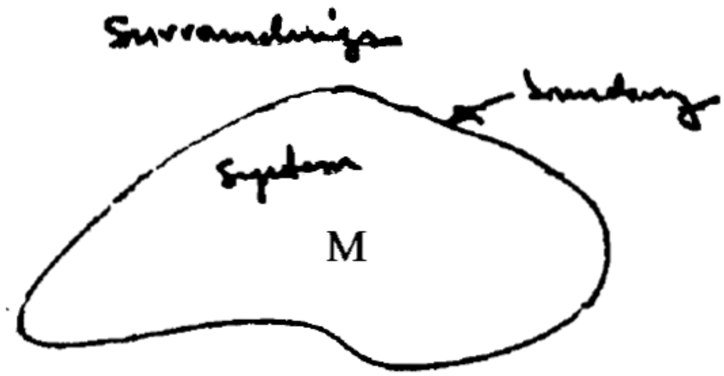
# Chapters 3 & 4: Integral Relations for a Control Volume and Differential Relations for Fluid Flow

Laws of mechanics are written for a system, i.e., a fixed amount of matter.




1. Conservation of mass: $\dfrac{dM}{dt} = 0$
2. Conservation of momentum: $\underline{F} = M\underline{a} = \dfrac{d(M\underline{V})}{dt}$
3. Conservation of energy: $\dfrac{dE}{dt} = \dot{Q} - \dot{W}$

   $\Delta E$=heat added – work done

Also

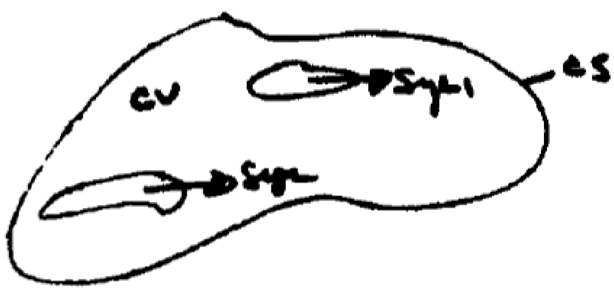
Conservation of angular momentum: $\dfrac{d\underline{H_G}}{dt} = \underline{M_G}$

Second Law of Thermodynamics: $\dfrac{dS}{dt} = \dfrac{\delta\dot{Q}}{T} + \dot{\sigma}$

$\dot{\sigma}$, entropy production due to system irreversibilities

$\dot{\sigma} \le 0$

In fluid mechanics we are usually interested in a region of space, i.e, control volume and not particular systems. Therefore, we need to transform GDE's from a system to a control volume, which is accomplished through the use of




RTT (actually derived in thermodynamics for CV forms of continuity and 1st and 2nd laws, but not in general form or referred to as RTT).

Note GDE's are of form:

$$\frac{d}{dt}\underbrace{(M, M\underline{V}, E)}_{} = RHS$$

system extensive properties $B_{sys}$ depend on mass

i.e., involve $\frac{dB_{sys}}{dt}$ which needs to be related to changes in CV. Recall, definition of corresponding system intensive properties

$$\beta = (1, \underline{V}, e) \qquad \text{independent of mass}$$

where

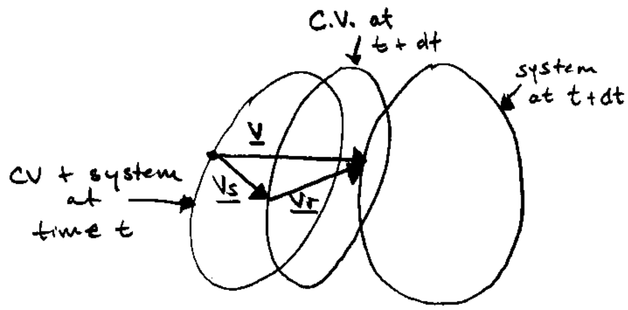
$$B = \int \beta dm = \int \beta \rho d\forall$$

i.e., $\beta = \frac{dB}{dm}$

# Reynolds Transport Theorem (RTT)

Need relationship between $\frac{d}{dt}\left(B_{sys}\right)$ and changes in

$B_{CV} = \int_{CV} \beta \, dm = \int_{CV} \beta\rho \, d\forall$.




Moving deforming CV:
$\underline{V}_r = \underline{V} - \underline{V}_s$

$\underline{V}$ = fluid velocity
$\underline{V}_s$ = CS defining CV velocity
$\underline{V}_r$ = relative velocity

all in same coordinate system

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0} \frac{(B_{cv} + \Delta B)_{t+\Delta t} - (B_{cv} + \Delta B)_t}{\Delta t}$$

$$= \underbrace{\lim_{\Delta t \to 0} \frac{B_{cv_{t+\Delta t}} - B_{cv_t}}{\Delta t}}_{1} + \underbrace{\lim_{\Delta t \to 0} \frac{\Delta B_{t+\Delta t} - \Delta B_t}{\Delta t}}_{2}$$

1 = time rate of change of B in CV = $\frac{dB_{CV}}{dt} = \frac{d}{dt}\int_{CV} \beta\rho \, d\forall$

2 = net outflux of B from CV across CS = $\int_{CS} \beta\rho \underline{V}_R \cdot \underline{n} \, dA$

As with Q and $\dot{m}$, $\Delta\dot{B}$ flux though A per unit time is:

$$dQ = \underline{V}_R . \underline{n} \, dA$$
$$d\dot{m} = \rho \underline{V}_R . \underline{n} \, dA$$
$$d\Delta\dot{B} = \beta\rho \underline{V}_R . \underline{n} \, dA$$

Therefore:

$$\frac{dB_{SYS}}{dt} = \frac{d}{dt}\int_{CV} \beta\rho \, d\forall + \int_{CS} \beta\rho \underline{V}_R \cdot \underline{n} \, dA$$

General form RTT for moving deforming control volume
Special Cases:

1) Non-deforming CV

$$\frac{dB_{SYS}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\beta\rho) d\forall + \int_{CS} \beta\rho \underline{V}_R \cdot \underline{n} \, dA$$

2) Fixed CV

$$\frac{dB_{SYS}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\beta\rho) d\forall + \int_{CS} \beta\rho \underline{V} \cdot \underline{n} \, dA$$

Greens Theorem: $\int_{CV} \nabla \cdot \underline{b} \, d\forall = \int_{CS} \underline{b} \cdot \underline{n} \, dA$

$$\frac{dB_{SYS}}{dt} = \int_{CV} \left[ \frac{\partial}{\partial t}(\beta\rho) + \nabla \cdot (\beta\rho \underline{V}) \right] d\forall$$

Since CV fixed and arbitrary $\lim_{d\forall \to 0}$ gives governing differential equation.

3) Uniform flow across discrete CS (steady or unsteady)

$$\int_{CS} \beta\rho \underline{V}_R \cdot \underline{n}\, dA = \sum_{CS} \beta\rho \underline{V}_R \cdot \underline{n}\, dA \qquad \textit{(- inlet, + outlet)}$$

or for fixed CV, $\underline{V}_R = \underline{V}$, $\underline{V}_S = 0$

4) Steady Flow: $\dfrac{\partial}{\partial t} = 0$

<u>Continuity Equation</u>:

B = M = mass of system
β = 1

$\dfrac{dM}{dt} = 0$ by definition, system = fixed amount of mass

Integral Form:

$$\frac{dM}{dt} = 0 = \frac{d}{dt}\int_{CV} \rho\, d\forall + \int_{CS} \rho \underline{V}_R \cdot \underline{n}\, dA$$

$$-\frac{d}{dt}\int_{CV} \rho\, d\forall = \int_{CS} \rho \underline{V}_R \cdot \underline{n}\, dA$$

*Rate of decrease of mass in CV = net rate of mass outflow across CS*

Note simplifications for 1) non-deforming and fixed CV ($\forall \neq \forall$ (t), $\underline{V}_S = 0$), 2) uniform flow across discrete CS ($\int = \sum$), 3) steady flow ($\frac{\partial}{\partial t} = 0$), and 4) incompressible fluid (ρ = constant ⇒ $-\frac{d}{dt}\int_{CV} d\forall = \int_{CS} \underline{V}_R \cdot \underline{n}\, dA$ : "conservation of volume")

1) Non-deforming and fixed CV

$$\int_{CV} \frac{\partial \rho}{\partial t} d\forall + \int_{CS} \rho \underline{V} \cdot \underline{n}\, dA = 0$$

2) and uniform flow over discrete inlet/outlet

$$\int_{CV} \frac{\partial \rho}{\partial t} d\forall + \sum \rho \underline{V} \cdot \underline{n} A = 0$$

3) and steady flow

$$\sum \rho \underline{V} \cdot \underline{n} A = 0$$

or

$$-\sum \left(\rho V A\right)_{in} + \sum \left(\rho V A\right)_{out} = 0$$

$$\rho Q = \dot{m} \Rightarrow \sum \left(\dot{m}\right)_{in} = \sum \left(\dot{m}\right)_{out}$$

4) and incompressible flow

$$-\sum Q_{in} + \sum Q_{out} = 0$$

if non-uniform flow over discrete inlet/outlet

$$Q_{CS_i} = \int_{CS} \underline{V} \cdot \underline{n}\, dA = \left(V_{av} A\right)_{CS_i} \qquad V_{av} = \frac{1}{A}\int_{CS} \underline{V} \cdot \underline{n}\, dA$$

Differential Form:

$$\frac{dM}{dt} = 0 = \int_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla . \left(\rho \underline{V}\right)\right] d\forall$$

$$\beta = 1$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \underline{V}\right) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \underline{V} + \underline{V} \cdot \nabla \rho = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{V} = 0$$

$$M = \rho \forall \Rightarrow \qquad dM = \rho\, d\forall + \forall\, d\rho = 0 \Rightarrow \; -\frac{d\forall}{\forall} = \frac{d\rho}{\rho}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{1}{\forall}\frac{D\forall}{Dt}$$

$$\underbrace{\frac{1}{\rho}\frac{D\rho}{Dt}}_{\substack{rate\ of\ change\ \rho \\ per\ unit\ \rho}} + \underbrace{\nabla \cdot \underline{V}}_{} = 0$$

$$\underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\forall}\frac{D\forall}{Dt}}_{\substack{rate\ of\ change\ \forall \\ per\ unit\ \forall}}$$

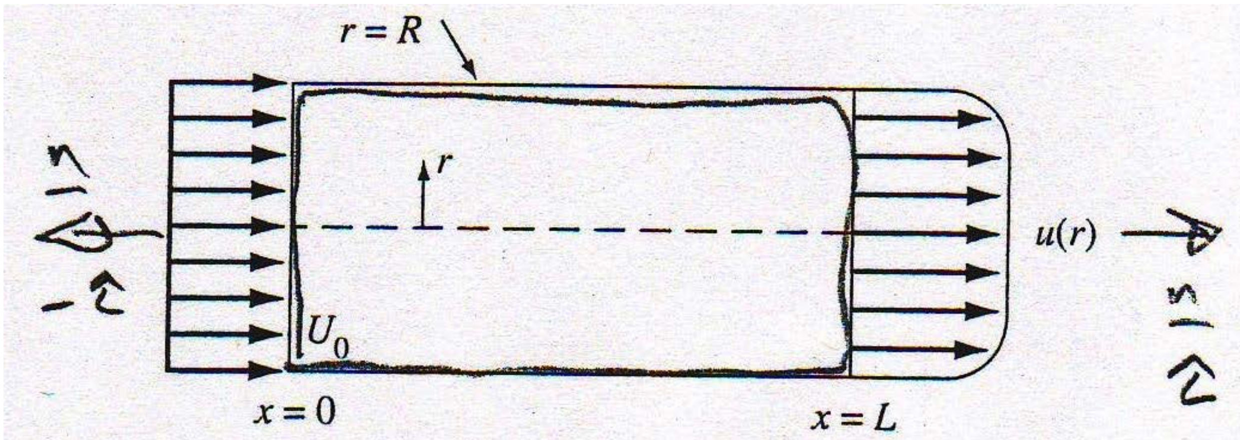
Called the continuity equation since the implication is that ρ and $\underline{V}$ are continuous functions of $\underline{x}$.

Incompressible Fluid: ρ = constant

$$\nabla \cdot \underline{V} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

P3.15 Water, assumed incompressible, flows steadily through the round pipe in Fig. P3.15. The entrance velocity is constant, $u = U_0$ , and the exit velocity approximates turbulent flow, $u = u_{\max}\left(1 - r/R\right)^{1/7}$. Determine the ratio $U_0/u_{max}$ for this flow.




P3.15

Steady flow, non-deforming, fixed CV, one inlet uniform flow and one outlet non-uniform flow

$$-\dot{m}_{in} + \dot{m}_{out} = 0; \;\; \rho = constant; \;\; -Q_{in} + Q_{out} = 0$$

$$0 = -U_0 \pi R^2 + \int_0^R u_{\max}\left(1 - r/R\right)^{1/7} 2\pi r dr$$

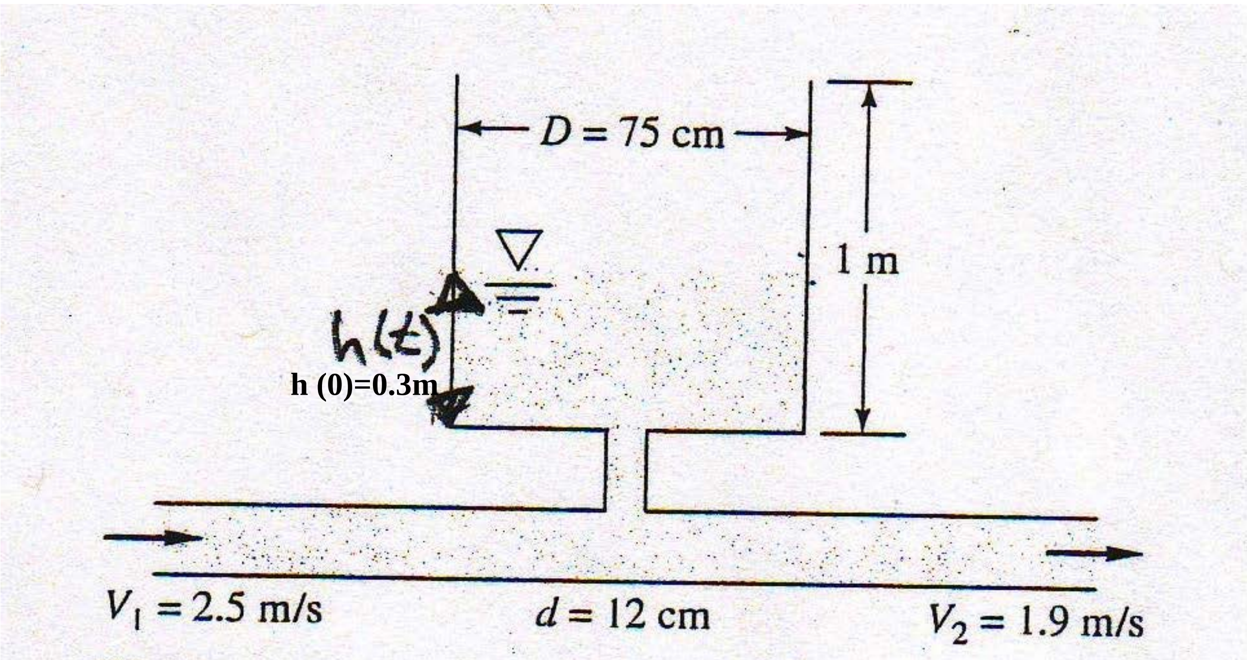
$$0 = -U_0 \pi R^2 + u_{\max}\frac{49\pi}{60}R^2$$

$$\frac{U_0}{u_{\max}} = \frac{49}{60}$$

$$2\pi u_{\max}\int_0^R \left(1 - \frac{r}{R}\right)^{1/7} r dr = 2\pi u_{\max}\left[\left[\frac{1}{R^{-2}\left(\frac{1}{7}+2\right)}\left(1 - r/R\right)^{15/7} - \frac{1}{R^{-2}\left(\frac{1}{7}+1\right)}\left(1 - r/R\right)^{8/7}\right]\right]_0^R$$

$$= 2\pi u_{\max} R^2\left[0 - \left(\frac{7}{15} - \frac{7}{8}\right)\right] = \pi u_{\max} R^2 \frac{49}{60}$$

P3.12 The pipe flow in Fig. P3.12 fills a cylindrical tank as shown. At time t=0, the water depth in the tank is 30cm. Estimate the time required to fill the remainder of the tank.




P3.12

Unsteady flow, deforming CV, one inlet one outlet uniform flow

$$0 = \frac{d}{dt}\int_{CV} \rho d\forall - \rho Q_1 + \rho Q_2$$

$$0 = \frac{d}{dt}\int_{CV} \rho d\forall - \rho V_1 \frac{\pi d^2}{4} + \rho V_2 \frac{\pi d^2}{4}$$

$$\forall(t) = h(t)\frac{\pi D^2}{4}$$

$$0 = \frac{\rho \pi D^2}{4}\frac{dh}{dt} + \rho \frac{\pi d^2}{4}(V_2 - V_1)$$

$$\frac{dh}{dt} = \left(\frac{d}{D}\right)^2 (V_1 - V_2) = 0.0153$$

$$dt = \frac{dh}{0.0153} = \frac{0.7}{0.0153} = 46s$$

Steady flow, fixed CV with one inlet and two exits with uniform flow

Note: $Q = \int_A \underline{V} \cdot \underline{n}\, dA = \frac{\forall}{dt} \quad \frac{L^3}{s}$

$$0 = -Q_1 + Q_2 + Q_3$$

$$Q_3 = \frac{\forall}{dt} = Q_1 - Q_2 = \frac{\pi d^2}{4}(V_1 - V_2)$$

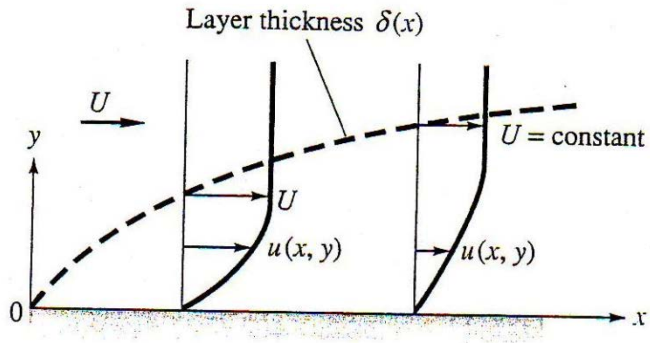
$$dt = \frac{\forall}{Q_3} = \frac{dh\,\frac{\pi D^2}{4}}{\frac{\pi d^2}{4}(V_1 - V_2)}$$

$$= \frac{dh\left(\frac{D}{d}\right)^2}{(V_1 - V_2)}$$

P4.17 A reasonable approximation for the two-dimensional incompressible laminar boundary layer on the flat surface in Fig.P4.17 is $u = U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)$ for $y \le \delta$, where $\delta = Cx^{1/2}$, $C = const$

(a) Assuming a no-slip condition at the wall, find an expression for the velocity component $v(x, y)$ for $y \le \delta$.

(b) Find the maximum value of $v$ at the station $x = 1m$, for the particular case of flow, when $U = 3m/s$ and $\delta = 1.1cm$.




**P4.17**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -U\left(-2y\delta^{-2} + 2y^2\delta^{-3}\right)\frac{\partial \delta}{\partial x}$$

$$v = 2U\delta_x \int_0^y \left(y\delta^{-2} - y^2\delta^{-3}\right)dy$$

(a) $v = 2U\delta_x\left(\frac{y^2}{2\delta^2} - \frac{y^3}{3\delta^3}\right)$ $\quad \delta = Cx^{1/2} \quad \delta_x = \frac{C}{2}x^{-1/2} = \frac{\delta}{2x}$

(b) Since $v_y = 0$ at $y = \delta$

$$v_{\max} = v(y = \delta) = \frac{2U\delta}{2x}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{U\delta}{6x} = \frac{3 \times 0.011}{6} = 0.0055\,m/s$$

## Momentum Equation:

B = M$\underline{V}$ = momentum, β = $\underline{V}$

Integral Form:

$$\frac{d(M\underline{V})}{dt} = \underbrace{\frac{d}{dt}\int_{CV}\underline{V}\rho\, d\forall}_{1} + \underbrace{\int_{CS}\underline{V}\rho\underline{V}_R \cdot \underline{n}\, dA}_{2} = \underbrace{\sum \underline{F}}_{3}$$

$\sum \underline{F}$ = vector sum of all forces acting on CV
= $\underline{F}_B + \underline{F}_s$

$\underline{F}_B$ = Body forces, which act on entire CV of fluid due to external force field such as gravity or electrostatic or magnetic forces. Force per unit volume.

$\underline{F}_s$ = Surface forces, which act on entire CS due to normal (pressure and viscous stress) and tangential (viscous stresses) stresses. Force per unit area.

