MPP-Based Dimension Reduction Method for RBDO Problems with Correlated Input Variables

Yoojeong Noh¹, K.K. Choi², and Ikjin Lee³

Department of Mechanical & Industrial Engineering College of Engineering, The University of Iowa
Iowa City, IA 52242, U.S.A.

In reliability-based design optimization (RBDO) problems with correlated input variables, a joint cumulative distribution function (CDF) needs to be obtained to transform, using the Rosenblatt transformation, the correlated input variables into independent standard Gaussian variables for the reliability analysis. However, a true joint CDF requires an infinite number of data to be obtained, so in this paper, a copula is used to model the joint CDF using marginal CDFs and correlation parameters obtained from samples, which are available in practical applications. Using the joint CDF modeled by the copula, the transformation can be carried out based on the first order reliability method (FORM), which has been commonly used in reliability analysis. However, the FORM may yield different reliability analysis results with some errors for different transformation ordering of input variables due to the nonlinearities of differently transformed constraint functions. For this, the most probable point (MPP) based dimension reduction method (DRM), which more accurately and efficiently calculates the probability of failure than the FORM and the second order reliability method (SORM), respectively, is proposed to use to reduce the effect of transformation ordering in the inverse reliability analysis, and thus RBDO. To study the effect of transformation ordering on RBDO results, several numerical examples are tested using two different reliability methods, the FORM and DRM.

Nomenclature

\[n\] = Number of random variables
\[ndv\] = Number of design variables
\[d\] = Vector of design variables, \(d = [d_1, \ldots, d_{ndv}]\)
\[X\] = Vector of random variables, \(X = [X_1, \ldots, X_n]\)
\[x\] = Realization of vector \(X\), \(x = [x_1, \ldots, x_n]\)
\[x^*\] = FORM-based MPP
\[x_{DRM}\] = DRM-based MPP
\[U\] = Vector of independent standard Gaussian variables, \(U = [U_1, \ldots, U_n]\)
\[u\] = Realization of vector \(U\), \(u = [u_1, \ldots, u_n]\)
\[Y\] = Vector of correlated standard Gaussian variables, \(Y = [Y_1, \ldots, Y_n]\)
\[y\] = Realization of vector \(Y\), \(y = [y_1, \ldots, y_n]\)
\[\rho_{ij}\] = Pearson’s correlation coefficient between \(X_i\) and \(X_j\)
\[P\] = Covariance matrix of \(X\), \(\{\rho_{ij}\}\)

¹Graduate Research Assistant, e-mail: noh@engineering.uiowa.edu.
²Roy J. Carver Professor, Corresponding Author, e-mail: kkchoi@engineering.uiowa.edu.
³Graduate Research Assistant, e-mail: ilee@engineering.uiowa.edu.

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\( \mathbf{P}' \) = Covariance matrix of \( \mathbf{y} \), \( \{\rho^y_{ij}\} \)

\( \Phi(\cdot) \) = Marginal Gaussian CDF

\( \Phi_p(\cdot|\mathbf{P}') \) = Multivariate Gaussian CDF with \( \mathbf{P}' \)

\( F_{x_i}(\cdot) \) = Marginal CDF of \( X_i \)

\( F_{x_1,\ldots,x_n}(\cdot) \) = Joint CDF of \( X_1,\ldots,X_n \)

\( f_{x_1,\ldots,x_n}(\cdot) \) = Joint PDF of \( X_1,\ldots,X_n \)

\( F_{x_i}(\cdot|\cdot) \) = Conditional CDF of \( X_i \)

\( \theta \) = Matrix of correlation parameters of \( X_1,\ldots,X_n \)

\( C(\cdot|\theta) \) = Copula with \( \theta \)

\( \tau \) = Kendall’s tau

I. Introduction

In many RBDO problems, input random variables, such as the material properties and fatigue parameters, are correlated.\(^1\)-\(^3\) For the RBDO problem with the correlated input variables, the joint CDF of the input variables should be available to transform the correlated input variables into the independent standard Gaussian variables, by using the Rosenblatt transformation,\(^4\) to carry out the inverse reliability analysis. However, in industrial applications, often only the marginal CDFs and limited paired sampled data are available using experimental testing, and the input joint CDF is very difficult to obtain. In this paper, a copula, which links the joint CDF and marginal CDFs, is used to model the joint CDF. Since the copula only requires marginal CDFs and correlation parameters, which are often available in industrial applications, to model the joint CDF, the joint CDF can be readily obtained. Thus, it is valuable to use the copula for modeling the joint CDFs in practical applications with correlated input variables.

