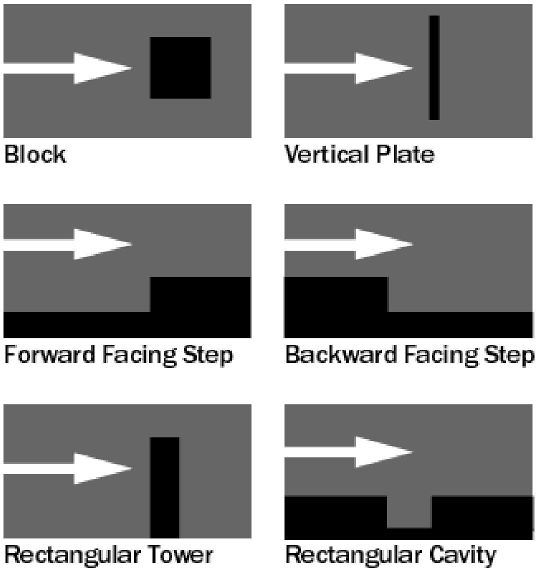
Laminar and Turbulent separation



Flow over cylinders: effect of Re

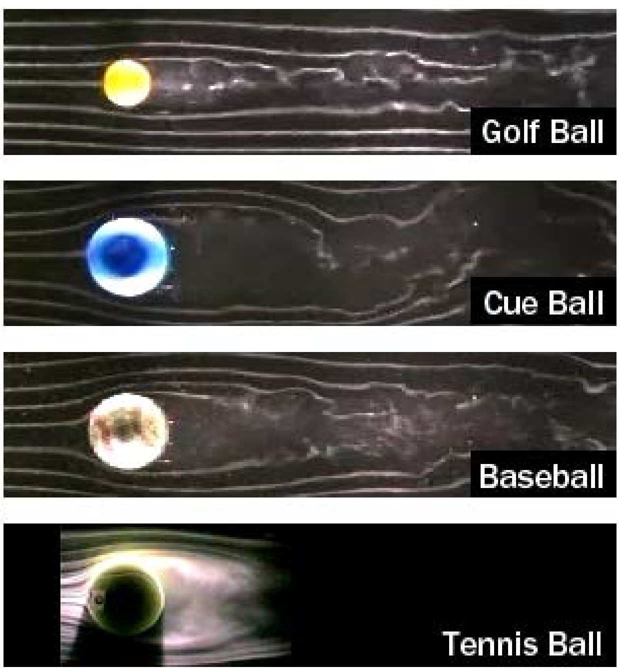


Flow over spheres: effect of Re



Flow over edges and blunt bodies

Flow over a truck



Effect of separation on sports balls