When CS cuts through solids $\underline{F}_s$ may also include $\underline{F}_R$ = reaction forces, e.g., reaction force required to hold nozzle or bend when CS cuts through bolts holding nozzle/bend in place.

1 = rate of change of momentum in CV
2 = rate of outflux of momentum across CS
3 = vector sum of all body forces acting on entire CV and surface forces acting on entire CS.

Many interesting applications of CV form of momentum equation: vanes, nozzles, bends, rockets, forces on bodies, water hammer, etc.

Differential Form:

$$\int_{CV}\left[\frac{\partial}{\partial t}\left(\underline{V}\rho\right)+\nabla\cdot\left(\underline{V}\rho\underline{V}\right)\right]d\forall=\sum\underline{F}$$

Where $\frac{\partial}{\partial t}\left(\underline{V}\rho\right)=\underline{V}\frac{\partial\rho}{\partial t}+\rho\frac{\partial\underline{V}}{\partial t}$

and $\underline{V}\rho\underline{V}=\rho\underline{V}\underline{V}=\rho u\hat{i}\,\underline{V}+\rho v\hat{j}\,\underline{V}+\rho w\hat{k}\,\underline{V}$ is a tensor

$$\nabla\cdot(\underline{V}\rho\underline{V})=\nabla\cdot(\rho\underline{V}\underline{V})=\frac{\partial}{\partial x}(\rho u\underline{V})+\frac{\partial}{\partial y}(\rho v\underline{V})+\frac{\partial}{\partial z}(\rho w\underline{V})$$

$$=\underline{V}\nabla\cdot(\rho\underline{V})+\rho\underline{V}\cdot\nabla\underline{V}$$

$$\int_{CV}\left[\underline{V}\underbrace{\left(\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\underline{V}\right)\right)}_{=0\text{ , continuity}}+\rho\left(\frac{\partial\underline{V}}{\partial t}+\underline{V}\cdot\nabla\underline{V}\right)\right]d\forall=\sum\underline{F}$$

Since $\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} = \frac{D\underline{V}}{Dt}$

$$\int_{CV} \rho \frac{D\underline{V}}{Dt} d\forall = \sum \underline{F}$$

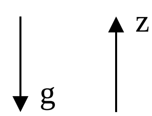
$$\rho \frac{D\underline{V}}{Dt} = \sum \underline{f} \qquad \textit{per elemental fluid volume}$$

$$\rho \underline{a} = \underline{f}_b + \underline{f}_s$$

$\underline{f}_b$ = body force per unit volume
$\underline{f}_s$ = surface force per unit volume

<u>Body forces</u> are due to external fields such as gravity or magnetic fields. Here we only consider a gravitational field; that is,

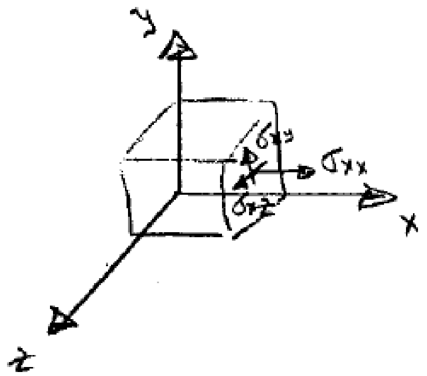
$$\sum \underline{F}_{body} = d\underline{F}_{grav} = \rho \underline{g}\, dxdydz$$

and $\underline{g} = -g\hat{k}$ for 


i.e. $\underline{f}_{body} = -\rho g \hat{k}$

<u>Surface Forces</u> are due to the stresses that act on the sides of the control surfaces

$$\sigma_{ij} = \underbrace{-p\delta_{ij}}_{\text{Normal pressure}} + \underbrace{\tau_{ij}}_{\text{Viscous stress}}$$




$$= \begin{bmatrix} -p+\tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & -p+\tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -p+\tau_{zz} \end{bmatrix}$$

Symmetric $\sigma_{ij} = \sigma_{ji}$

2nd order tensor

As shown before, for p alone it is not the stresses themselves that cause a net force but their gradients.

Symmetry condition from requirement that for elemental fluid volume, stresses themselves cause no rotation.

$$\underline{f_s} = \underline{f_p} + \underline{f_\tau}$$

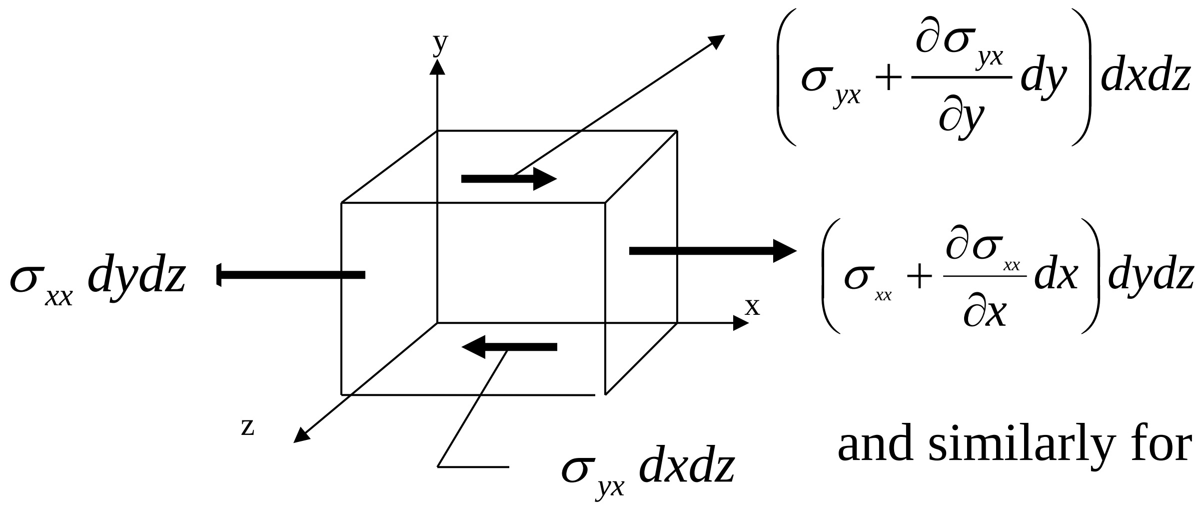
Recall $\underline{f_p} = -\nabla p$ based on 1st order TS. $\underline{f_\tau}$ is more complex since $\tau_{ij}$ is a 2nd order tensor, but similarly as for p, the force is due to stress gradients and are derived based on 1st order TS.

$$\underline{\sigma_x} = \sigma_{xx}\hat{i} + \sigma_{xy}\hat{j} + \sigma_{xz}\hat{k}$$

$$\underline{\sigma_y} = \sigma_{yx}\hat{i} + \sigma_{yy}\hat{j} + \sigma_{yz}\hat{k}$$

$$\underline{\sigma_z} = \sigma_{zx}\hat{i} + \sigma_{zy}\hat{j} + \sigma_{zz}\hat{k}$$

Resultant stress on each face




and similarly for z

$$\underline{F_s} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dxdydz\,\hat{\imath}$$
$$+ \left[\frac{\partial}{\partial x}(\sigma_{xy}) + \frac{\partial}{\partial y}(\sigma_{yy}) + \frac{\partial}{\partial z}(\sigma_{zy})\right] dxdydz\,\hat{\jmath}$$
$$+ \left[\frac{\partial}{\partial x}(\sigma_{xz}) + \frac{\partial}{\partial y}(\sigma_{yz}) + \frac{\partial}{\partial z}(\sigma_{zz})\right] dxdydz\,\hat{k}$$

$$\underline{F_s} = \left[\frac{\partial}{\partial x}(\underline{\sigma_x}) + \frac{\partial}{\partial y}(\underline{\sigma_y}) + \frac{\partial}{\partial z}(\underline{\sigma_z})\right] dxdydz$$

Divided by the volume:

$$\underline{f}_s = \frac{\partial}{\partial x}(\underline{\sigma_x}) + \frac{\partial}{\partial y}(\underline{\sigma_y}) + \frac{\partial}{\partial z}(\underline{\sigma_z})$$

$$\underline{f}_s = \nabla \cdot \sigma_{ij} = \frac{\partial}{\partial x_j}\sigma_{ij}$$

Putting together the above results,

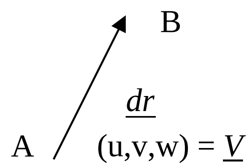
$$\rho \underline{a} = \rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} + \nabla \cdot \sigma_{ij}$$

Inertial force

body force due to gravity

surface force = p + viscous terms (due to stress gradients)

Next, we need to relate the stresses $\sigma_{ij}$ to the fluid motion, i.e. the velocity field. To this end, we examine the relative motion between two neighboring fluid particles.




@ B: $\underline{V} + \underline{dV} = \underline{V} + \underline{dr} \cdot \nabla \underline{V}$ 1st order Taylor Series

$$\underline{dV} = \underline{dr} \cdot \nabla \underline{V} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = e_{ij}\, dx_j$$

relative motion

deformation rate tensor = $e_{ij}$

$$e_{ij} = \frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\substack{symmetric\ part \\ \varepsilon_{ij}=\varepsilon_{ji}}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\substack{anit-symmetric\ part \\ \omega_{ij}=-\omega_{ji}}} = \varepsilon_{ij} + \omega_{ij}$$

$$\omega_{ij} = \begin{bmatrix} 0 & \frac{1}{2}(u_y - v_x) & \overbrace{\frac{1}{2}(u_z - w_x)}^{\eta} \\ \underbrace{\frac{1}{2}(v_x - u_y)}_{\zeta} & 0 & \frac{1}{2}(v_z - w_y) \\ \frac{1}{2}(w_x - u_z) & \underbrace{\frac{1}{2}(w_y - v_z)}_{\xi} & 0 \end{bmatrix} = \begin{matrix} rigid\ body\ rotation \\ of\ \ fluid\ element \end{matrix}$$

*where* $\xi$ = *rotation about x axis*
$\eta$ = *rotation about y axis*
$\varsigma$ = *rotation about z axis*

Note that the components of $\omega_{ij}$ are related to the <u>vorticity vector define by</u>:

$$\underline{\omega} = \nabla \times \underline{V} = \underbrace{(w_y - v_z)}_{2\xi}\hat{i} + \underbrace{(u_z - w_x)}_{2\eta}\hat{j} + \underbrace{(v_x - u_y)}_{2\zeta}\hat{k}$$

$$= 2 \times \text{angular velocity of fluid element}$$

$$\varepsilon_{ij} = rate\ of\ strain\ tensor$$

$$= \begin{bmatrix} u_x & \frac{1}{2}(u_y + v_x) & \frac{1}{2}(u_z + w_x) \\ \frac{1}{2}(v_x + u_y) & v_y & \frac{1}{2}(v_z + w_y) \\ \frac{1}{2}(w_x + u_z) & \frac{1}{2}(w_y + v_z) & w_z \end{bmatrix}$$

$$u_x + v_y + w_z = \nabla \cdot \underline{V} = \textit{elongation (or volumetric dilatation)}$$

$$\textit{of fluid element} = \frac{1}{\forall}\frac{D\forall}{Dt}$$

$\frac{1}{2}(u_y + v_x)$ = distortion wrt (x,y) plane

$\frac{1}{2}(u_z + w_x)$ = distortion wrt (x,z) plane

$\frac{1}{2}(v_z + w_y)$ = distortion wrt (y,z) plane

Thus, general motion consists of:

1) pure translation described by $\underline{V}$
2) rigid-body rotation described by $\underline{\omega}$
3) volumetric dilatation described by $\nabla \cdot \underline{V}$
4) distortion in shape described by $\varepsilon_{ij}$ $\quad i \neq j$

It is now necessary to make certain postulates concerning the relationship between the fluid stress tensor ($\sigma_{ij}$) and rate-of-deformation tensor ($e_{ij}$). These postulates are based on physical reasoning and experimental observations and have been verified experimentally even for extreme conditions. For a Newtonian fluid:

1) When the fluid is at rest the stress is hydrostatic and the pressure is the thermodynamic pressure

2) Since there is no shearing action in rigid body rotation, it causes no shear stress.

3) $\tau_{ij}$ is linearly related to $\varepsilon_{ij}$ and only depends on $\varepsilon_{ij}$.

4) There is no preferred direction in the fluid, so that the fluid properties are point functions (condition of isotropy).

Using statements 1-3

$$\sigma_{ij} = -p\delta_{ij} + k_{ijmn}\varepsilon_{ij}$$

$k_{ijmn}$ = 4th order tensor with 81 components such that each stress is linearly related to all nine components of $\varepsilon_{ij}$.

However, statement (4) requires that the fluid has no directional preference, i.e. $\sigma_{ij}$ is independent of rotation of coordinate system, which means $k_{ijmn}$ is an isotropic tensor = even order tensor made up of products of $\delta_{ij}$.

$$k_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm}$$

$$(\lambda, \mu, \gamma) = scalars$$

Lastly, the symmetry condition $\sigma_{ij} = \sigma_{ji}$ requires:

$$k_{ijmn} = k_{jimn} \quad \rightarrow \quad \gamma = \mu = \text{viscosity}$$

$$\sigma_{ij} = -p\delta_{ij} + \mu\delta_{im}\delta_{jn}\varepsilon_{ij} + \mu\delta_{in}\delta_{jm}\varepsilon_{ij} + \lambda\delta_{ij}\delta_{mn}\varepsilon_{ij}$$

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\varepsilon_{ij} + \lambda \underbrace{\varepsilon_{mm}}_{\nabla\cdot\underline{V}}\delta_{ij}$$

λ and μ can be further related if one considers mean normal stress vs. thermodynamic p.

$$\sigma_{ii} = -3p + (2\mu + 3\lambda)\nabla \cdot \underline{V}$$

$$p = \underbrace{-\frac{1}{3}\sigma_{ii}}_{\bar{p} = \text{mean normal stress}} + \left(\frac{2}{3}\mu + \lambda\right)\nabla \cdot \underline{V}$$

$$p - \bar{p} = \left(\frac{2}{3}\mu + \lambda\right)\nabla \cdot \underline{V}$$

Incompressible flow: $p = \bar{p}$ and absolute pressure is indeterminant since there is no equation of state for p. Equations of motion determine $\nabla p$.

Compressible flow: $p \neq \bar{p}$ and λ = bulk viscosity must be determined; however, it is a very difficult measurement requiring large $\nabla \cdot \underline{V} = -\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\forall}\frac{D\forall}{Dt}$, e.g., within shock waves.

Stokes Hypothesis also supported kinetic theory monotonic gas.

$$\lambda = -\tfrac{2}{3}\mu$$

$$p = \bar{p}$$

$$\sigma_{ij} = -\left( p + \frac{2}{3}\mu \nabla \cdot \underline{V} \right)\delta_{ij} + 2\mu\varepsilon_{ij}$$

Generalization $\tau = \mu \dfrac{du}{dy}$ for 3D flow.

$$\sigma_{ij} = \mu\left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad i \neq j \qquad \textit{relates shear stress to strain rate}$$

$$\sigma_{ii} = -p - \frac{2}{3}\mu\nabla \cdot \underline{V} + 2\mu\left( \frac{\partial u_i}{\partial x_i} \right) = -p + \underbrace{2\mu\left[ -\frac{1}{3}\nabla \cdot \underline{V} + \frac{\partial u_i}{\partial x_i} \right]}_{\textit{normal viscous stress}}$$

Where the normal viscous stress is the difference between the extension rate in the $x_i$ direction and average expansion at a point. Only differences from the average = $\frac{1}{3}\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$ generate normal viscous stresses. For incompressible fluids, average = 0 i.e. $\nabla \cdot \underline{V} = 0$.

Non-Newtonian fluids:

$\tau_{ij} \propto \varepsilon_{ij}$ for small strain rates $\dot{\theta}$, which works well for air, water, etc. Newtonian fluids

$$\tau_{ij} \propto \underbrace{\varepsilon_{ij}^n}_{\textit{non–linear}} + \underbrace{\frac{\partial}{\partial t}\varepsilon_{ij}}_{\textit{history effect}}$$

Non-Newtonian

Viscoeslastic materials

Non-Newtonian fluids include:

(1) Polymer molecules with large molecular weights and form long chains coiled together in spongy ball shapes that deform under shear.

(2) Emulsions and slurries containing suspended particles such as blood and water/clay

Navier Stokes Equations:

$$\rho \underline{a} = \rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} + \nabla \cdot \sigma_{ij}$$

$$\rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} - \nabla p + \frac{\partial}{\partial x_j}\left[2\mu\varepsilon_{ij} - \frac{2}{3}\mu \nabla \cdot \underline{V}\ \delta_{ij}\right]$$

Recall μ = μ(T) μ increases with T for gases, decreases with T for liquids, but if it is assumed that μ = constant:

$$\rho \frac{D\underline{V}}{Dt} = -\rho g \hat{k} - \nabla p + 2\mu \frac{\partial}{\partial x_j}\varepsilon_{ij} - \frac{2}{3}\mu \frac{\partial}{\partial x_j}\nabla \cdot \underline{V}$$

$$2\frac{\partial}{\partial x_j}\varepsilon_{ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \nabla^2 u_i = \nabla^2 \underline{V}$$

$$\rho \frac{D\underline{V}}{Dt} = -\rho g\,\hat{k} - \nabla p + \mu\left[\nabla^2 \underline{V} - \frac{2}{3}\frac{\partial}{\partial x_j}\nabla\cdot\underline{V}\right]$$

For incompressible flow $\nabla\cdot\underline{V} = 0$

$$\rho\frac{D\underline{V}}{Dt} = \underbrace{-\rho g\hat{k} - \nabla p}_{\substack{-\nabla\hat{p} \text{ where } \hat{p}=p+\gamma z \\ \text{piezometric pressure}}} + \mu\nabla^2\underline{V}$$

For μ = 0

$$\rho\frac{D\underline{V}}{Dt} = -\rho g\,\hat{k} - \nabla p \qquad \text{Euler Equation}$$

NS equations for ρ, μ constant

$$\rho\frac{D\underline{V}}{Dt} = -\nabla\hat{p} + \mu\nabla^2\underline{V}$$

$$\rho\left[\frac{\partial\underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right] = -\nabla\hat{p} + \mu\nabla^2\underline{V}$$

$$\left[\frac{\partial\underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right] = -\frac{1}{\rho}\nabla\hat{p} + \nu\nabla^2\underline{V} \qquad \nu = \frac{\mu}{\rho} \text{ kinematic viscosity}$$

*Non-linear 2nd order PDE, as is the case for ρ, μ not constant*

Combine with $\nabla\cdot\underline{V}$ for 4 equations for 4 unknowns $\underline{V}$, p and can be, albeit difficult, solved subject to initial and boundary conditions for $\underline{V}$, p at t = $t_0$ and on all boundaries i.e. “well posed” IBVP.

Application of CV Momentum Equation:

$$\underbrace{\sum \underline{F}}_{net\ force\ on\ CV} = \underbrace{\frac{d}{dt}\int_{CV} \underline{V}\rho\, d\forall}_{\substack{time\ rate\ of\ change \\ of\ momentum\ in\ CV}} + \underbrace{\int_{CS} \underline{V}\rho \underline{V}_R . n\, dA}_{\substack{net\ momentum \\ outflux}}$$

$\underline{F} = \underline{F}_B + \underline{F}_S$ ($\underline{F}_S$ includes reaction forces)

Note:

1. Vector equation

2. $\underline{n}$ = outward unit normal: $\underline{V}_R \cdot \underline{n}$ < 0 inlet, > 0 outlet

3. 1D Momentum flux, fixed CV

$$\int_{CS} \underline{V}\rho\underline{V}\cdot\underline{n}\, dA = \sum\left(\dot{m}_i \underline{V}_i\right)_{out} - \sum\left(\dot{m}_i \underline{V}_i\right)_{in}$$

Where $\underline{V}_i$, $\rho_i$ are assumed uniform over fixed discrete inlets and outlets

$$\dot{m}_i = \rho_i V_{ni} A_i$$

$$\sum \underline{F} = \frac{d}{dt}\int_{CV} \underline{V}\rho\, d\forall + \underbrace{\sum\left(\dot{m}_i\ \underline{V_i}\right)_{out}}_{\substack{outlet\ momentum \\ flux}} - \underbrace{\sum\left(\dot{m}_i\ \underline{V_i}\right)_{in}}_{inlet\ momentum\ flux}$$

## 4. Momentum flux correlation factors

$$\int u\rho \underline{V}.\underline{n}\, dA = \underbrace{\rho\int u^2\, dA}_{\substack{axial\ flow\ with\\ non-uniform\\ velocity\ profile}} = \rho\beta A V_{av}^2 = \dot{m}\beta V_{av}$$

Where $\beta = \dfrac{1}{A}\displaystyle\int_{CS}\left(\frac{u}{V_{av}}\right)^2 dA$

$$V_{av} = \frac{1}{A}\int_{CS} u\, dA = Q/A$$

Laminar pipe flow:

$$u = U_0\left(1-\frac{r^2}{R^2}\right) \approx U_0\left(1-\frac{r}{R}\right)^{\frac{1}{2}}$$

$$V_{av} = .53U_0 \qquad \beta = \frac{4}{3} = 1.33$$

Turbulent pipe flow:

$$u = U_0\left(1-\frac{r}{R}\right)^m \qquad 1/9 \le m \le 1/5$$

$$V_{av} = U_0\frac{2}{(1+m)(2+m)}: \text{ for } m = 1/7, \quad V_{av} = .82U_0$$

$$\beta = \frac{(1+m)^2(2+m)^2}{2(1+2m)(2+2m)}: \text{ for } m=1/7, \quad \beta = 1.02$$

5. Constant p causes no force; Therefore,

Use $p_{gage} = p_{atm} - p_{absolute}$

$$\underline{F}_p = -\int_{CS} p\underline{n}\, dA = -\int_{CV} \nabla p\, d\forall = 0 \quad \text{for p = constant}$$

6. For jets open to atmosphere: $p = p_a$, i.e. $p_{gage} = 0$.

7. Choose CV carefully with CS normal to flow (if possible) and indicating coordinate system and $\sum \underline{F}$ on CV similar as free body diagram used in dynamics.