Once the joint CDF is obtained using the copula, the Rosenblatt transformation can be utilized to transform the original random variables into the independent standard Gaussian variables for the inverse reliability analysis. For the inverse reliability analysis, the FORM is most often used. On the other hand, depending on the types of the joint input CDF, if a different order of Rosenblatt transformation is used, even the constraint function that was mildly nonlinear with respect to the original random variables could become highly nonlinear in terms of the independent standard Gaussian variables. First, obviously, if the input variables are independent (i.e., the joint CDF is a simple multiplication of the marginal CDFs), there is no effect of transformation ordering. Second, if the input variables have the joint CDF modeled by a Gaussian copula, the effect of transformation ordering still does not exist because the Gaussian copula makes the Rosenblatt transformation become linear, which is independent of orderings. However, if the input variables have a non-Gaussian joint CDF modeled by a non-Gaussian copula, which often occurs in industrial applications,\(^5\), since the Rosenblatt transformation becomes highly nonlinear, the different ordering can significantly affect the nonlinearity of the transformed constraints. In this case, if the FORM is used, the inverse reliability analysis results could be very different for the different ordering since the FORM uses a linear approximation of the constraint to estimate the probability of failure. This effect of transformation ordering in RBDO will be unacceptable and make the user significantly concerned.

To reduce the dependency of the inverse reliability analysis result and thus the RBDO result on the ordering of the Rosenblatt transformation, it is proposed to use the MPP-based DRM,\(^6\) for the inverse reliability analysis in this paper. With the accuracy of the inverse reliability analysis using the DRM even for highly nonlinear constraint functions, it is shown that the RBDO results are becoming less dependent on the Rosenblatt transformation ordering of the input variables.

II. Modeling of Joint CDF using Copula

As mentioned earlier, if the input variables are correlated, it is often too difficult to obtain the true joint CDF in practical industrial applications with only limited experimental data. In this paper, a copula is used to model the joint CDF using marginal CDFs and correlation measures that are calculated from the experimental data. The definition of copula and the correlation measures associated with copulas are explained in this section.
A. Definition of Copula

Copula is originated from a Latin word for “link” or “tie” that connects two different things. In statistics, the definition of copula is stated by Roser (1999): “Copulas are functions that join or couple multivariate distribution functions to their one-dimensional marginal distribution functions. Alternatively, copulas are multivariate distribution functions whose one-dimensional margins are uniform on the interval [0, 1].”

According to Sklar’s theorem, if the random variables have a joint distribution \( F_{x_1\ldots x_n}(x_1,\ldots,x_n) \) with marginal distributions, \( F_{x_1}(x_1),\ldots,F_{x_n}(x_n) \), then there exists an n-dimensional copula \( C \) such that

\[
F_{x_1\ldots x_n}(x_1,\ldots,x_n) = C\left(F_{x_1}(x_1),\ldots,F_{x_n}(x_n)|\theta\right)
\]

where \( \theta \) is the matrix of the correlation parameters of \( x_1,\ldots,x_n \). If marginal distributions are all continuous, then \( C \) is unique. Conversely, if \( C \) is an n-dimensional copula and \( F_{x_1}(x_1),\ldots,F_{x_n}(x_n) \) are the marginal distributions, then \( F_{x_1\ldots x_n}(x_1,\ldots,x_n) \) is the joint distribution. By taking the derivative of Eq. (1), the joint probability density function (PDF) \( f_{x_1\ldots x_n}(x_1,\ldots,x_n) \) is obtained as

\[
f_{x_1\ldots x_n}(x_1,\ldots,x_n) = c\left(F_{x_1}(x_1),\ldots,F_{x_n}(x_n)|\theta\right)\prod_{i=1}^{n} f_{x_i}(x_i)
\]

where \( c(u_1,\ldots,u_n) = \frac{\partial^n C(u_1,\ldots,u_n)}{\partial u_1\ldots\partial u_n} \) with \( u_i = F_{x_i}(x_i) \), and \( f_{x_i}(x_i) \) is the marginal PDF for \( i = 1,\ldots,n \). A copula only requires marginal CDFs and correlation parameters to model a joint CDF, so the joint CDF can be readily obtained from limited data. In addition, since the copula decouples marginal CDFs from the joint CDF, the joint CDF modeled by the copula can be expressed in terms of any type of marginal CDF. That is, having marginal Gaussian CDFs does not mean that the joint CDF is Gaussian. Thus, it is desirable to be able to model the joint CDF of correlated input variables with mixed types of marginal CDFs, which can often occur in industrial applications. To model the joint CDF using the copula, the correlation parameters need to be obtained from experimental data as seen in Eqs. (1) and (2). Since various types of copulas have their own correlation parameters, it is desirable to have a common correlation measure to obtain the correlation parameters from the experimental data.