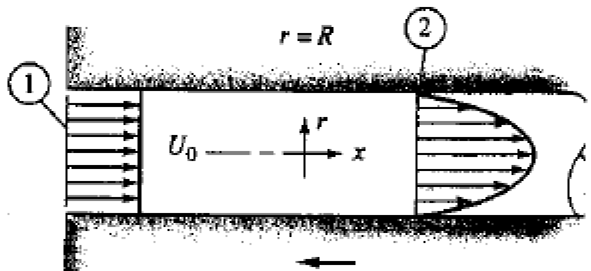
8. Many applications, usually with continuity and energy equations. Careful practice is needed for mastery.
   a. Steady and unsteady developing and fully developed pipe flow
   b. Emptying or filling tanks
   c. Forces on transitions
   d. Forces on fixed and moving vanes
   e. Hydraulic jump
   f. Boundary Layer and bluff body drag
   g. Rocket or jet propulsion
   h. Nozzle
   i. Propeller
   j. Water-hammer

**P3.53** Consider incompressible flow in the entrance of a circular tube, as in Fig. P3.53. The inlet flow is uniform, $u_1 = U_0$. The flow at section 2 is developed pipe flow.

Find the wall drag force $F$ as a function of ($p_1$, $p_2$, $\rho$ $U_0$, $R$) if the flow at section 2 is

(*a*) Laminar: $u_2 = u_{max}\left(1 - \frac{r^2}{R^2}\right) \approx u_{max}\left(1 - r/R\right)^{1/2}$

(*b*) Turbulent: $u_2 \approx u_{max}\left(1 - \frac{r}{R}\right)^{1/7}$




**P3.53** Friction drag on fluid

First relate $u_{max}$ to $U_0$ using continuity equation

$$-Q_{in} + Q_{out} = 0 \Rightarrow \quad Q_{in} = Q_{out} = Q \quad \Rightarrow V_{av,in} = V_{av,out}; \quad V_{av} = \frac{Q}{A}$$

$$U_0 \pi R^2 = \int_0^R u_{max}\left(1 - r/R\right)^m 2\pi r\, dr$$

$$U_0 = \frac{1}{\pi R^2}\int_0^R u_{max}\left(1 - r/R\right)^m 2\pi r\, dr = V_{av}$$

$$V_{av} = u_{max}\frac{2}{(1+m)(2+m)}$$

m = 1/2 $\quad V_{av} = .53u_{max} \quad \rightarrow \quad u_{max} = V_{av}/.53$

m = 1/7 $\quad V_{av} = .82u_{max} \quad \rightarrow \quad u_{max} = V_{av}/.82$

Second, calculate F using momentum equation:

F = wall drag force = $\tau_w 2\pi R dx$ (force fluid on wall)
-F = force wall on fluid

$$\sum F_x = (p_1 - p_2)\pi R^2 - F = \int_0^R u_2(\rho u_2 2\pi r\, dr) - U_0(\rho \pi R^2 U_0)$$

$$F = (p_1 - p_2)\pi R^2 + \rho U_0^2 \pi R^2 - \underbrace{\int_0^R \rho u_2^2 2\pi r\, dr}_{\beta \rho A V_{av}^2}$$

= $U_0^2$ from continuity

$$F = (p_1 - p_2)\pi R^2 + \underbrace{\rho U_0^2 \pi R^2 - \beta_2 \rho A V_{av}^2}_{\rho U_0^2 \pi R^2 (1-\beta_2)}$$

$$\underbrace{\beta = \frac{1}{A}\int \left(\frac{u}{V_{av}}\right)^2 dA}_{momentul\ flux\ correction\ factor}$$

= 4/3 laminar flow
= 1.02 turbulent flow

$$F_{lam} = (p_1 - p_2)\pi R^2 - \frac{1}{3}\rho U_0^2 \pi R^2$$

$$F_{turb} = (p_1 - p_2)\pi R^2 - .02\rho U_0^2 \pi R^2$$

Complete analysis using CFD!

Reconsider the problem for fully developed flow:

Continuity:

$$-\dot{m}_{in} + \dot{m}_{out} = 0$$
$$\dot{m} = \dot{m}_{in} = \dot{m}_{out}$$

or Q = constant

Momentum:

$$\sum F_x = (p_1 - p_2)\pi R^2 - F = \rho \int_{in} u(\underline{V} \cdot \underline{n})\, dA + \rho \int_{out} u(\underline{V} \cdot \underline{n})\, dA$$
$$= -\rho(\beta A V_{ave}^2)_{in} + \rho(\beta A V_{ave}^2)_{out}$$
$$= \rho Q V_{ave}(\beta_{out} - \beta_{in})$$
$$= 0$$

$$(p_1 - p_2)\pi R^2 - \tau_w 2\pi R \mathrm{dx} = 0$$

$$\Delta p \pi R^2 - \tau_w 2\pi R \mathrm{dx} = 0$$

Since $\Delta p = p_1 - p_2 = -dp = -(p_2 - p_1)$

$\tau_w = \dfrac{R}{2}\left(-\dfrac{dp}{dx}\right)$ or for smaller CV r < R, $\tau = \dfrac{r}{2}\left(-\dfrac{dp}{dx}\right)$

*(valid for laminar or turbulent flow, but assume laminar)*

$$\tau = \mu \frac{du}{dy} = -\mu \frac{du}{dr} = \frac{r}{2}\left(-\frac{dp}{dx}\right)$$

y = R-r (*wall coord.)*

$$\frac{du}{dr} = -\frac{r}{2\mu}\left(-\frac{dp}{dx}\right)$$

$$u = -\frac{r^2}{4\mu}\left(-\frac{dp}{dx}\right) + c$$

$$u(r = R) = 0 \quad \rightarrow \quad c = \frac{R^2}{4\mu}\left(-\frac{dp}{dx}\right)$$

$$u(r) = \frac{R^2 - r^2}{4\mu}\left(-\frac{dp}{dx}\right)$$ (If $\frac{dp}{dx} < 0$ flow moves from left to right)

$$u_{\max} = \frac{R^2}{4\mu}\left(-\frac{dp}{dx}\right) \qquad u(r) = u_{\max}\left(1 - \frac{r^2}{R^2}\right)$$

$$Q = \int_0^R u(r) 2\pi r \, dr = \frac{\pi R^4}{8\mu}\left(-\frac{dp}{dx}\right)$$

$$V_{ave} = \frac{Q}{A} = \frac{R^2}{8\mu}\left(-\frac{dp}{dx}\right) = u_{\max}/2$$

$$\tau_w = \frac{R}{2}\left(-\frac{dp}{dx}\right) = \frac{R}{2}\left(\frac{8\mu V_{ave}}{R^2}\right) = \frac{4\mu V_{ave}}{R}$$

$$f = \frac{8\tau_w}{\rho V_{ave}^2} = \frac{32\mu}{\rho R V_{ave}} = \frac{64\mu}{\rho V_{ave} D} = \frac{64}{\mathrm{Re}}$$

$$\mathrm{Re} = \frac{V_{ave} D}{\nu}$$

Piezometric head

$$\mathrm{h} = \mathrm{z} + \frac{\mathrm{p}}{\gamma}$$

For a horizontal pipe

$$\Delta p = \gamma \Delta h \,,\ \Delta z = 0$$

$$\frac{2\ dx\ \tau_w}{R} = -dp = \Delta p = \frac{2\ L\ \tau_w}{R} \,, \quad f = \frac{8\tau_w}{\rho V_{av}^2}$$

$$\Delta p = \frac{2L\rho V_{av}^2 f}{8R} = \frac{L\rho V_{av}^2 f}{2D}$$

Dividing by $\gamma$

$$\frac{\Delta p}{\gamma} = \frac{L\rho V_{av}^2 f}{2D\gamma} = f\,\frac{L}{D}\frac{V_{av}^2}{2g}$$

More generally

$$\Delta h = f\,\frac{L}{D}\frac{V_{av}^2}{2g}$$ Darcy–Weisbach equation

Exact solution of NS for laminar fully developed pipe flow

# Application of relative inertial coordinates for a moving but non-deforming control volume (CV)

The CV moves at a constant velocity $\underline{V}_{CS}$ with respect to the absolute inertial coordinates. If $\underline{V}_R$ represents the velocity in the relative inertial coordinates that move together with the CV, then:

$$\underline{V}_R = \underline{V} - \underline{V}_{CS}$$

Reynolds transport theorem for an arbitrary moving deforming CV:

$$\frac{dB_{SYS}}{dt} = \frac{d}{dt}\int_{CV} \beta\rho \, d\forall + \int_{CS} \beta\rho \underline{V}_R \cdot \underline{n} \; dA$$

For a non-deforming CV moving at constant velocity, RTT for incompressible flow:

$$\frac{dB_{syst}}{dt} = \rho\int_{CV} \frac{\partial\beta}{\partial t} d\forall + \rho\int_{CS} \beta\underline{V}_R \cdot \underline{n} dA$$

## 1. Conservation of mass

$B_{syst} = M$, and $\beta = 1$:

$$\frac{dM}{dt} = \rho\int_{CS} \underline{V}_R \cdot \underline{n} dA$$

For steady flow:

$$\boxed{\int_{CS} \underline{V}_R \cdot \underline{n} dA = 0}$$

# 2. Conservation of momentum

$$B_{syst} = M\left(\underline{V}_R + \underline{V}_{CS}\right) \text{ and } \beta = d\underline{B}_{syst}/dM = \underline{V}_R + \underline{V}_{CS}$$

$$\frac{d[M\left(\underline{V}_R + \underline{V}_{CS}\right)]}{dt} = \sum \underline{F} = \rho \int_{CV} \frac{\partial\left(\underline{V}_R + \underline{V}_{CS}\right)}{\partial t} d\forall + \rho \int_{CS} \left(\underline{V}_R + \underline{V}_{CS}\right)\underline{V}_R \cdot \underline{n} dA$$

For steady flow with the use of continuity:

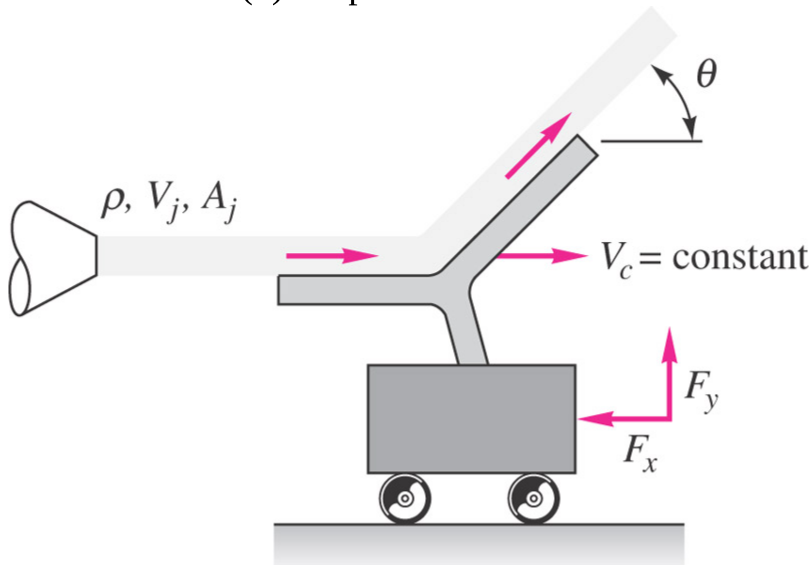
$$\sum \underline{F} = \rho \int_{CS} \left(\underline{V}_R + \underline{V}_{CS}\right)\underline{V}_R \cdot \underline{n} dA$$

$$= \rho \int_{CS} \underline{V}_R \underline{V}_R \cdot \underline{n} dA + \cancelto{0}{\rho \underline{V}_{CS} \int_{CS} \underline{V}_R \cdot \underline{n} dA}$$

$$\boxed{\sum \underline{F} = \rho \int_{CS} \underline{V}_R \underline{V}_R \cdot \underline{n} dA}$$

# Example (use relative inertial coordinates):

A jet strikes a vane which moves to the right at constant velocity $V_C$ on a frictionless cart. Compute (a) the force $F_x$ required to restrain the cart and (b) the power $P$ delivered to the cart. Also find the cart velocity for which (c) the force $F_x$ is a maximum and (d) the power $P$ is a maximum.




**Solution:**

Assume relative inertial coordinates with non-deforming CV i.e. CV moves at constant translational non-accelerating

$$\underline{V_{CS}} = u_{CS}\hat{\imath} + v_{CS}\hat{\jmath} + w_{CS}\hat{k} = V_C\hat{\imath}$$

then $\underline{V}_R = \underline{V} - \underline{V}_{CS}$ . Also assume steady flow $\underline{V} \neq \underline{V}(t)$ with $\rho = constant$ and neglect gravity effect.

Continuity:

$$0 = \rho \int_{CS} \underline{V_R} \cdot \underline{n} dA$$

$$-\rho V_{R1}A_1 + \rho V_{R2}A_2 = 0$$

$$V_{R1}A_1 = V_{R2}A_2 = \underbrace{(V_j - V_C)}_{V_{R1}=V_{R_x1}=V_j-V_C} A_j$$

Bernoulli without gravity:

$$\cancel{p_1}^{0} + \frac{1}{2}\rho V_{R1}^2 = \cancel{p_2}^{0} + \frac{1}{2}\rho V_{R2}^2$$

$$V_{R1} = V_{R2}$$

$$A_1 = A_2 = A_j$$

Momentum:

$$\sum \underline{F} = \rho \int_{CS} \underline{V_R}\, \underline{V_R} \cdot \underline{n} dA$$

$$\sum F_x = -F_x = \rho \int_{CS} V_{Rx} \underline{V_R} \cdot \underline{n} dA$$

$$-F_x = \rho V_{R_x 1}(-V_{R1} A_1) + \rho V_{R_x 2}(V_{R2} A_2)$$

$$-F_x = \rho(V_j - V_C)[-(V_j - V_C)A_j] + \rho(V_j - V_C)\cos\theta\,(V_j - V_C)A_j$$

$$F_x = \rho(V_j - V_C)^2 A_j[1 - \cos\theta]$$

$$Power = V_C F_x = V_C \rho(V_j - V_C)^2 A_j(1 - \cos\theta)$$

$$F_{x_{max}} = \rho V_j^2 A_j(1 - \cos\theta), \qquad V_C = 0$$

$$P_{max} \Rightarrow \frac{dP}{dV_C} = 0$$

$$P = V_C \rho(V_j^2 - 2V_C V_j + V_C^2)A_j(1 - \cos\theta)$$

$$= \rho(V_j^2 V_C - 2V_C^2 V_j + V_C^3)A_j(1 - \cos\theta)$$

$$\frac{dP}{dV_C} = \rho(V_j^2 - 4V_C V_j + 3V_C^2)A_j(1 - \cos\theta) = 0$$

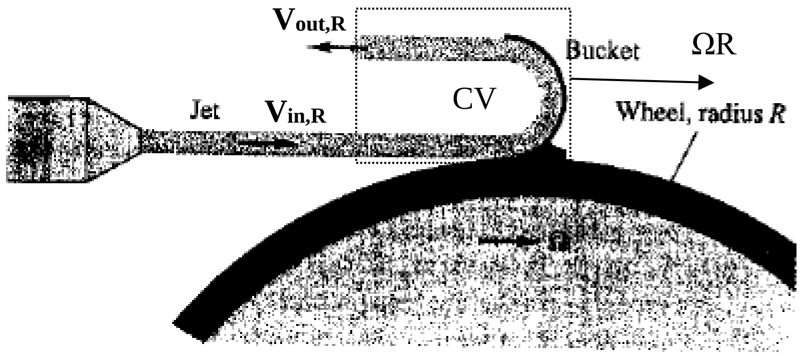
$$3V_C^2 - 4V_j V_C + V_j^2 = 0$$

$$V_C = \frac{+4V_j \pm \sqrt{16V_j^2 - 12V_j^2}}{6} = \frac{4V_j \pm 2V_j}{6}$$

For $V_C = \frac{V_j}{3}$: $P_{max} = \frac{V_j}{3}\rho\left(\frac{2V_j}{3}\right)^2 A_j(1 - \cos\theta) = \frac{4}{27}V_j^3 \rho A_j(1 - \cos\theta)$

*Example (use absolute inertial and relative inertial coordinates)*

**P3.51** A liquid jet of velocity $V_j$ and area $A_j$ strikes a single 180° bucket on a turbine wheel rotating at angular velocity $\Omega$,




**P3.51**

as in Fig. P3.51. Derive an expression for the power $P$ delivered to this wheel at this instant as a function of the system parameters. At what angular velocity is the maximum power delivered? How would your analysis differ if there were many, many buckets on the wheel, so that the jet was continually striking at least one bucket?

Assume gravity force is negligible and the cross section area of the jet does not change after striking the bucket. Taking moving CV at speed $\underline{V}_s = \Omega R\, \hat{i}$ enclosing jet and bucket:

*Solution 1 (relative inertial coordinates)*

Continuity: $-\dot{m}_{in,R} + \dot{m}_{out,R} = 0$

$$\dot{m}_R = \dot{m}_{in,R} = \dot{m}_{out,R} = \rho \int_{CS} \underline{V}_R \cdot \underline{n}\, dA$$

Bernoulli without gravity:

$$\cancel{p_1}^{0} + \frac{1}{2}\rho V_{in,R}^2 = \cancel{p_2}^{0} + \frac{1}{2}\rho V_{out,R}^2$$

$$V_{in,R} = V_{out,R}$$

Inlet $\underline{V_{in,R}} = (V_j - \Omega R)\hat{\imath}$

Outlet $\underline{V_{out,R}} = -(V_j - \Omega R)\hat{\imath}$

Since $-\rho V_{in,R} A_1 + \rho V_{out,R} A_2 = 0$

$$A_1 = A_2 = A_j$$

Momentum:

$$\sum F_X = -F_{bucket} = \dot{m}_R V_{out,R} - \dot{m}_R V_{in,R}$$

$$F_{bucket} = -\dot{m}_R\left[-(V_j - \Omega R) - (V_j - \Omega R)\right]$$

$$= 2\dot{m}_R (V_j - \Omega R)$$

$$= 2\rho A_j (V_j - \Omega R)^2$$

$$\dot{m}_R = \rho A_j (V_j - \Omega R)$$

$$P = \Omega R F_{bucket} = 2\rho A_j \Omega R (V_j - \Omega R)^2$$

$$\frac{dP}{d\Omega} = 2\rho A_j R (V_j - \Omega R)^2 - 2\rho A_j \Omega R 2 (V_j - \Omega R) R$$

$$= 2\rho A_j R\left[(V_j - \Omega R)^2 - 2R\Omega(V_j - \Omega R)\right]$$

$$= 2\rho A_j R (V_j - \Omega R)\left[V_j - \Omega R - 2R\Omega\right]$$

$$\frac{dP}{d\Omega} = 0 \;\rightarrow\; V_j - 3\Omega R = 0 \;\rightarrow\; \frac{V_j}{3} = \Omega R$$

$$P_{max} = 2\rho A_j \frac{V_j}{3}\left(V_j - \frac{V_j}{3}\right)^2 = 2\rho A_j \frac{V_j}{3}\frac{4V_j^2}{9} = \underbrace{\frac{8}{27}}_{0.296} \rho A_j V_j^3$$

If infinite number of buckets: $\dot{m}_R = \rho A_j V_j$

$$F_{bucket} = 2\rho A_j V_j (V_j - \Omega R)$$

all jet mass flow result in work.

$$P = 2\rho A_j V_j \Omega R (V_j - \Omega R)$$

$$\frac{dP}{d\Omega} = 0 \quad for \quad \Omega R = \frac{V_j}{2} \qquad P_{\max} = \frac{1}{2}\rho A_j V_j^3$$

*Solution 2 (absolute inertial coordinates)*

$$\underline{V}_R = \underline{V} - \underline{V}_{CS} \quad \rightarrow \quad \underline{V} = \underline{V}_R + \underline{V}_{CS}$$

$$\underline{V}_{in} = V_j \, \hat{\imath}$$

$$\underline{V}_{out} = -\left(V_j - \Omega R\right) \hat{\imath} + \Omega R \, \hat{\imath} = -\left(V_j - 2\Omega R\right) \hat{\imath}$$

Continuity: from solution 1

$$-V_{in,R} + V_{out,R} = 0$$

express in the absolute inertial coordinates: $\underline{V}_R = \underline{V} - \underline{V}_{CS}$

$$-\left(V_j - \Omega R\right) \hat{\imath} + \left(V_j + 2\Omega R - \Omega R\right) \hat{\imath} = 0$$

Momentum:

$$\sum F_x = -F_{bucket} = \dot{m}(V_{out} - V_{in})$$

$$= \rho A_j \left(V_j - \Omega R\right)\left[-\left(V_j - 2\Omega R\right) - V_j\right]$$

$$F_{bucket} = 2\rho A_j \left(V_j - \Omega R\right)^2$$

Same as Solution 1.

Application of CV continuity equation for steady incompressible flow, fixed CV, one inlet and outlet with A = constant

$$\rho \int_{in} \underline{V} \cdot \underline{n} dA = \rho \int_{out} \underline{V} \cdot \underline{n} dA = \dot{m} = \rho Q$$

$$Q_{in} = Q_{out}$$

$$\left(V_{ave} A\right)_{in} = \left(V_{ave} A\right)_{out}$$

For A = constant $\left(V_{ave}\right)_{in} = \left(V_{ave}\right)_{out}$

$$\sum \underline{F} = \rho \int_{in} \underline{V}\left(\underline{V} \cdot \underline{n}\right) dA + \rho \int_{out} \underline{V}\left(\underline{V} \cdot \underline{n}\right) dA$$

Pipe:

$$\sum F_x = \rho \int_{in} u\left(\underline{V} \cdot \underline{n}\right) dA + \rho \int_{out} u\left(\underline{V} \cdot \underline{n}\right) dA$$

$$= -\rho\left(\beta A V_{ave}^2\right)_{in} + \rho\left(\beta A V_{ave}^2\right)_{out}$$

$$= \rho Q V_{ave}\left(\beta_{out} - \beta_{in}\right)$$

change in shape u

Vane:

$$\sum \underline{F} = \dot{m}\left(\underline{V_{out}} - \underline{V_{in}}\right); \quad |V_{out}| = |V_{in}|$$

If θ=180:

$$\sum F_x = \dot{m}\left(u_{out} - u_{in}\right) = \dot{m}\left(-2u_{in}\right)$$

For arbitrary θ:

$$\sum F_x = \dot{m}(u_{out}\cos\theta - u_{in}) = \dot{m}u_{in}(\cos\theta - 1)$$

change in direction u

Application of differential momentum equation:

1. NS valid both laminar and turbulent flow; however, many order of magnitude difference in temporal and spatial resolution, i.e. turbulent flow requires very small time and spatial scales

2. Laminar flow $\mathrm{Re}_{crit} = \frac{U\delta}{\nu} \le 2000$

   $\mathrm{Re} > \mathrm{Re}_{crit}$ instability

3. Turbulent flow $\mathrm{Re}_{transition} \geq 10$ or $20\ \mathrm{Re}_{crit}$

   Random motion superimposed on mean coherent structures.

   Cascade: energy from large scale dissipates at smallest scales due to viscosity.
   Kolmogorov hypothesis for smallest scales

4. No exact solutions for turbulent flow: RANS, DES, LES, DNS (all CFD)

5. 80 exact solutions for simple laminar flows are mostly linear $\underline{V} \cdot \nabla \underline{V} = 0$

   a. Couette flow = shear driven
   b. Steady duct flow = Poiseuille flow
   c. Unsteady duct flow
   d. Unsteady moving walls
   e. Asymptotic suction
   f. Wind-driven flows
   g. Similarity solutions. etc.

6. Also many exact solutions for low Re Stokes and high Re BL approximations

7. Can also use CFD for non simple laminar flows

8. AFD or CFD requires well posed IBVP; therefore, exact solutions are useful for setup of IBVP, physics, and verification CFD since modeling errors yield $U_{SM} = 0$ and only errors are numerical errors $U_{SN}$, i.e., assume analytical solution = truth, called analytical benchmark

# Energy Equation:

B = E = energy
β = e = dE/dm = energy per unit mass

# Integral Form (fixed CV):

$$\underset{\substack{\uparrow \\ \text{Rate of} \\ \text{change E}}}{\frac{dE}{dt}} = \underbrace{\int_{CV} \frac{\partial}{\partial t}(e\rho)\, d\forall}_{\substack{\text{rate of change} \\ \text{E in CV}}} + \underbrace{\int_{CS} e\rho \underline{V} \cdot \underline{n}\, dA}_{\substack{\text{rate of outflux} \\ \text{E across CS}}} = \underset{\substack{\uparrow \\ \text{Rate of heat} \\ \text{added CV}}}{\dot{Q}} - \underset{\substack{\nwarrow \\ \text{Rate work} \\ \text{done by CV}}}{\dot{W}}$$

$$e = \hat{u} + \frac{1}{2}v^2 + gz = \text{internal} + KE + PE$$

$$\dot{Q} = \text{conduction} + \text{convection} + \text{radiation}$$

$$\dot{W} = \underbrace{\dot{W}_{shaft}}_{\text{pump/turbine}} + \underbrace{\dot{W}_p}_{\text{pressure}} + \underbrace{\dot{W}_v}_{\text{viscous}}$$

$$d\dot{W}_p = \left(p\,\underline{n} dA\right) \cdot \underline{V} \qquad \text{- pressure force} \times \text{velocity}$$

$$\dot{W}_p = \int_{CS} p\left(\underline{V} \cdot \underline{n}\right) dA$$

$$d\dot{W}_v = -\underline{\tau}\, dA \cdot \underline{V}$$ *- viscous force × velocity*

$$\dot{W}_v = -\int_{CS} \underline{\tau} \cdot \underline{V}\, dA$$

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \int_{CV} \frac{\partial}{\partial t}(e\rho)\, d\forall + \int_{CS} \left(e + p/\rho\right) \rho \underline{V} \cdot \underline{n}\, dA$$

For our purposes, we are interested in steady flow one inlet and outlet. Also $\dot{W}_v \approx 0$ in most cases; since, $\underline{V} = 0$ at solid surface; on inlet and outlet only $\tau_n \sim 0$ since its perpendicular to flow; or for $\underline{V} \neq 0$ and $\tau_{streamline} \sim 0$ if outside BL.

$$\dot{Q} - \dot{W}_S = \int_{inlet\&outlet} \left(\hat{u} + \frac{1}{2}V^2 + gz + p/\rho\right) \rho \underline{V} \cdot \underline{n}\, dA$$

Assume parallel flow with $\underbrace{p/\rho + gz}$ and $\hat{u}$ constant over inlet and outlet.

= constant ie hydrostatic pressure variation

$$\dot{Q} - \dot{W}_S = \left(\hat{u} + p/\rho + gz\right) \int_{inlet\&outlet} \rho \underline{V} \cdot \underline{n}\, dA + \frac{\rho}{2} \int_{inlet\&outlet} V^2 (\underline{V} \cdot \underline{n})\, dA$$

$$\dot{Q} - \dot{W}_S = \left(\hat{u} + p/\rho + gz\right)_{in} (-\dot{m}_{in}) - \frac{\rho}{2} \int_{in} V_{in}^{\ 3}\, dA_{in}$$

$$+ \left(\hat{u} + p/\rho + gz\right)_{out} (\dot{m}_{out}) + \frac{\rho}{2} \int_{out} V_{out}^{\ 3}\, dA_{out}$$

Define kinetic energy correction factor

$$\alpha = \frac{1}{A}\int_A \left(\frac{V}{V_{ave}}\right)^3 dA \quad \rightarrow \quad \frac{\rho}{2}\int_A V^2 (\underline{V}\cdot\underline{n})\, dA = \alpha \frac{V_{ave}^2}{2}\dot{m}$$

Laminar flow: $u = U_0\left(1-\left(\frac{r}{R}\right)^2\right)$

$V_{ave}=0.5$ $\beta = 4/3$ $\alpha=2$

Turbulent flow: $u = U_0\left(1-\frac{r}{R}\right)^m$

$$\alpha = \frac{(1+m)^3(2+m)^3}{4(1+3m)(2+3m)}$$

*m=1/7* $\alpha=1.058$ *as with β, α~1 for turbulent flow*

$$\frac{\dot{Q}}{\dot{m}} - \frac{\dot{W}_s}{\dot{m}} = (\hat{u} + p/\rho + gz + \alpha\frac{V_{ave}^2}{2})_{out} - (\hat{u} + p/\rho + gz + \alpha\frac{V_{ave}^2}{2})_{in}$$

Let in = 1, out = 2, $V = V_{\text{ave}}$, and divide by g

$$\frac{p_1}{\rho g} + \frac{\alpha_1}{2g}V_1^2 + z_1 + h_p = \frac{p_2}{\rho g} + \frac{\alpha_2}{2g}V_2^2 + z_2 + h_t + h_L$$

$$\frac{\dot{W}_s}{g\dot{m}} = \frac{\dot{W}_t}{g\dot{m}} - \frac{\dot{W}_p}{g\dot{m}} = h_t - h_p$$

$$h_L = \frac{1}{g}(u_2 - u_1) - \frac{\dot{Q}}{\dot{m}g}$$

*$h_L$ = thermal energy (other terms represent mechanical energy*

$$\dot{m} = \rho A_1 V_1 = \rho A_2 V_2$$

Assuming no heat transfer mechanical energy converted to thermal energy through viscosity and can not be recovered; therefore, it is referred to as head loss $\geq 0$, which can be shown from 2nd law of thermodynamics.

1D energy equation can be considered as modified Bernoulli equation for $h_p$, $h_t$, and $h_L$.

Application of 1D Energy equation fully developed pipe flow without $h_p$ or $h_t$.

Recall the horizontal pipe flow using continuity and momentum (page 32): $\tau_w = \frac{R}{2}\left(-\frac{dp}{dx}\right)$, i.e. $-\frac{dp}{dx} = \frac{2\tau_w}{R}$

Similarly, for non-horizontal pipe: $-\frac{d}{dx}(p+\gamma z) = \frac{2\tau_w}{R}$

Using energy equation, $L = dx$ and $\hat{p} = p + \gamma z$:

$$h_L = \frac{p_1 - p_2}{\rho g} + (z_1 - z_2) = \frac{L}{\rho g}\left[-\frac{d}{dx}(p+\gamma z)\right]$$

$$h_L = \frac{L}{\rho g}\left(-\frac{d\hat{p}}{dx}\right) = \frac{L}{\rho g}\left(\frac{2\tau_w}{R}\right)$$ (If $\frac{d\hat{p}}{dx} < 0$ flow moves from left to right)

Where $\tau_w = \frac{1}{8} f \rho V_{ave}^2$

$$h_L = h_f = f\frac{L}{D}\frac{V_{ave}^2}{2g}$$ Darcy-Weisbach Equation (valid for laminar or Turbulent

Where $h_f$ is the friction loss

Also recall from page 33 that $\tau_w = \frac{4\mu V_{ave}}{R}$

For laminar flow,

$$f = \frac{8\tau_w}{\rho V_{ave}^2} = \frac{32\mu}{\rho R V_{ave}}$$

$$h_L = \frac{32\mu L V_{ave}}{\gamma D^2} \quad \propto \quad V_{ave}$$ exact solution!

For turbulent flow, $Re_{crit} \sim 2000$, $Re_{trans} \sim 3000$

$f = f(Re, k/D)$ $Re = \underline{V}_{ave} D/\nu$, $k$ = roughness

$$h_L \propto V_{ave}^2$$

Pipe with minor losses,

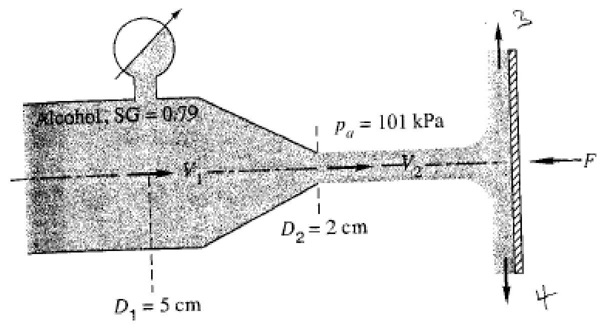
$$h_L = h_f + \Sigma h_m$$

where $h_m = K\dfrac{V^2}{2g}$

$K = loss\ coefficient$

$h_m$ = "so called" minor losses, e.g. entrance/exit, expansion/contraction, bends, elbows, tees, other fitting, and valves.

**P3.149** A jet of alcohol strikes the vertical plate in Fig. P3.149. A force $F \approx 425$ N is required to hold the plate stationary. Assuming there are no losses in the nozzle, estimate (*a*) the mass flow rate of alcohol and (*b*) the absolute pressure at section 1.




**P3.149**

(a) First suppose 2D problem: $D_1$ and $D_2$ denotes width in y instead of diameter and we take unit in z (span-wise) direction

$$\sum F_x = -F = -\dot{m}V_2 \Rightarrow \underbrace{.79 * 989}_{\rho} \times \underbrace{0.02 \times 1}_{A_2} \times V_2^2 = 425\,N$$

$$V_2 = 5.22\ m/s, \quad \dot{m} = 81.6\ kg/s$$

Continuity equation between points 1 and 2

$$V_1 A_1 = V_2 A_2 \Rightarrow V_1 = V_2 \frac{D_2}{D_1} = 2.09 m/s$$

Bernoulli neglect g, $p_2 = p_a$

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 \qquad h_L = 0,\ z = constant$$

$$p_1 = p_2 + \frac{1}{2}\rho\left(V_2^2 - V_1^2\right) \rightarrow \quad p_1 = 101{,}000 + \frac{.79 \times 998}{2}(5.22^2 - 2.09^2)$$

$$\boxed{p_1 = 110{,}020\ Pa}$$

Note:
$$p_2 + \frac{\rho}{2}V_2^2 = p_3 + \frac{\rho}{2}V_3^2 = p_4 + \frac{\rho}{2}V_4^2$$

$$p_2 = p_3 = p_4 = p_a \rightarrow \quad V_2 = V_3 = V_4$$

$$0 = \sum_{CS} \rho \underline{V} \cdot \underline{A} \rightarrow \quad A_2 V_2 = A_3 V_3 + A_4 V_4$$

$$A_2 = A_3 + A_4$$

$$\sum F_y = 0 = \sum_{CS} \rho V \underline{V} \cdot \underline{A} = \rho V_3 V_3 A_3 + \rho(-V_4) V_4 A_4$$

$$= \rho V_3^2 A_3 - \rho V_4^2 A_4 \rightarrow \qquad A_3 = A_4$$

(b) For the round jet implied in the problem statement

$$\sum F_x = -F = -\dot{m} V_2 \Rightarrow \underbrace{.79 * 989}_{\rho} \underbrace{\frac{\pi}{4} .02^2}_{A_2} V_2^2 = 425\, N$$

$$V_2 = 41.4\, m/s, \quad \dot{m} = 10.3\, kg/s$$

Continuity equation between points 1 and 2

$$V_1 A_1 = V_2 A_2 \Rightarrow V_1 = V_2 \left( \frac{D_2}{D_1} \right)^2$$

$$V_1 = 41.4 \left( \frac{2}{5} \right)^2 \qquad \boxed{V_1 = 6.63\, m/s}$$

Bernoulli neglect g, $p_2 = p_a$

$$p_1 + \frac{1}{2} \rho V_1^2 = p_2 + \frac{1}{2} \rho V_2^2 \qquad h_L = 0,\ z = constant$$

$$p_1 = p_2 + \frac{1}{2} \rho \left( V_2^2 - V_1^2 \right) \rightarrow \quad p_1 = 101{,}000 + \frac{.79 \times 998}{2} (41.4^2 - 6.63^2)$$

$$\boxed{p_1 = 760{,}000\ Pa}$$

**Example 7.9**

Water is being discharged from a large tank open to the atmosphere through a vertical tube, as shown in Fig. 7.5. The tube is 10 m long, 1 cm in diameter, and its inlet is 1 m below the level of the water in the tank. Find the velocity and the volumetric flowrate in the pipe, assuming:

a. Frictionless flow.
b. Laminar viscous flow.

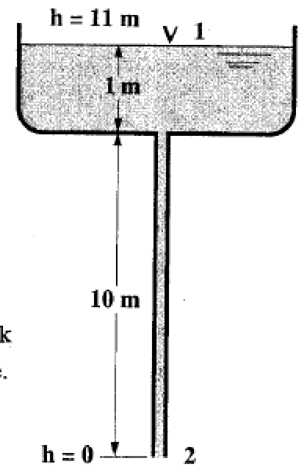



**Figure 7.5** Flow from a water tank through a vertical tube.

(a) $z_1 = \frac{V_2^2}{2g} + z_2$ $\alpha_2 = 1, h_L = 0, z_1 = 11, z_2 = 0$

Torricelli's expression for speed of efflux from reservoir

$$V_2 = \sqrt{2g(z_1 - z_2)} = \sqrt{2*9.81*11} = \boxed{14.7\ m/s}$$

$$Q_2 = A_2V_2 = \frac{\pi}{4}(.01)^2 *14.7*3600 = \boxed{4.16\ m^3/h}$$

$$\mathrm{Re} = \frac{VD}{\nu} = \frac{14.7 \times 0.01}{10^{-6}} = 1.5 \times 10^5$$

(b) $z_1 = \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L$ $\alpha_2 = 2,\ h_L = \frac{32VL\mu}{D^2 \rho g},\ \nu = 10^{-6}\ m^2/s$

$$V_2^2 + 3.2V_2 - 107.8 = 0$$

$$\boxed{\begin{array}{l} V_2 = 8.9\ m/s \\ Q = 2.516\ m^3/h \end{array}}$$

$$Re=89{,}000=8.9*10^4 >>2000$$

(c) $$z_1 = \alpha_2 \frac{V_2^2}{2g} + z_2 + f\,\frac{L}{D}\frac{V_2^2}{2g} \qquad \alpha_2 = 1$$

$$z_1 - z_2 = \frac{V_2^2}{2g}\left(1 + fL/D\right)$$

$$V_2 = \left[2g(z_1 - z_2)/(1 + fL/D)\right]^{1/2}$$

$$V_2 = \left[216/(1 + f*1000)\right]^{1/2} \qquad f = f(\text{Re}),\ \text{Re} = \frac{VD}{\nu}$$

guess f = 0.015 (smooth pipe Moody diagram)

$$V_2 = 3.7\ m/s \rightarrow \text{Re} = 3.7x10^4, \quad f = .024$$

$$V_2 = 2.94\ m/s \rightarrow \text{Re} = 2.9x10^4, \quad f = .025$$

$$V_2 = 2.88\ m/s \rightarrow \text{Re} = 2.9x10^4$$

(d) $$\text{Re} = \frac{VD}{\nu} = 2000 \qquad D = \frac{2000\nu}{V}$$

$$(z_1 - z_2) = \alpha_2 \frac{V_2^2}{2g} + \frac{32\nu L V_2}{g\,\frac{2000^2\nu^2}{V_2^2}}$$

$$(z_1 - z_2) = \alpha_2 \frac{V_2^2}{2g} + \frac{32\nu L V_2^3}{2000^2 \nu g}$$

$$\frac{32LV_2^3}{2000^2\nu g} + \frac{V_2^2}{g} - 11 = 0 \qquad \boxed{V_2 = 1.1\ m/s}$$

$$\boxed{D = 0.00182\ m}$$

Low U and small D to actually have laminar flow

# Differential Form of Energy Equation:

$$\frac{dE}{dt} = \int_{CV} \left[ \underbrace{\frac{\partial}{\partial t}(e\rho) + \nabla \cdot (e\rho \underline{V})} \right] d\forall = \dot{Q} - \dot{W}$$

$$\rho \frac{\partial e}{\partial t} + \underbrace{e \frac{\partial \rho}{\partial t} + e\nabla.\left(\rho \underline{V}\right)}_{=0} + \rho \underline{V}.\nabla e = \rho \frac{De}{Dt} = \rho \left( \frac{\partial e}{\partial t} + \underline{V}.\nabla e \right)$$

$$e = \hat{u} + \frac{1}{2}V^2 + gz = \hat{u} + \frac{1}{2}V^2 - \underline{g} \cdot \underline{r}$$

$$\rho \frac{De}{Dt} = (\dot{Q} - \dot{W}) / \forall = \dot{q} - \dot{w} = \rho \left( \frac{D\hat{u}}{Dt} + V \frac{DV}{Dt} - \underline{g} \cdot \underline{V} \right)$$

$$\dot{q} = -\nabla \cdot \underline{q} = \nabla \cdot (k \nabla T) \qquad \text{Fourier's Law}$$

$$\dot{w} = -\nabla \cdot \left( \underline{V} \cdot \sigma_{ij} \right) = -\underline{V} \cdot \underbrace{\left( \nabla \cdot \sigma_{ij} \right)}_{\substack{\rho\left(\frac{D\underline{V}}{Dt} - \underline{g}\right) \\ momentum \\ equation}} - \sigma_{ij} \frac{\partial u_i}{\partial x_j}$$

First term for $\dot{w}$

$$-\underline{V} \cdot \left( \nabla \cdot \sigma_{ij} \right) = -\underline{V} \cdot \rho \left( \frac{D\underline{V}}{Dt} - \underline{g} \right) = -\rho \left( \underline{V} \cdot \frac{D\underline{V}}{Dt} - \underline{V} \cdot \underline{g} \right)$$

Where

$$\underline{V} \cdot \frac{D\underline{V}}{Dt} = \underline{V} \cdot \left( \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right) = \frac{\partial V^2}{\partial t} + V^2 \nabla \underline{V} = V \frac{DV}{Dt}$$

Therefore

$$-\underline{V} \cdot \left( \nabla \cdot \sigma_{ij} \right) = -\rho \left( V \frac{DV}{Dt} - \underline{V} \cdot \underline{g} \right)$$

And

$$\dot{w} = -\rho\left(V\frac{DV}{Dt} - \underline{V}\cdot\underline{g}\right) - \sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

Substitute equation for $\dot{q}$ and $\dot{w}$

$$\dot{q} - \dot{w} = -\nabla\cdot(k\nabla\mathrm{T}) + \rho\left(\cancel{V\frac{DV}{Dt} - \underline{V}\cdot\underline{g}}\right) + \sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

$$= \rho\left(\frac{D\hat{u}}{Dt} + \cancel{V\frac{DV}{Dt} - \underline{V}\cdot\underline{g}}\right)$$

$$\rho\frac{D\hat{u}}{Dt} = -\nabla\cdot(k\nabla\mathrm{T}) + \sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

Second term on right hand side

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j} = (\tau_{ij} - p\delta_{ij})\frac{\partial u_i}{\partial x_j} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - p\nabla\cdot\mathrm{V}$$

From continuity

$$\frac{D\rho}{Dt} + \rho\nabla.\underline{V} = 0 \rightarrow \nabla.\underline{V} = -\frac{1}{\rho}\frac{D\rho}{Dt}$$

$$-p\nabla.\underline{V} = \frac{p}{\rho}\frac{D\rho}{Dt} = -\rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) + \frac{Dp}{Dt}$$

Therefore

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) + \frac{Dp}{Dt}$$

And

$$\rho\frac{D\hat{u}}{Dt} = -\nabla\cdot(k\nabla\mathrm{T}) + \tau_{ij}\frac{\partial u_i}{\partial x_j} - \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) + \frac{Dp}{Dt}$$

Rearranging equation and substituting dissipation function $\Phi = \tau_{ij}\frac{\partial u_i}{\partial x_j} \geq 0$

$$\rho\frac{D}{Dt}\left(u + \frac{p}{\rho}\right) = -\nabla\cdot(k\nabla\mathrm{T}) + \frac{Dp}{Dt} + \Phi$$

Summary GDE for compressible non-constant property fluid flow

Continuity: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$

Momentum: $\rho \frac{D\underline{V}}{Dt} = \rho \underline{g} - \nabla p + \nabla . \tau_{ij}$ $\tau_{ij} = \mu \epsilon_{ij} + \lambda \nabla . \underline{V} \delta_{ij}$; $\underline{g} = -g\hat{k}$

Energy $\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (k \nabla T) + \Phi$

Primary variables: p, $\underline{V}$, T

Auxiliary relations: ρ = ρ (p,T) μ = μ (p,T)
(equations of state) h = h (p,T) k = k (p,T)

Restrictive Assumptions:
1) Continuum
2) Newtonian fluids
3) Thermodynamic equilibrium
4) $\underline{g} = -g\hat{k}$
5) heat conduction follows Fourier's law
6) no internal heat sources

For incompressible constant property fluid flow

$$d\hat{u} = c_v \, dT \qquad c_v, \mu, k, \rho \sim constant$$

$$\rho c_v \frac{DT}{Dt} = k\nabla^2 T + \Phi$$

For static fluid or $\underline{V}$ small

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T \qquad \textit{heat conduction equation (also valid for solids)}$$

Summary GDE for incompressible constant property fluid flow ($c_v \sim c_p$)

$$\nabla \cdot \underline{V} = 0$$

$$\rho \frac{D\underline{V}}{Dt} = -\rho g\hat{k} - \nabla p + \mu\nabla^2 \underline{V} \qquad \text{“elliptic”}$$

$$\rho c_p \frac{DT}{Dt} = k\nabla^2 T + \Phi \qquad \text{where } \Phi = \tau_{ij}\frac{\partial u_i}{\partial x_j}$$

Continuity and momentum uncoupled from energy; therefore, solve separately and use solution post facto to get T.

For compressible flow, ρ solved from continuity equation, T from energy equation, and p = (ρ,T) from equation of state (eg, ideal gas law). For incompressible flow, ρ = constant and T uncoupled from continuity and momentum equations, the latter of which contains $\nabla p$ such that reference p is arbitrary and specified post facto (i.e. for incompressible flow, there is no connection between p and ρ). The connection is between $\nabla p$ and $\nabla \cdot \underline{V} = 0$, i.e. a solution for p requires $\nabla \cdot \underline{V} = 0$.

NS $\frac{D\underline{V}}{Dt} = -\frac{1}{\rho}\nabla\hat{p} + \nu\nabla^2\underline{V}$ $\hat{p} = p + \gamma z$

$\nabla \cdot (NS)$ (See derivation details on p.87)

$$\left(\frac{D}{Dt} - \nu\nabla^2\right)\nabla \cdot \underline{V} = -\frac{1}{\rho}\nabla^2 p + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

For $\nabla \cdot \underline{V} = 0$:

$$\nabla^2 p = -\rho\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

Poisson equation determines pressure up to additive constant.

<u>Approximate Models</u>:

1) Stokes Flow

For low $\mathrm{Re} = \frac{UL}{\nu} << 1, \quad \underline{V} \cdot \nabla\underline{V} \sim 0$

$$\nabla \cdot \underline{V} = 0$$
$$\frac{\partial \underline{V}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{V}$$

Linear, "elliptic"
Most exact solutions NS; and for steady flow superposition, elemental solutions and separation of variables

$$\nabla \cdot (NS) \Rightarrow \nabla^2 p = 0$$

## 2) Boundary Layer Equations

For high Re >> 1 and attached boundary layers or fully developed free shear flows (wakes, jets, mixing layers), $v$<<U, $\frac{\partial}{\partial x} << \frac{\partial}{\partial y}$, $p_y = 0$, and for free shear flows $p_x = 0$.

$$u_x + v_y = 0$$

$$u_t + uu_x + vu_y = -\hat{p}_x + \nu u_{yy} \quad \textit{non-linear, “parabolic”}$$

$$p_y = 0 \Rightarrow -\hat{p}_x = U_t + UU_x$$

Many exact solutions; similarity methods

## 3) Inviscid Flow

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho \underline{V} \right) = 0$$

$$\rho \frac{D\underline{V}}{Dt} = \rho \underline{g} - \nabla p \qquad \textit{Euler Equation, nonlinear, "hyperbolic"}$$

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \nabla \cdot (k \nabla T) \qquad p, \underline{V}, T \textit{ unknowns} \quad \textit{and } \rho, h, k = f(p, T)$$

4) Inviscid, Incompressible, Irrotational

$$\nabla \times \underline{V} = 0 \rightarrow \underline{V} = \nabla\varphi$$
$$\nabla . \underline{V} = 0 \rightarrow \nabla^2\varphi = 0 \quad \mathit{linear}\ \text{elliptic}$$

$\int$ Euler Equation → Bernoulli Equation:

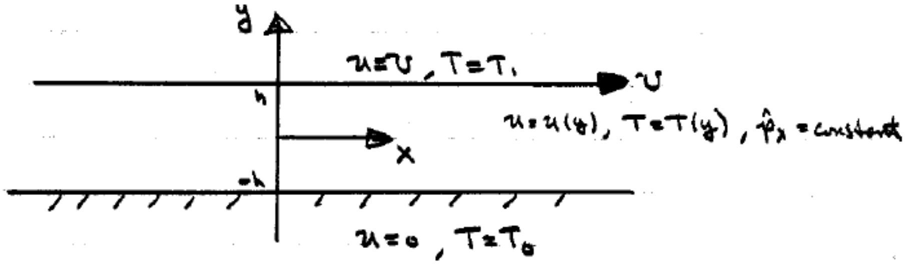
$$p + \frac{\rho}{2}V^2 + \rho g z = const$$

Many elegant solutions: Laplace equation using superposition elementary solutions, separation of variables, complex variables for 2D, and Boundary Element methods.

Couette Shear Flows: 1-D shear flow between surfaces of like geometry (parallel plates or rotating cylinders).

Steady Flow Between Parallel Plates: *Combined Couette and Poiseuille Flow.*

Since b.c. are independent of x,z solution in 1-D




$$\nabla\cdot\underline{V}=0$$

$$u_x+v_y+w_z=0$$

$$u_x=0$$

$$\rho\frac{D\underline{V}}{Dt}=-\nabla\hat{p}+\mu\nabla^2\underline{V}\qquad\qquad \frac{\partial u}{\partial t}+uu_x+vu_y+wu_z=0$$

$$0=-\hat{p}_x+\mu u_{yy}$$

$$\rho c_p\frac{DT}{Dt}=k\nabla^2T+\Phi\qquad\qquad \frac{\partial T}{\partial t}+uT_x+vT_y+wT_z=0$$

$$0=kT_{yy}+\mu u_y^2$$

$$\begin{aligned}\Phi=\mu\Big[&2u_x^2+2v_y^2+2w_z^2\\&+(v_x+u_y)^2+(w_y+v_z)^2+(u_z+w_x)^2\Big]\\&+\lambda(u_x+v_y+w_z)\\&=\mu u_y^2\end{aligned}$$

*(note: inertia terms vanish identically and ρ is absent from equations)*

*Non-dimensionalize equations, but drop **

$$u^* = u/U \qquad T^* = \frac{T - T_0}{T_1 - T_0} \qquad y^* = y/h$$

$$u_x = 0 \tag{1}$$

$$u_{yy} = \frac{h^2}{\mu U}\hat{p}_x = -B = cons. \tag{2}$$

$$T_{yy} = \underbrace{\frac{\mu U^2}{k(T_1 - T_0)}}_{\Pr Ec}\left[-u_y^2\right] \tag{3}$$

B.C. y = 1 u = 1 T = 1

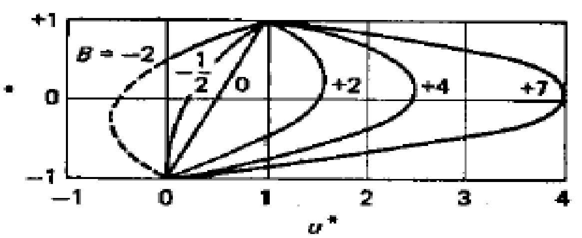
y = -1 u = 0 T = 0

(1) is consistent with 1-D flow assumption. Simple form of (2) and (3) allow for solution to be obtained by double integration.

$$\Rightarrow \quad u = \underbrace{\frac{1}{2}(1+y)}_{} + \underbrace{\frac{1}{2}B(1-y^2)}_{} \qquad y=y/h$$

Linear flow due to U

Parabolic flow due to $p_x$




Note: linear superposition since $\underline{V} \cdot \nabla \underline{V} = 0$

Solution depends on $B = -\frac{h^2}{\mu U}\hat{p}_x$:

B < 0 $\hat{p}_x$ is opposite to U

B < -0.5 backflow occurs near lower wall

|B| >> 1 flow approaches parabolic profile

$$T=\frac{1}{2}(1+y)+\frac{\Pr E_c}{8}(1-y^2)\overbrace{-\frac{\Pr E_c B}{6}(y-y^3)+\frac{\Pr E_c B^2}{12}(1-y^4)}^{\text{Pressure gradient effect}}$$

Pure conduction

T rises due to viscous dissipation

Dominant term for B→ ∞

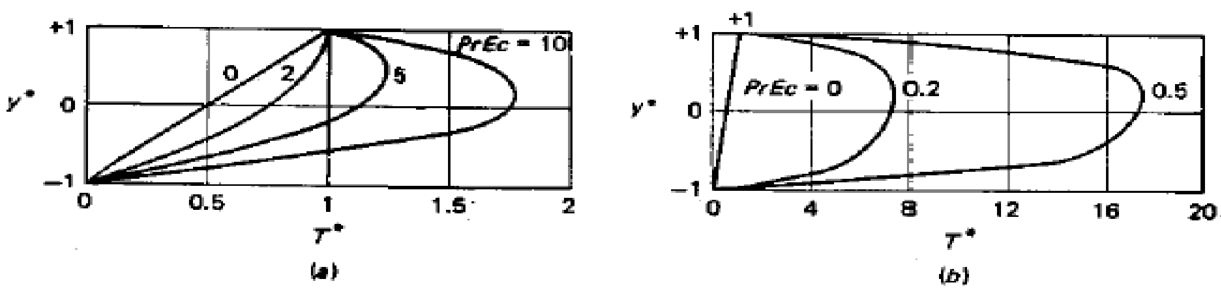



**FIGURE 3-3**
**Temperature distributions for flow between parallel plates, Eq. (3-12): (*a*) pure Couette flow: $B = 0$; (*b*) mostly Poiseuille flow: $B = 20$.**

*Note: usually $PrE_c$ is quite small*

| *Substance* | *$PrE_c$* | *dissipation* |
|---|---|---|
| Air | 0.001 | very small |
| Water | 0.02 | |
| Crude oil | 20 | large |

$$Br = \Pr E_c$$
$$= Brinkman\ \#$$

Shear Stress

1) $\hat{p}_x = 0$ i.e. pure Couette Flow

$$B = -\frac{h^2}{\mu U}\hat{p}_x = 0$$

Using solution shown previously

$$u^* = \frac{1}{2}(1+y^*) + \frac{1}{2}B\left(1-{y^*}^2\right) = \frac{1}{2}(1+y^*)$$

Calculating wall shear stress

$$\frac{u}{U} = \frac{1}{2}\left(1+\frac{y}{h}\right)$$

$$\frac{\partial\left(\frac{u}{U}\right)}{\partial\left(\frac{y}{h}\right)} = \frac{1}{2}$$

$$\tau_w = \mu \frac{du}{dy}\bigg]_{y=-1} = \frac{\mu U}{2h}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{\frac{\mu U}{2h}}{\frac{1}{2}\rho U^2} = \frac{\mu}{\rho U h}$$

Since $Re_h = \rho U h/\mu$

$$C_f = \frac{1}{Re_h}$$

$P_0 = C_f Re = 1$: Better for non-accelerating flows since $\rho$ is not in equations and $P_0$ = pure constant

2) U = 0 i.e. pure Poiseuille Flow

$$u^* = \frac{1}{2}B(1-y^{*2}) \qquad u^*_{y^*} = -By^* \qquad u_y = -\frac{BU}{h^2}y \qquad V_{ave} = \bar{u}$$

Where $B = \frac{-h}{\mu U}\hat{p}_x = \frac{2u_{\max}}{U}$

Dimensional form $u = \underbrace{-\frac{1}{2}\frac{h^2}{\mu}\hat{p}_x}_{u_{\max}}\left(1-\left(y/h\right)^2\right) \quad Q = \int_{-h}^{h} u\,dy = \frac{4}{3}hu_{\max}$

$$\bar{u} = \frac{Q}{2h} = \frac{2}{3}u_{\max} = V_{ave}$$

Remember that for laminar pipe flow, $V_{ave} = \frac{1}{2}u_{max}$

$$\tau_w = \mu u_y\Big|_{y=\pm h} = -\mu\frac{BU}{h} \qquad upper$$

$$= +\mu\frac{BU}{h} \qquad lower$$

$$|\tau_w| = \mu\frac{BU}{h} = \mu\frac{2u_{\max}}{h} = \mu 3\bar{u}/h \qquad \begin{array}{l} \propto \bar{u} \quad lam. \\ \propto \rho\bar{u}^2 \quad turb. \end{array}$$

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{6\mu}{\rho\bar{u}h} = \frac{6}{\mathrm{Re}_h} \quad or \quad P_0 = C_f\,\mathrm{Re}_h = 6$$

Remember that for laminar pipe flow, $C_f = \frac{16}{Re_D}$ and $\tau_w = \frac{\mu 8V_{ave}}{D}$, i.e. Except for numerical constants same as for circular pipe.

Rate of heat transfer at the walls:

$$q_w = kT_y\Big|_{y\pm h} = \frac{k}{2h}(T_1 - T_0) \pm \mu\frac{U^2}{4h}$$ + = upper, - = lower

Heat transfer coefficient:

$$\varsigma = q_w \big/ (T_1 - T_0)$$

$$Nu = \frac{2h\varsigma}{k} = 1 \pm Br/2$$

For Br >> 2, both upper & lower walls must be cooled to maintain $T_1$ and $T_0$

Conservation of Angular Momentum: moment form of momentum equation (not new conservation law!)

$B = \underline{H}_0 = \int_{sys} \underline{r} \times \underline{V}\, dm =$ *angular momentum of system about inertial coordinate system 0 (extensive property)*

$\beta = \frac{dB}{dM} = \underline{r} \times \underline{V}$ *(intensive property)*

$$\underbrace{\frac{d\underline{H}_0}{dt}}_{\substack{\text{Rate of}\\ \text{change of}\\ \text{angular}\\ \text{momentum}}} = \frac{d}{dt}\int_{CV} (\underline{r} \times \underline{V})\rho\, d\forall + \int_{CS} (\underline{r} \times \underline{V})\rho\, \underline{V}_R . \underline{n}\, dA$$

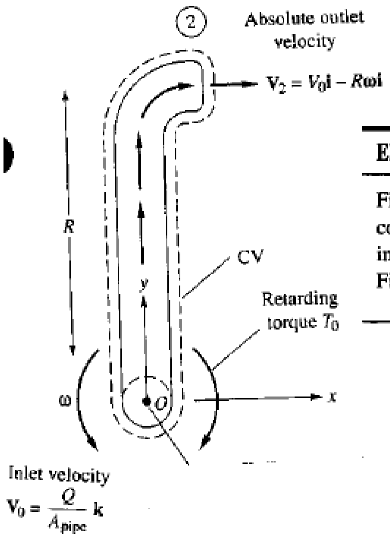
$= \sum \underline{M}_0 =$ *vector sum all external moments applied on CV due to both* $\underline{F}_B$ *and* $\underline{F}_S$*, including reaction forces*

For uniform flow across discrete inlet/outlet:

$$\int_{CS} (\underline{r} \times \underline{V})\rho\, \underline{V}_R . \underline{n}\, dA = \sum (\underline{r} \times \underline{V})_{out} \dot{m}_{out} - \sum (\underline{r} \times \underline{V})_{in} \dot{m}_{in}$$

$$\underline{M}_0 = \underbrace{\int_{CS} \underline{\tau} \cdot dA \times \underline{r}}_{surface\ force\ moment} + \underbrace{\int_{CV} (\rho \underline{g}\, d\forall) \times \underline{r}}_{body\ force\ moment} + \underline{M}_R$$

$\underline{M}_R =$ *moment of reaction forces*




**EXAMPLE 3.15**

Figure 3.14 shows a lawn sprinkler arm viewed from above. The arm rotates about $O$ at constant angular velocity $\omega$. The volume flux entering the arm at $O$ is $Q$, and the fluid is incompressible. There is a retarding torque at $O$, due to bearing friction, of amount $-T_o\mathbf{k}$. Find an expression for the rotation $\omega$ in terms of the arm and flow properties.

**Fig. 3.14** View from above of a single arm of a rotating lawn sprinkler.

Take inertial frame 0 as fixed to earth such that CS moving at $\underline{V}_S = -R\omega\,\hat{\imath}$

$$\underline{V} = \underline{V}_R + \underline{V}_S$$

$$\underline{V}_2 = V_0\hat{\imath} - R\omega\hat{\imath} = (V_0 - R\omega)\hat{\imath} \qquad \underline{r}_2 = R\,\hat{\jmath}$$

$$\underline{V}_1 = V_0\hat{k} \qquad \underline{r}_1 = 0\,\hat{\jmath}$$

$$V_0 = Q/A_{pipe}$$

Retarding torque due to bearing friction

$$\sum \underline{M}_Z = 0 = -T_0\hat{k} = (\underline{r}_2 \times \underline{V}_2)\dot{m}_{out} - (\underline{r}_1 \times \underline{V}_1)\dot{m}_{in}$$

$$\dot{m}_{out} = \dot{m}_{in} = \rho Q \qquad -T_o\hat{k} = R(V_0 - R\omega)(-\hat{k})\rho Q$$

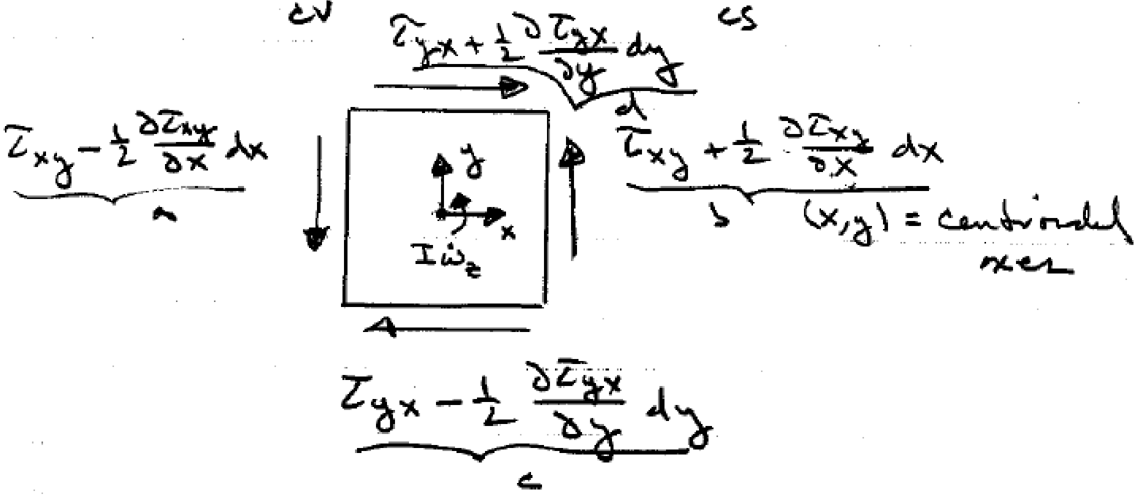
$$\omega = \frac{V_0}{R} - \frac{T_0}{\rho Q R^2}$$ → *interestingly, even for $T_0=0$, $\omega_{max}=V_0/R$*

(limited by ratio such that large R small ω; large $V_0$ large ω)

# Differential Equation of Conservation of Angular Momentum:

Apply CV form for fixed CV:

$$\Sigma \underline{M}_o = \frac{d}{dt}\int_{CV} (\underline{r}\times\underline{V})\,\rho\, d\forall + \int_{CS} (\underline{r}\times\underline{V})\,\rho\, \underline{V}\cdot\underline{n}\, dA$$




$\dot{\omega}_z$ = angular acceleration
$I$ = moment of inertia

$$I\dot{\omega}_z = a\,dy\frac{dx}{2} + b\,dy\frac{dx}{2} - c\,dx\frac{dy}{2} - d\,dx\frac{dy}{2}$$

$$I\dot{\omega}_z = \left(\tau_{xy} - \tau_{yx}\right) dxdy$$

Since $I = \frac{\rho}{12}\left[dxdy^3 + dydx^3\right] = \frac{\rho}{12}dxdy\left[dx^2 + dy^2\right]$

$$\frac{\rho}{12}\left[dx^2 + dy^2\right]\dot{\omega}_z = \tau_{xy} - \tau_{yx}$$

$$\lim_{dx\to 0, dy\to 0} \tau_{xy} = \tau_{yx}, \text{ similarly} \quad \tau_{xz} = \tau_{zx}, \quad \tau_{yz} = \tau_{zy}$$

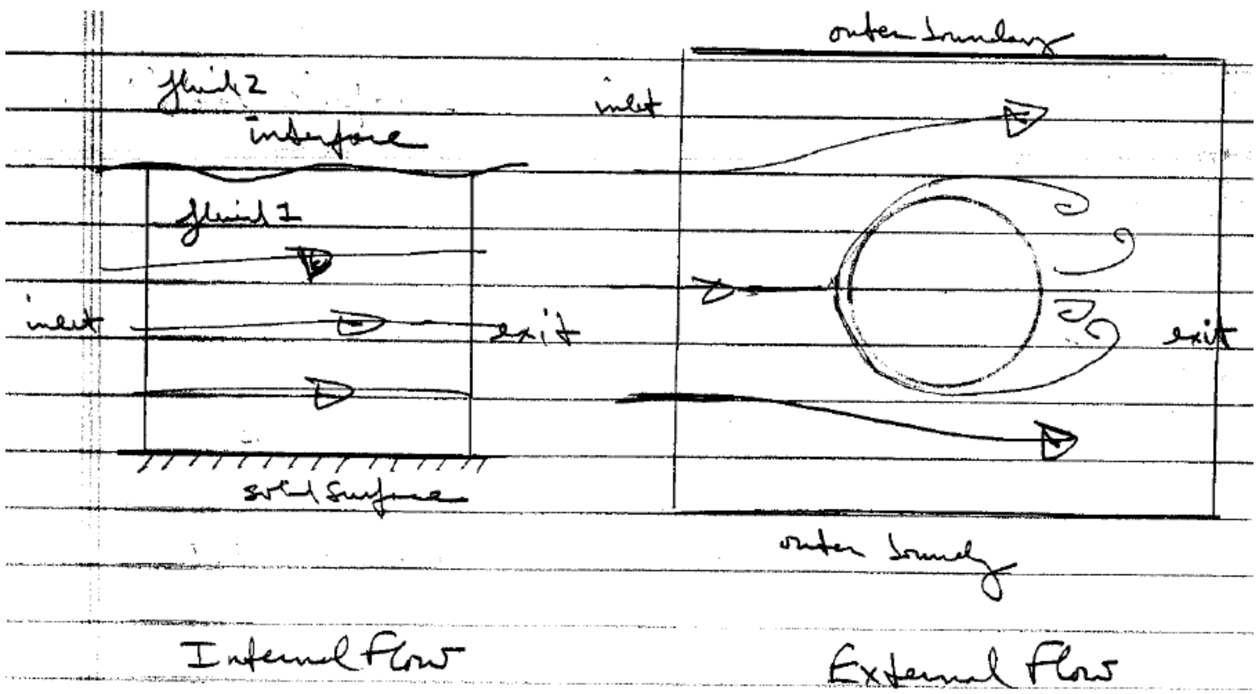
i.e $\tau_{ij} = \tau_{ji}$ *stress tensor is symmetric (stresses themselves cause no rotation)*

## Boundary Conditions for Viscous-Flow Problem

The GDE to be discussed next constitute an IBVP for a system of 2nd order nonlinear PDE, which require IC and BC for their solution, depending on physical problem and appropriate approximations.

Types of Boundaries:

1. Solid Surface
2. Interface
3. Inlet/exit/outer



1. Solid Surface

a. Liquid

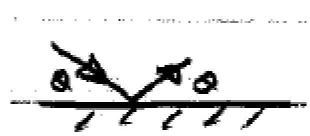
ℓ = mean free path << fluid motion; therefore, maroscopic view is "no slip" condition, i.e. no relative motion or temperature difference between liquid and solid.

$$\underline{V}_{liquid} = \underline{V}_{solid} \qquad\qquad T_{liquid} = T_{solid}$$

Exception is for contact line for which analysis is similar to that for gas.

b. Gas

Smooth wall



Specular reflection
Conservation of tangential momentum
$u_w$=0=fluid velocity at wall

Rough wall

Diffuse reflection.
Lack of reflected tangential momentum balanced by $u_w$

$$u_w = l\left.\frac{du}{dy}\right|_w$$

$$\tau_w = \mu \frac{du}{dy}\bigg|_w \qquad l = \frac{\mu}{\frac{2}{3}\rho a} \qquad \textit{low density limit}$$

$$u_w = \frac{3}{2}\frac{\mu}{\rho a}\frac{\tau_w}{\mu} \qquad Ma = U/a \qquad C_f = \tau_w \bigg/ \frac{1}{2}\rho U^2$$

$$u_w / U = .75 Ma\, C_f$$

High Re: $C_f \sim 0.005$, Say $Ma \sim 20$ $\longrightarrow$ $\frac{u_w}{U} < 0.01$

Low Re: $C_f \sim .6 Re_x^{-1/2}$ $\qquad Re_x = Ux/\upsilon$

$$\frac{u_w}{U} = \frac{.4 Ma}{\mathrm{Re}_x^{\frac{1}{2}}}$$

Significant slip possible at low Re, high Ma:
"Hypersonic LE Problem"

Similar for T:

High Re: $T_{gas} = T_w$

Low Re $\qquad \frac{T_{gas} - T_w}{\underbrace{(T_r - T_w)}_{=\text{ driving } \Delta T}} = .87 Ma C_f \qquad$ *air*

Ref. T

*Where*
$$\underbrace{C_f = 2C_h}_{\text{Reynolds Analogy}} = 2\underbrace{\frac{q_w}{\rho C_p U(T_r - T_w)}}_{C_h \text{ = Stanton number, i.e. wall heat transfer coefficient}}$$

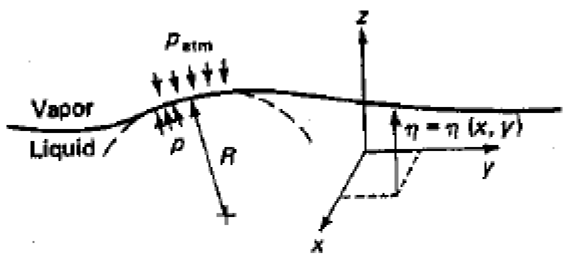


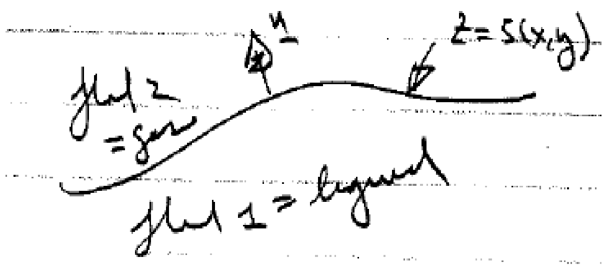

**FIGURE 1-29**
Conditions at an ideal free surface.

2. <u>Idealized gas/liquid interface</u> (free surface problems since interface is unknown and part of the solution, but effect gas on liquid idealized).

Kinematic FSBC: free surface is stream surface

$$F = \zeta(x, y) - z = surface\ \ function$$

$$\underline{n} = \nabla F / |\nabla F| = (\zeta_x, \zeta_y, -1) / \left[\zeta_x^2 + \zeta_y^2 + 1\right]^{\frac{1}{2}}$$




$$\frac{DF}{Dt} = 0 = \frac{\partial F}{\partial t} + \underline{V} \cdot \nabla F$$

$$\frac{1}{|\nabla F|}\frac{\partial F}{\partial t} + \underline{V} \cdot \underline{n} = 0$$

# Dynamic FSBC: stress continuous across free surface (similarly for mass and heat flux)

$$\tau_{ij} n_j = \tau_{ij}^{*} n_j - p_\gamma \delta_{ij}$$

Fluid 1 stress    Fluid 2 stress    Surface tension pres.

*(vector whose components are stress in direction of coordinate axes on surface with normal $n_j$)*

$$\tau_{ij} = -p\delta_{ij} + \mathrm{Re}^{-1}(U_{i,j} + U_{j,i})$$

$$\tau_{ij}^{*} = \left[-p\delta_{ij} + \mathrm{Re}^{-1}(U_{i,j} + U_{j,i})\right]_{fluid\ 2}$$ *eg* $= p_a \delta_{ij}$ *for air if neglecting* $\mu_{air}$

Atmospheric pressure

$$p_\gamma = We^{-1}\left(K_{SN} + K_{tN}\right)$$

$$K_{SN} = \underline{\hat{n}} \cdot \frac{\partial \hat{e}_S}{\partial s}$$

$$K_{tN} = \underline{\hat{n}} \cdot \frac{\partial \hat{e}_t}{\partial t}$$

*Curvature F for two mutually perp. directions.*

Note: $\hat{e}_S$ and $\hat{e}_t$ normal to $\underline{n}$ = $\hat{e}_n$

$$We = \rho U^2 L / \sigma = Weber\ Number$$

Surface tension

(2) $\tau_x = \tau_{11} n_1 + \tau_{12} n_2 + \tau_{13} n_3 = (p_a - p_\gamma) n_1$

(3) $\tau_y = \tau_{21} n_1 + \tau_{22} n_2 + \tau_{23} n_3 = (p_a - p_\gamma) n_2$

(4) $\tau_z = \tau_{31} n_1 + \tau_{32} n_2 + \tau_{33} n_3 = (p_a - p_\gamma) n_3$

$(5)\nabla\cdot\underline{V}=0=U_x+V_y+W_z$ *incompressible flow*

1+3+1=5 conditions for 5 unknowns = ($\underline{V}$, p, ζ)
The first 4 conditions nonlinear

-Also need conditions for turbulence variables

Many approximations, eg, inviscid approximation:

$$p_a = p_\gamma = 0$$

small slope: $\zeta_x \sim \zeta_y \sim 0$
small normal velocity gradient: $W_x \sim W_y \sim W_z = 0$

$$\frac{\partial}{\partial z}(U,V)=0 \qquad W_z=-U_x-V_y \qquad or\ \ W_z=0$$

$$p = 0 \quad or \quad \hat{p}=\rho g z \qquad \hat{p} = \text{piezometric pres.}$$

3) <u>Inlet/exit/outer</u>

*a) inlet: $\underline{V}$, p, T specified*
*b) outer: $\underline{V}$, p, T specified*

eg. constant Temp., uniform stream:
$\underline{V} = U\,\hat{\imath}$ , p = 0 , T = $T_{i,o}$

*c) exit: depends on the problem, but often use $U_{xx}=0$,*
*(i.e. zero stream wise diffusion for external flow and periodic for fully developed internal flow).*

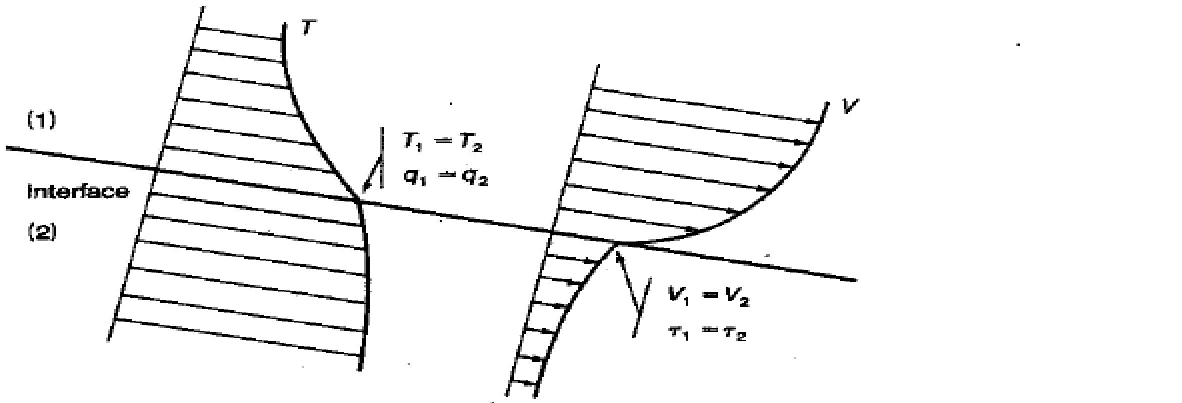



**FIGURE 1-30**
Conditions at an actual fluid interface. Velocities and temperatures match, but their slopes do not because of differing $k$ and $\mu$. Pressures also match except for the surface-tension effect of Eq. (1-106).

Interface Velocity Condition

Just as with solid surface, there can be no relative velocity across interface (i.e. exact condition for liquid/liquid and gas/gas or gas/liquid non-mixing fluids).

$$\underline{V}_1 = \underline{V}_2$$

$$V_{n1} = V_{n2} \quad \text{required by KFSBC}$$

$$\underline{V}_1 \cdot \underline{n} = \underline{V}_2 \cdot \underline{n} = -\frac{1}{|\nabla F|}\frac{\partial F}{\partial t}$$

Tangential should also match, but usually due to different approximations used in fluid 1 or 2, (eg fluid 1 liquid and fluid 2 gas do not). Often, in fact, motions in gas are neglected and therefore V is not continuous.

Also liquid/liquid interfaces are not stable for large Re and one must consider "turbulent interface".

# *Table 7. Boundary Conditions*

| | *IBTYP* | *Description* | *U* | *V* | *W* | *P* | *k* | *ω* | $\nu_t$ |
|---|---|---|---|---|---|---|---|---|---|
| Domain Truncation Boundaries | 10 | Inlet | UINF | VINF | WINF | $\partial P/\partial\xi_i=0$ | $k_{fs}=1\times10^{-7}$ | $\omega_{fs}=9.0$ | $\nu_{t,fs}$ |
| | 11 | Exit | $\partial^2U/\partial\xi_i^2=0$ | $\partial^2V/\partial\xi_i^2=0$ | $\partial^2W/\partial\xi_i^2=0$ | $\partial P/\partial\xi_i=0$ | $\partial k/\partial\xi_i=0$ | $\partial\omega/\partial\xi_i=0$ | $\partial\nu_t/\partial\xi_i=0$ |
| | 12 | Far-field #1 | UINF | $\partial V/\partial\xi_i=0$ | $\partial W/\partial\xi_i=0$ | 0 | $\partial k/\partial\xi_i=0$ | $\partial\omega/\partial\xi_i=0$ | $\partial\nu_t/\partial\xi_i=0$ |
| | 13 | Far-field #2 | UINF | VINF | WINF | $\partial P/\partial\xi_i=0$ | $\partial k/\partial\xi_i=0$ | $\partial\omega/\partial\xi_i=0$ | $\partial\nu_t/\partial\xi_i=0$ |
| | 14 | Prescribed | * | * | * | * | * | * | * |
| Physical Boundaries | 20 | Absolute-frame no-slip | 0 | 0 | 0 | $\partial P/\partial\xi_i=0$ | 0 | $60/\mathrm{Re}\,\beta\Delta y^2$ | 0 |
| | 22 | Relative-frame no-slip | $\dot{x}$ | $\dot{y}$ | $\dot{z}$ | $\partial P/\partial\xi_i=0$ | 0 | $60/\mathrm{Re}\,\beta\Delta y^2$ | 0 |
| | 27 | Impermeable slip (calculate forces) | Eq. (78) | Eq. (78) | Eq. (78) | $\partial P/\partial\xi_i=0$ | $\partial k/\partial\xi_i=0$ | $\partial\omega/\partial\xi_i=0$ | $\partial\nu_t/\partial\xi_i=0$ |
| | 28 | Impermeable slip (no forces) | Eq. (78) | Eq. (78) | Eq. (78) | $\partial P/\partial\xi_i=0$ | $\partial k/\partial\xi_i=0$ | $\partial\omega/\partial\xi_i=0$ | $\partial\nu_t/\partial\xi_i=0$ |
| | 30 | Free surface | Eq. (34) | Eq. (34) | Eq. (35) | Eq. (33) | $\partial k/\partial\xi_i=0$ | $\partial\omega/\partial\xi_i=0$ | $\partial\nu_t/\partial\xi_i=0$ |
| Computational Boundaries | 40 | Zero gradient | $\partial U/\partial\xi_i=0$ | $\partial V/\partial\xi_i=0$ | $\partial W/\partial\xi_i=0$ | $\partial P/\partial\xi_i=0$ | $\partial k/\partial\xi_i=0$ | $\partial\omega/\partial\xi_i=0$ | $\partial\nu_t/\partial\xi_i=0$ |
| | 41 | Translational periodicity, w/ ghost cells | * | * | * | * | * | * | * |
| | 42 | Translational periodicity, w/o ghost cells | * | * | * | * | * | * | * |
| | 43 | Pole (I-around) | Eq. (80) | Eq. (80) | Eq. (80) | Eq. (80) | Eq. (80) | Eq. (80) | Eq. (80) |
| | 44 | Pole (j-around) | Eq. (80) | Eq. (80) | Eq. (80) | Eq. (80) | Eq. (80) | Eq. (80) | Eq. (80) |
| | 45 | Pole (k around) | Eq. (80) | Eq. (80) | Eq. (80) | Eq. (80) | Eq. (80) | Eq. (80) | Eq. (80) |
| | 50 | Cylindrical zero gradient | * | * | * | * | * | * | * |
| | 51 | Rotational periodicity, w/ ghost cells | * | * | * | * | * | * | * |
| | 52 | Rotational periodicity, w/o ghost cells | * | * | * | * | * | * | * |
| | 60 | No-slip/centerplane | * | * | * | * | * | * | * |
| | 61 | x-axis symmetry | 0 | $\partial V/\partial\xi_i=0$ | $\partial W/\partial\xi_i=0$ | $\partial P/\partial\xi_i=0$ | $\partial k/\partial\xi_i=0$ | $\partial\omega/\partial\xi_i=0$ | $\partial\nu_t/\partial\xi_i=0$ |
| | 62 | y-axis symmetry | $\partial U/\partial\xi_i=0$ | 0 | $\partial W/\partial\xi_i=0$ | $\partial P/\partial\xi_i=0$ | $\partial k/\partial\xi_i=0$ | $\partial\omega/\partial\xi_i=0$ | $\partial\nu_t/\partial\xi_i=0$ |
| | 63 | z-axis symmetry | $\partial U/\partial\xi_i=0$ | $\partial V/\partial\xi_i=0$ | 0 | $\partial P/\partial\xi_i=0$ | $\partial k/\partial\xi_i=0$ | $\partial\omega/\partial\xi_i=0$ | $\partial\nu_t/\partial\xi_i=0$ |
| | 91 | Multi-block w/ ghost cells | * | * | * | * | * | * | * |
| | 92 | Multi-block w/o ghost cells | * | * | * | * | * | * | * |
| | 99 | Blanked out points | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

* See text for detailed description

## Vorticity Theorems

The incompressible flow momentum equations focus attention on $\underline{V}$ and p and explain the flow pattern in terms of inertia, pressure, gravity, and viscous forces. Alternatively, one can focus attention on $\underline{\omega}$ and explain the flow pattern in terms of the rate of change, deforming, and diffusion of $\underline{\omega}$ by way of the vorticity equation. As will be shown, the existence of $\underline{\omega}$ generally indicates the viscous effects are important since fluid particles can only be set into rotation by viscous forces. Thus, the importance of this topic is to demonstrate that under most circumstances, an inviscid flow can also be considered irrotational.

### 1. Vorticity Kinematics

$$\underline{\omega} = \nabla \times \underline{V} = (w_y - v_z)\hat{i} + (u_z - w_x)\,\hat{j} + (v_x - u_y)\,\hat{k}$$

$$\omega_i = \varepsilon_{ijk}\frac{\partial u_j}{\partial x_k} = \left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k}\right)$$

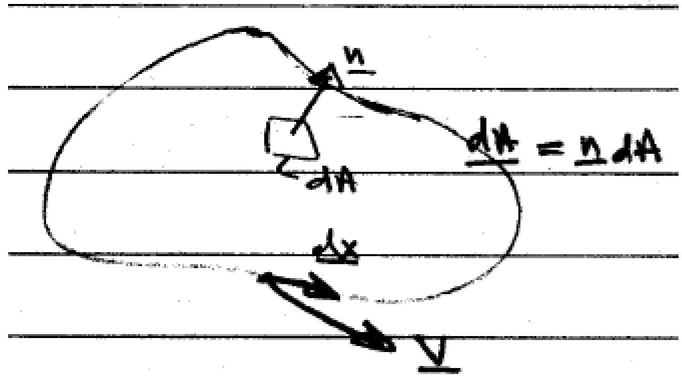
$$\underbrace{\begin{aligned} &\varepsilon_{123} = \varepsilon_{321} = \varepsilon_{231} = 1 \\ &\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1 \\ &\varepsilon_{ijk} = 0 \; otherwise \end{aligned}}_{alternating\ \ tensor}$$

*= 2 × the angular velocity of the fluid element*

(i, j, k cyclic)

A quantity intimately tied with vorticity is the circulation:

$$\Gamma = \oint \underline{V} \cdot \underline{dx}$$



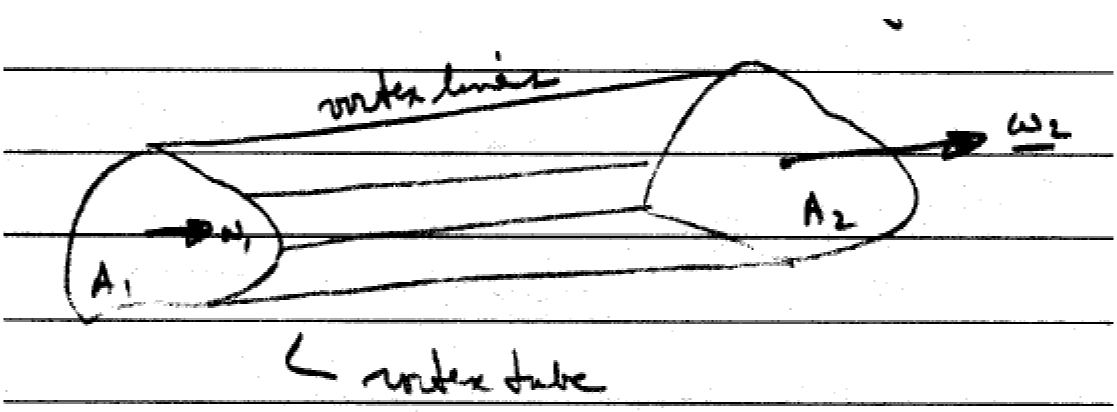

Stokes Theorem:

$$\oint \underline{a} \cdot \underline{dx} = \int_A \nabla \times \underline{a} \cdot \underline{dA}$$

$$\therefore \Gamma = \oint \underline{V} \cdot \underline{dx} = \int_A \nabla \times \underline{V} \cdot \underline{dA} = \int_A \underline{\omega} \cdot \underline{n} dA$$

Which shows that if $\underline{\omega}$ =0 (i.e., if the flow is irrotational, then $\Gamma = 0$ also.

Vortex line = lines which are everywhere tangent to the vorticity vector.



Next, we shall see that vorticity and vortex lines must obey certain properties known as the Helmholtz vorticity theorems, which have great physical significance.

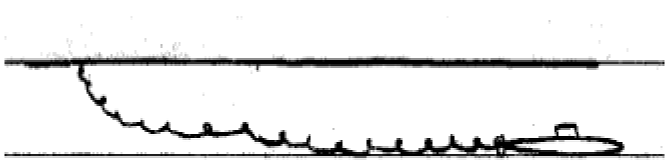
The first is the result of its very definition:

$$\underline{\omega} = \nabla \times \underline{V}$$

$$\nabla \cdot \underline{\omega} = \nabla \cdot (\nabla \times \underline{V}) = 0 \qquad \text{Vector identity}$$

i.e. the vorticity is divergence-free, which means that there can be no sources or sinks of vorticity within the fluid itself.

<u>Helmholtz Theorem #1</u>: a vortex line cannot end in the fluid. It must form a closed path (smoke ring), end at a boundary, solid or free surface, or go to infinity.



Propeller vortex is known to drift up towards the free surface

The second follows from the first and using the divergence theorem:

$$\int_{\forall} \nabla \cdot \underline{\omega}\, d\forall = \int_{A} \underline{\omega} \cdot \underline{n}\, dA = 0$$

Application to a vortex tube results in the following

Minus sign due to outward normal

$$\underbrace{\int_{A1} \underline{\omega}\cdot\underline{n}\,dA}_{-\Gamma_1}+\underbrace{\int_{A2} \underline{\omega}\cdot\underline{n}\,dA}_{\Gamma_2}=0$$

Or $\Gamma_1 = \Gamma_2$

Helmholtz Theorem #2:

The circulation around a given vortex line (i.e., the strength of the vortex tube) is constant along its length.

This result can be put in the form of a simple one-dimensional incompressible continuity equation. Define $\omega_1$ and $\omega_2$ as the average vorticity across $A_1$ and $A_2$, respectively

$$\omega_1 A_1 = \omega_2 A_2$$

which relates the vorticity strength to the cross sectional area changes of the tube.

2. Vortex dynamics

Consider the substantial derivative of the circulation assuming incompressible flow and conservative body forces

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint \underline{V}\cdot \underline{dx}$$
$$= \oint \frac{D\underline{V}}{Dt}\cdot dx + \oint \underline{V}\cdot \frac{D}{Dt}\underline{dx}$$

From the N-S equations we have

$$\frac{D\underline{V}}{Dt} = \frac{1}{\rho}\underline{f} - \frac{\nabla p}{\rho} + \nu\nabla^2\underline{V}$$
$$= -\nabla\left(F + {p}/{\rho}\right) + \nu\nabla^2\underline{V}$$

Define $\underline{f} = -\nabla F$ for the gravitational body force F=ρgz.

Also, $\frac{D}{Dt}\underline{dx} = d\,\frac{D\underline{x}}{Dt} = \underline{dV}$

$$\frac{D\Gamma}{Dt} = \underbrace{\oint\left[-\nabla\left(F + p/\rho\right)\right]\cdot d\,\underline{x}}_{-\oint dF - \oint \frac{dp}{\rho}} + \oint\left[\nu\nabla^2\underline{V}\right]\cdot d\,\underline{x} + \underbrace{\oint \underline{V}\cdot d\underline{V}}_{\frac{1}{2}\oint d(\underline{V}\cdot\underline{V})}$$

$$= \underbrace{\oint\left[-dF - \frac{dp}{\rho} + \frac{1}{2}dV^2\right]} + \nu\oint\nabla^2\underline{V}\cdot d\,\underline{x}$$

*=0 since integration is around a closed contour and F,p, & V are single valued!*

$$\frac{D\Gamma}{Dt} = \nu\oint\nabla^2\underline{V}\cdot dx = -\nu\oint\nabla\times\underline{\omega}\cdot dx$$

$$\nabla\times\underbrace{\left(\nabla\times\underline{V}\right)}_{\underline{\omega}} = \underbrace{\nabla\left(\nabla\cdot\underline{V}\right)}_{=0} - \nabla^2\underline{V}$$

Implication: The circulation around a material loop of particles changes only if the net viscous force on those particles gives a nonzero integral.

If $\nu = 0$ or $\underline{\omega} = 0$ (i.e., inviscid or irrotational flow, respectively) then

$$\frac{D\Gamma}{Dt} = 0$$

The circulation of a material loop never changes

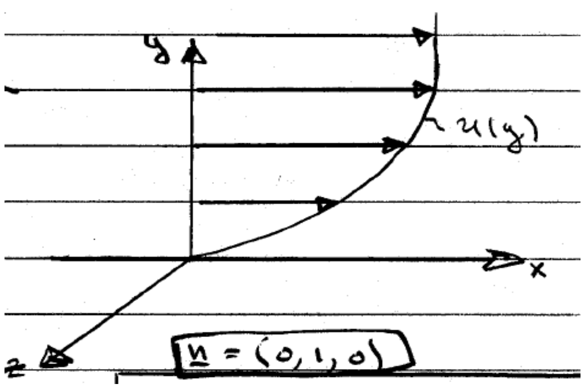
<u>Kelvins Circulation Theorem</u>: for an ideal fluid (i.e. inviscid, incompressible, and irrotational) acted upon by conservative forces (e.g., gravity) the circulation is constant about any closed material contour moving with the fluid, which leads to:

<u>Helmholtz Theorem #3</u>: No fluid particle can have rotation if it did not originally rotate. Or, equivalently, in the absence of rotational forces, a fluid that is initially irrotational remains irrotational. In general, we can conclude that vortices are preserved as time passes. Only through the action of viscosity can they decay or disappear.

Kelvins Circulation Theorem and Helmholtz Theorem #3 are very important in the study of inviscid flow. The important conclusion is reached that a fluid that is initially irrotational remains irrotational, which is the justification for ideal-flow theory.

In a real viscous fluid, vorticity is generated by viscous forces. Viscous forces are large near solid surfaces as a result of the no-slip condition. On the surface there is a direct relationship between the viscous shear stress and the vorticity.

Consider a 1-D flow near a wall:




The viscous stresses are given by:

$\tau_{ij} n_j$ where $\tau_{ij} = \mu \varepsilon_{ij}$

$$\tau_{11} n_1 + \tau_{12} n_2 + \tau_{13} n_3 = \tau_x$$

$$\tau_{21} n_1 + \tau_{22} n_2 + \tau_{23} n_3 = \tau_y$$

$$\tau_{31} n_1 + \tau_{32} n_2 + \tau_{33} n_3 = \tau_z$$

$$\tau_{12} = \mu \varepsilon_{12} = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mu \frac{\partial u}{\partial y}$$

$$\tau_{22} = \mu \varepsilon_{22} = 2\mu \frac{\partial v}{\partial y} = 0$$

$$\tau_{32} = \mu \varepsilon_{32} = \mu \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) = 0$$

NOTE: the only component of $\underline{\omega}$ is $\omega_z$. Actually, this is a general result in that it can be shown that $\underline{\omega}_{surface}$ is perpendicular to the limiting streamline.

Which shows that

$$\tau_x = \mu \frac{\partial u}{dy} \qquad \tau_y = \tau_z = 0$$

However from the definition vorticity we also see that

$$\tau_x = \mu \frac{\partial u}{\partial y} = -\mu \omega_z$$

i.e., the wall vorticity is directly proportional to the wall shear stress. This analysis can be extended for general 3D flow.

Rotation tensor

$$\tau_{ij} n_j = -\mu \omega_{ij} n_j \quad \text{at a fixed solid wall}$$

True since at a wall with coordinate $x_2$, $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_3} = 0$ and from continuity $\frac{\partial v}{\partial x_2} = 0$

Once vorticity is generated, its subsequent behavior is governed by the vorticity equation.

N-S $$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} = -\nabla \left( p / \rho \right) + \nu \nabla^2 \underline{V}$$ *neglect* $\underline{f}$

Or $$\frac{\partial \underline{V}}{\partial t} + \nabla \left( \frac{1}{2} \underline{V} \cdot \underline{V} \right) - \underline{V} \times \underline{\omega} = -\nabla \left( p / \rho \right) + \nu \nabla^2 \underline{V}$$

The vorticity equation is obtained by taking the curl of this equation. (Note $\nabla \times (\nabla \theta) = 0$).

$$\frac{\partial \underline{\omega}}{\partial t} - \underbrace{\nabla \times \left( \underline{V} \times \underline{\omega} \right)} = \nu \nabla^2 \underline{\omega}$$

Rate of change of $\underline{\omega}$ =

$$= \underline{V}(\nabla \cdot \underline{\omega}) - \underline{\omega}(\nabla \cdot \underline{V}) - (\underline{V} \cdot \nabla)\underline{\omega} + (\underline{\omega} \cdot \nabla)\underline{V}$$

Therefore, the transport Eq. for $\underline{\omega}$ is

$$\underbrace{\frac{\partial \underline{\omega}}{\partial t} + (\underline{V} \cdot \nabla)\underline{\omega}}_{\frac{D\underline{\omega}}{Dt}} = \left( \underline{\omega} \cdot \nabla \right) \underline{V} + \nu \nabla^2 \underline{\omega}$$

Rate of deforming vortex lines

Rate of viscous diffusion of $\underline{\omega}$

$$\frac{\partial \underline{\omega}}{\partial t} + \left( u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) \underline{\omega} = \left( \omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z} \right) \underline{V} + \nu \nabla^2 \underline{\omega}$$

$$\frac{\partial \omega_x}{\partial t} + u \frac{\partial \omega_x}{\partial x} + v \frac{\partial \omega_x}{\partial y} + w \frac{\partial \omega_x}{\partial z} = \underbrace{\omega_x \frac{\partial u}{\partial x}}_{\textit{Stretching}} + \underbrace{\omega_y \frac{\partial u}{\partial y} + \omega_z \frac{\partial u}{\partial z}}_{\textit{turning}} + \nu \nabla^2 \omega_x$$

$$\frac{\partial \omega_y}{\partial t} + u \frac{\partial \omega_y}{\partial x} + v \frac{\partial \omega_y}{\partial y} + w \frac{\partial \omega_y}{\partial z} = \omega_x \frac{\partial v}{\partial x} + \omega_y \frac{\partial v}{\partial y} + \omega_z \frac{\partial v}{\partial z} + \nu \nabla^2 \omega_y$$

$$\frac{\partial \omega_z}{\partial t} + u \frac{\partial \omega_z}{\partial x} + v \frac{\partial \omega_z}{\partial y} + w \frac{\partial \omega_z}{\partial z} = \omega_x \frac{\partial w}{\partial x} + \omega_y \frac{\partial w}{\partial y} + \omega_z \frac{\partial w}{\partial z} + \nu \nabla^2 \omega_z$$

Note: (1) Equation does not involve p explicitly
(2) for 2-D flow $(\underline{\omega} \cdot \nabla)\underline{V} = 0$ since $\underline{\omega}$ is perp. to $\underline{V}$ and there can be no deformation of $\underline{\omega}$, ie

$$\frac{D\underline{\omega}}{Dt} = \nu \nabla^2 \underline{\omega}$$

In order to determine the pressure field in terms of the vorticity, the divergence of the N-S equation is taken.

$$\nabla \cdot \left[ \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} = -\nabla\left(p/\rho\right) + \nu \nabla^2 \underline{V} \right]$$

$$\nabla^2 (p/\rho) = -\nabla \cdot \left[\underline{V} \cdot \nabla \underline{V}\right]$$ *Poisson Eq. for p*

$$= -\frac{1}{2}\nabla^2\left(\underline{V} \cdot \underline{V}\right) + \underline{V} \cdot \nabla^2 \underline{V} + \underline{\omega} \cdot \underline{\omega}$$

does not depend explicitly on ν

<u>Derivation of pressure Poisson equation</u>:

Three vector identities to be used:

(1) $\mathbf{V} \cdot \nabla \mathbf{V} = \frac{1}{2}\nabla\left(\mathbf{V} \cdot \mathbf{V}\right) - \mathbf{V} \times \left(\nabla \times \mathbf{V}\right)$

(2) $\nabla \cdot \left(\mathbf{a} \times \mathbf{b}\right) = \mathbf{b} \cdot \left(\nabla \times \mathbf{a}\right) - \mathbf{a} \cdot \left(\nabla \times \mathbf{b}\right)$

(3) $\nabla \times \left(\nabla \times \mathbf{a}\right) = -\nabla^2 \mathbf{a} + \nabla\left(\nabla \cdot \mathbf{a}\right)$

Pressure Poisson equation in vector form:

$$\nabla^2 \left(\frac{p}{\rho}\right) = -\nabla \cdot \left(\mathbf{V} \cdot \nabla \mathbf{V}\right)$$

$$= -\nabla \cdot \left(\frac{1}{2}\nabla\left(\mathbf{V} \cdot \mathbf{V}\right) - \mathbf{V} \times \left(\nabla \times \mathbf{V}\right)\right)$$

$$= -\frac{1}{2}\nabla^2\left(\mathbf{V} \cdot \mathbf{V}\right) + \nabla \cdot \left(\mathbf{V} \times \boldsymbol{\omega}\right)$$

$$=-\frac{1}{2}\nabla^2(\mathbf{V}\cdot\mathbf{V})+\boldsymbol{\omega}\cdot(\nabla\times\mathbf{V})-\mathbf{V}\cdot(\nabla\times\boldsymbol{\omega})$$

$$=-\frac{1}{2}\nabla^2(\mathbf{V}\cdot\mathbf{V})+\boldsymbol{\omega}\cdot\boldsymbol{\omega}-\mathbf{V}\cdot(\nabla\times(\nabla\times\mathbf{V}))$$

$$=-\frac{1}{2}\nabla^2(\mathbf{V}\cdot\mathbf{V})+\boldsymbol{\omega}\cdot\boldsymbol{\omega}-\mathbf{V}\cdot\left[-\nabla^2\mathbf{V}+\nabla\left(\cancel{\nabla\cdot\mathbf{V}}\right)\right]$$

$$=-\frac{1}{2}\nabla^2(\mathbf{V}\cdot\mathbf{V})+\mathbf{V}\cdot\nabla^2\mathbf{V}+\boldsymbol{\omega}\cdot\boldsymbol{\omega}$$

Pressure Poisson equation in tensor form:

$$\nabla^2\left(\frac{p}{\rho}\right)=-\frac{1}{2}\nabla^2(\mathbf{V}\cdot\mathbf{V})+\mathbf{V}\cdot\nabla^2\mathbf{V}+\boldsymbol{\omega}\cdot\boldsymbol{\omega}$$

$$=-\frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_i}\left[(u_je_j)\cdot(u_ke_k)\right]+(u_ie_i)\cdot\frac{\partial^2(u_ke_k)}{\partial x_j\partial x_j}+(\nabla\times\mathbf{V})\cdot(\nabla\times\mathbf{V})$$

$$=-\frac{1}{2}\frac{\partial^2}{\partial x_i\partial x_i}(u_ju_k\delta_{jk})+u_i\delta_{ik}\cdot\frac{\partial^2u_k}{\partial x_j\partial x_j}+\left(\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}e_i\right)\cdot\left(\varepsilon_{lmn}\frac{\partial u_n}{\partial x_m}e_l\right)$$

$$=-\frac{1}{2}\frac{\partial^2(u_ju_j)}{\partial x_i\partial x_i}+u_i\frac{\partial^2u_i}{\partial x_j\partial x_j}+\left(\varepsilon_{ijk}\varepsilon_{lmn}\frac{\partial u_k}{\partial x_j}\frac{\partial u_n}{\partial x_m}\right)(e_i\cdot e_l)$$

$$=-\frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i}(u_ju_j)\right)+u_i\frac{\partial^2u_i}{\partial x_j\partial x_j}+\left(\varepsilon_{ijk}\varepsilon_{lmn}\frac{\partial u_k}{\partial x_j}\frac{\partial u_n}{\partial x_m}\right)\delta_{il}$$

$$=-\frac{1}{2}\frac{\partial}{\partial x_i}\left(2u_j\frac{\partial u_j}{\partial x_i}\right)+u_i\frac{\partial^2u_i}{\partial x_j\partial x_j}+(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km})\frac{\partial u_k}{\partial x_j}\frac{\partial u_n}{\partial x_m}$$

$$=-\frac{\partial}{\partial x_i}\left(u_j\frac{\partial u_j}{\partial x_i}\right)+u_i\frac{\partial^2u_i}{\partial x_j\partial x_j}+\delta_{jm}\delta_{kn}\frac{\partial u_k}{\partial x_j}\frac{\partial u_n}{\partial x_m}-\delta_{jn}\delta_{km}\frac{\partial u_k}{\partial x_j}\frac{\partial u_n}{\partial x_m}$$

$$=-\left(\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_i}+u_j\frac{\partial^2u_j}{\partial x_i\partial x_i}\right)+u_i\frac{\partial^2u_i}{\partial x_j\partial x_j}+\frac{\partial u_k}{\partial x_j}\frac{\partial u_k}{\partial x_j}-\frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_k}$$

$$=-\frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_k}$$

# 3. Kinematic Decomposition of flow fields

Previously, we discussed the decomposition of fluid motion into translation, rotation, and deformation. This was done locally for a fluid element. Now we shall see that a global decomposition is possible.

Helmholtz's Decomposition: any continuous and finite vector field can be expressed as the sum of the gradient of a scalar function $\phi$ plus the curl of a zero-divergence vector A. The vector A vanishes identically if the original vector field is irrotational.

$$\underline{V} = \underline{V}^{\omega} + \underline{V}^{\phi}$$

Where

$$\underline{\omega} = \nabla \times \underline{V}^{\omega}$$

$$0 = \nabla \times V^{\phi}$$

The irrotational part of the velocity field can be expressed as the gradient of a scalar

$$\rightarrow \quad \underline{V}^{\phi} = \nabla \phi$$

If $\quad \nabla \cdot \underline{V} = \nabla \cdot \underline{V}^{\omega} + \nabla \cdot \underline{V}^{\phi} = 0$

Then $\quad \nabla^2 \phi = 0 \quad$ *The GDE for $\phi$ is the Laplace Eq.*

And $\quad \underline{V}^{\omega} = \nabla \times \underline{A} \quad$ *Since* $\nabla \cdot \left(\nabla \times \underline{A}\right) = 0$

$$\nabla \times \underline{V}^{\omega} = \underline{\omega} = \nabla \times \nabla \times \underline{A}$$

$$= -\nabla^2 \underline{A} + \nabla(\nabla \cdot \underline{A})$$

Again, by vector identity

i.e $\nabla^2 \underline{A} = -\underline{\omega}$

The solution of this equation is $\underline{A} = \frac{1}{4\pi}\int \frac{\underline{\omega}}{|\underline{R}|}\, d\forall$

Thus $\underline{V}^{\omega} = -\frac{1}{4\pi}\int \frac{\underline{R}\times\underline{\omega}}{|\underline{R}|^3}\, d\forall$

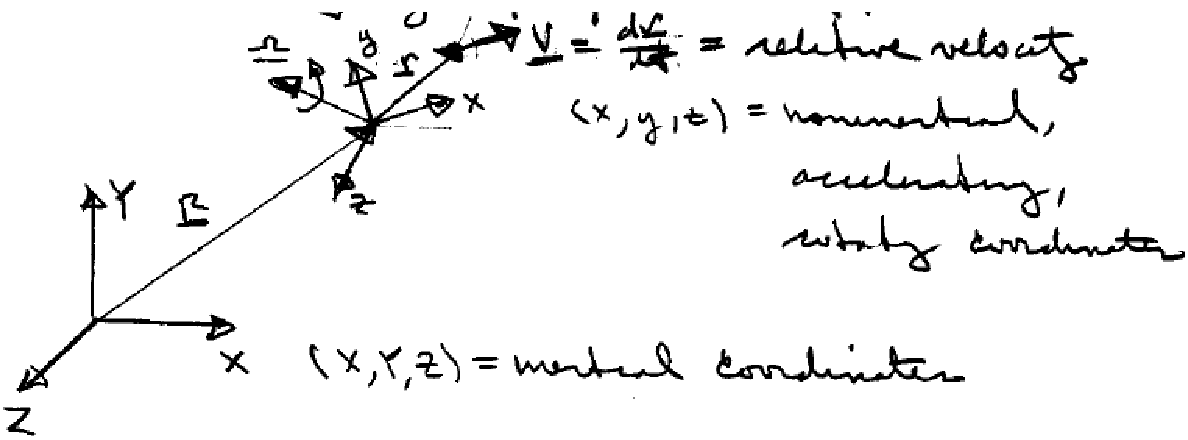
Which is known as the Biot-Savart law.

The Biot-Savart law can be used to compute the velocity field induced by a known vorticity field. It has many useful applications, including in ideal flow theory (e.g., when applied to line vortices and vortex sheets it forms the basis of computing the velocity field in vortex-lattice and vortex-sheet lifting-surface methods).

The important conclusion from the Helmholtz decomposition is that any incompressible flow can be thought of as the vector sum of rotational and irrotational components. Thus, a solution for irrotational part $\underline{V}^{\phi}$ represents at least part of an exact solution. Under certain conditions, high Re flow about slender bodies with attached thin boundary layer and wake, $\underline{V}^{\omega}$ is small over much of the flow field such that $\underline{V}^{\phi}$ is a good approximation to $\underline{V}$. This is probably the strongest justification for ideal-flow theory. (*incompressible, inviscid, and irrotational flow).*

Non-inertial Reference Frame

Thus far we have assumed use of an inertial reference frame (i.e. fixed with respect to the distant stars in deriving the CV and differential form of the momentum equation). However, in many cases non-inertial reference frames are useful (e.g. rotational machinery, vehicle dynamics, geophysical applications, etc).




$$\underline{a}_i = \frac{D\underline{V}}{Dt} + \underline{a}_{rel}$$

$$\sum \underline{F} = m\underline{a}_i = m\left(\frac{D\underline{V}}{Dt} + \underline{a}_{rel}\right)$$

$$\sum \underline{F} - m\underline{a}_{rel} = m\frac{D\underline{V}}{Dt}$$

i.e Newton's law applies to non-inertial frame with addition of known inertial force terms

$$\underline{S}_i = \underline{R} + \underline{r}$$

$$V_i = \underline{V} + \frac{d\underline{R}}{dt} + \underline{\Omega} \times \underline{r}$$

$$\underline{a}_i = \frac{D\underline{V}}{Dt} + \frac{d^2\underline{R}}{dt^2} + \frac{d\underline{\Omega}}{dt} \times \underline{r} + 2\underline{\Omega} \times \underline{V} + \underline{\Omega} \times (\underline{\Omega} \times \underline{r})$$

$$= \frac{D\underline{V}}{Dt} + \underline{a}_{rel}$$

*3rd term from fact that (x,y,z) rotating at Ω(t).*

$\frac{d^2 \underline{R}}{dt^2}$ = *acceleration (x,y,z)*

$\frac{d\underline{\Omega}}{dt} \times \underline{r}$ = *angular acceleration (x,y,z)*

$2\underline{\Omega} \times \underline{V}$ = *Coriolis acceleration*

$\underline{\Omega} \times (\underline{\Omega} \times \underline{r})$ = *centripetal acceleration (=-$\Omega^2 L$, where L = normal distance from $\underline{r}$ to axis of rotation $\underline{\Omega}$).*

Since $\underline{R}$ and $\underline{\Omega}$ assumed known, although more complicated, we are simply adding known inhomogeneities to the momentum equation.

CV form of Momentum equation for non-inertial coordinates:

$$\sum \underline{F} - \int_{CV} \underline{a}_{rel}\rho\, d\forall = \frac{d}{dt}\int_{CV} \underline{V}\rho\, d\forall + \int_{CS} \underline{V}\rho\underline{V}_R \cdot \underline{n}\, dA$$

where $\underline{V}_R$ is the velocity of the CV relative to the non-inertial coordinates ($x$,$y$,$z$).

Differential form of momentum equation for non-inertial coordinates:

$$\rho\left[\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right] = \underbrace{-\rho\underline{a}_{rel}}_{body\quad force} - \nabla\left(p+\gamma z\right) + \mu\nabla^2\underline{V}$$

where

$$\underline{a}_{rel} = \underline{\ddot{R}} + 2\underline{\Omega}\times\underline{V} + \underline{\Omega}\times\left(\underline{\Omega}\times\underline{r}\right) + \underline{\dot{\Omega}}\times\underline{r}$$

All terms in $\underline{a}_{rel}$ seldom act in unison (e.g. geophysical flows):

$\underline{\ddot{R}} \sim 0$ earth not accelerating relative to distant stars

$\underline{\dot{\Omega}} \sim 0$ for earth

$\underline{\Omega}\times\left(\underline{\Omega}\times\underline{r}\right) \sim 0$ g nearly constant with latitude

$\therefore 2\underline{\Omega}\times\underline{V}$ most important!

$$\underline{a}_i = \frac{D\underline{V}}{Dt} + R_0^{-1}\left(2\underline{\Omega}\times\underline{V}\right) \qquad \underline{V} = \frac{\underline{V}}{V_0},\ t = \frac{tV_0}{L}$$

$$R_0 = Rossby\ \# = \frac{V_0^2/L}{\Omega V_0} = \frac{V_0}{\Omega L}$$

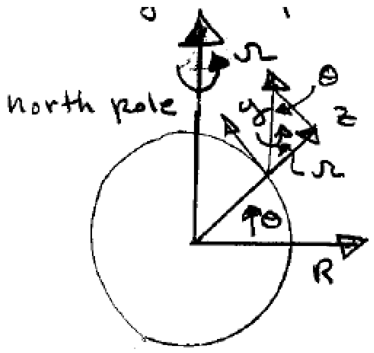
if L is large, i.e., comparable to the order of magnitude of the earth radius, $R_0<1$, then Coriolis term is larger than the inertia terms and is important.

Example of Non-inertial Coordinates:
Geophysical fluids dynamics

Atmosphere and oceans are naturally studied using non-inertial coordinate system rotating with the earth. Two primary forces are Coriolis force and buoyancy force due to density stratification ρ = ρ(T). Both are studied using Boussinesq approximations (ρ = constant, except $-\rho(T)g\hat{k}$ term; and μ, k, $C_p$ = constant) and thin layer on rotating surface assumption $\left(\frac{W}{U} \sim \frac{H}{L}\right)$.

Differences between atmosphere and oceans: lateral boundaries (continents) in oceans; currents in ocean (gulf and Kuroshio stream) along western boundaries; clouds and latent heat release in atmosphere due to moisture condensation; $V_{ocean}$ = 0.1~1 or 2 m/s and $V_{atmosphere}$ 10~20 m/s

H << L = 0 (radius of earth = 6371 km)
Therefore, one can neglect curvature of earth and replace spherical coordinates by local Cartesian tangent plane coordinates.




x = eastward
y = northward
z = upward

$\Omega = 2\pi \text{ rad/day} = .73\times10^{-4}\ \text{s}^{-1}$

θ = latitude
>0 northern hemisphere
<0 Southern hemisphere

$\Omega_x = 0$
$\Omega_y = \Omega\cos\theta$
$\Omega_z = \Omega\sin\theta$

Coriolis force = $2\underline{\Omega}\times\underline{V}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Omega_x & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix}$$

0 since w << v

$$= 2\Omega\left[\hat{i}\left(\cancel{w\cos\theta} - v\sin\theta\right) + \hat{j}\,u\sin\theta - \hat{k}\,u\cos\theta\right]$$

$$= -fv\,\hat{i} + fu\,\hat{j} - 2\Omega\cos\theta\,u\,\hat{k} \qquad f = 2\Omega\sin\theta$$

= planetary vorticity
= 2 * vertical component Ω

f > 0 northern hemisphere
f < 0 southern hemisphere
f = ±Ω at poles
f = 0 at equator

Person spins at Ω

Person translates with inertial period $T_i = \dfrac{2\pi}{f}$

## Equations of Motion

$$\nabla \cdot \underline{V} = 0$$

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \upsilon \nabla^2 u$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \upsilon \nabla^2 v$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{\rho g}{\rho_0} + \upsilon \nabla^2 w$$

*vertical component Ω negligible due to thin layer assumption, i.e., magnitude of* $2\Omega \cos\theta u <<$ *other terms*

$$\rho = \rho_0 \left[1 - \alpha (T - T_0)\right]$$

p, ρ = perturbation from hydrostatic condition

## Geostrophic Flow: quasi-steady, large-scale motions in atmosphere or ocean far from boundaries

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \qquad fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{D\underline{V}}{Dt} \sim 0\left(\frac{U^2}{L}\right) \qquad f\underline{V} \sim 0\,(fU)$$

U,L = horizontal scales

$$\text{Rossby number} = \frac{U}{fL}$$

Atmosphere: U ~ 10 m/s; f = $10^{-4}$ Hz; L ~ 1000 km; and $R_0$ = 0.1

Ocean: U ~ 0.1 m/s; f = $10^{-4}$ Hz; L ~ 1000 km; and $R_0$ = 0.01

Therefore, neglect $\frac{D\underline{V}}{Dt}$ and since there are no boundaries, neglect $\nu\nabla^2\underline{V}$.

Z momentum → $\frac{\partial p}{\partial z} = -\rho g$ baroclinic (i.e. p = p(T))
and can be used to eliminate p in above equations whereby (u,v) = f(T(z)), which is called thermal wind but not considered here.
If we neglect ρ=ρ(T) effects, (u,v) = f(p) and can be determined from measured p(x,y). Not valid near the equator ($\pm$ 3°) where f is small.

$$\left(u\,\hat{i} + v\,\hat{j}\right)\cdot\nabla p = \frac{1}{\rho_0 f}\left(-\frac{\partial p}{\partial y}\hat{i} + \frac{\partial p}{\partial x}\,\hat{j}\right)\cdot\left(\frac{\partial p}{\partial x}\,\hat{i} + \frac{\partial p}{\partial y}\,\hat{j}\right)$$

$$= 0$$

i.e $\underline{V}$ is perpendicular to $\nabla p$ → horizontal velocity is along (and not across) lines of constant horizontal pressure, which is reason isobars and stream lines coincide on a weather map!

# Ekman Layer on Free Surface: effects of friction near boundaries

Viscous layers:

Sudden acceleration flat plate: $u_t = \nu u_{yy}$ $\quad \delta = 3.64\sqrt{\nu t}$

$$u(y,0) = 0$$
$$u(0,t) = U$$
$$u(\infty,t) = 0$$

Oscillating flat plate: $u_t = \upsilon u_{yy}$ $\quad \delta = 6.5\sqrt{\nu/\omega}$

$$u(0,t) = U_0 \cos \omega t$$
$$u(\infty,t) = 0$$

Flat plate boundary layer:

$$u_x + v_y = 0$$
$$uu_x + vu_y = \upsilon u_{yy}$$
$$\delta = 4.9\sqrt{\nu x/U} \qquad u(x,0) = 0$$
$$u(x,\infty) = U$$

For Ekman layer viscous effects due to wind shear $\tau(x)$. Assume horizontal uniformity (i.e $p_x = p_y = 0$), which is justified for L ~ 100 km and H ~ 50 m. However, can be included easily if assume $p \neq p(z)$ such that geostropic solution is additive and combined solution recovers former for large depths $z/\delta \to -\infty$.

$$-fv = \nu u_{ZZ} \qquad\qquad fu = \nu v_{ZZ}$$

$\mu u_z = \tau$ at z = 0

$v_z = 0$ at z = 0

$(u,v) = 0$ at z =-∞

$\underline{\tau} = \tau \hat{i}$

$\tau = .002\rho_{air}(v_{wind} - u(0))$

Multiply v-equation by $i = \sqrt{-1}$ and add to u-equation:

$$\frac{d^2V}{dz^2} = \frac{i\,f}{\nu}V \qquad \begin{aligned} V &= u + i\,v \\ &= complex\ velocity \end{aligned} \qquad z = x + i\,y$$

$$V = Ae^{(1+i)z/\delta} + Be^{-(1+i)z/\delta}$$

$\delta = \sqrt{\dfrac{2\nu}{f}} =$ Ekman layer thickness

B = 0 for u(-∞), v(-∞) = 0

$\mu \dfrac{dV}{dz} = \tau$ at z = 0

$$\rightarrow A = \frac{\tau\delta(1-i)}{2\rho\nu}$$

i.e. $u = \dfrac{\tau/\rho}{\sqrt{f\upsilon}} e^{z/\delta} \cos\left(-\dfrac{z}{\delta} + \dfrac{\pi}{4}\right)$ and

$$v = \frac{\tau / \rho}{\sqrt{f\upsilon}} e^{z/\delta} \sin\left( -\frac{z}{\delta} + \frac{\pi}{4} \right)$$

F. Nansen (1902) observed drifting arctic ice drifted 20-40$^0$ to the right of the wind, which he attributed to Coriolis acceleration. His student Ekman (1905) derived the solution.

Recall f < 0 in southern hemisphere, so the drift is to the left of τ.

$\int_{-\infty}^{0} u\,dz = 0$ $\int_{-\infty}^{0} v\,dz = -\frac{\tau}{\rho f}$ net transport is to right of $\tau$ and independent $\nu$

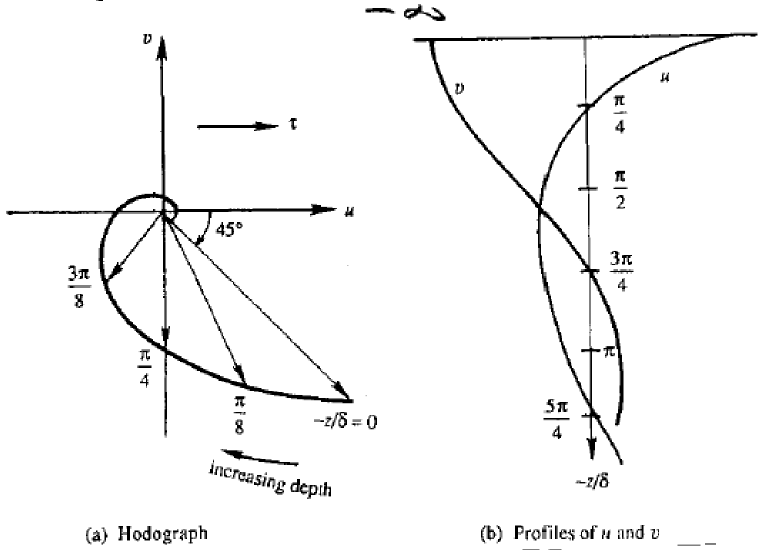



Fig. 13.7 Ekman layer at a free surface. The left panel shows velocity vectors at various depths; values of $-z/\delta$ are indicated along the curve traced out by the tip of the velocity vectors. The right panel shows vertical distributions of $u$ and $v$.

Similar solution for impulsive wind:

$$u_t = \upsilon u_{zz}, \ \mu u_z = \tau \quad z = 0, \ u = 0 \quad z = -\infty, u(z,0) = 0$$

$$u_0 = \frac{2\tau}{\mu}\sqrt{\frac{\upsilon t}{\pi}}$$

laminar solution*:*

$u_0(V_{wind} = 6\ m/s, T = 20^0 C) =$ *0.6 m/s after one min., 2.3 m/s after one hour*

turbulent $\nu_t$ solution*:(more realistic)*
*$u_0$=0.2 m/s after 1 hr (3 % $v_{wind}$)*

For Ekman layer similar conditions θ = $40^0$ N ,

Laminar solution $u_0$ = 2.7 m/s at D = 45 cm, which are too high/low; however, using turbulent $\nu_t$, $u_0$ = 2 cm/s and D = 100 m, which is more realistic.