B. Correlation Measures

To measure the correlation between two random variables, Pearson’s rho and Kendall’s tau can be used. Pearson’s rho was first discovered by Bravais in 1846, and was developed by Pearson in 1896. Pearson’s rho indicates the degree of linear relationship between two random variables as

\[
\rho_{xy} = \frac{\text{Cov}(X,Y)}{\sigma_x \sigma_y}
\]

where \( \sigma_x \) and \( \sigma_y \) are standard deviations of \( X \) and \( Y \), respectively, and \( \text{Cov}(X,Y) \) is the covariance between \( X \) and \( Y \). Since Pearson’s rho only indicates the linear relationship between two random variables, it is valid only when the joint CDF is Gaussian. Pearson’s rho also can be used as correlation measure in the joint CDF modeled by Gaussian copula, because the Gaussian copula is originated from a joint Gaussian CDF. If the marginal CDFs are Gaussian, then the joint CDF modeled by the Gaussian copula is the joint Gaussian CDF. The Gaussian copula allows generating a joint Gaussian CDF with non-marginal Gaussian CDFs as

\[
C_{\Phi}(u_1,\ldots,u_n|\mathbf{P}) = \Phi_{\mathbf{P}}(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n)|\mathbf{P}), \quad \mathbf{u} \in I^n
\]
where \( u_i = F_{X_i}(x_i) \) is the marginal CDF of \( X_i \) for \( i = 1, \ldots, n \), \( \mathbf{P} \) is the covariance matrix consisting of correlation coefficients, Pearson’s rho, between correlated input variables. \( \Phi(\cdot) \) represents the marginal standard Gaussian CDF, \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \), and \( \Phi_\rho(\cdot) \) is the joint Gaussian CDF defined as
\[
\Phi_\rho(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} (\mathbf{x} - \mathbf{\mu})^T \mathbf{P}^{-1} (\mathbf{x} - \mathbf{\mu}) \right)
\]
for \( \mathbf{x} = [x_1, \ldots, x_n]^T \) with a mean vector \( \mathbf{\mu} = [\mu_1, \ldots, \mu_n]^T \). However, Pearson’s rho cannot be a good measure for a nonlinear relationship between two random variables, which often occurs in practical engineering applications. If the given data follows a joint non-Gaussian CDF modeled by a non-Gaussian copula, another correlation measure is necessary.

Unlike Pearson’s rho, Kendall’s tau does not require the assumption that the relationship between two random variables is linear. Since the Kendall’s tau measures the correspondence of rankings between correlated random variable, it is called a rank correlation coefficient. The Kendall’s tau was first introduced by Kendall in 1938,\(^{10}\) and is defined as
\[
\tau = 4 \int_{I^*} C(u, v | \theta) dC(u, v) - 1
\]
where \( I^* = I \times I (I = [0,1]) \) and Eq. (5) is the population version of Kendall’s tau. The sample version of Kendall’s tau is
\[
t = \left( \frac{c - d}{c + d} \right) \left( \frac{ns}{2} \right)
\]
where \( c \) represents the number of concordant pairs, \( d \) is the number of discordant pairs, and \( ns \) is the number of samples. Using the estimated Kendall’s tau, the correlation parameter of the copula, \( \theta \), can be calculated because Kendall’s tau can be expressed as a function of the correlation parameter as shown in Eq. (5). The explicit functions of Eq. (5) for some copulas are presented in Ref. 11.

Consider a non-Gaussian copula, which uses a rank correlation coefficient such as Kendall’s tau as the correlation measures. Unlike the Gaussian copula, the Archimedean copula is constructed in a completely different way. An important component of constructing Archimedean copula is a generator function \( \varphi_\theta \) with a correlation parameter \( \theta \). If \( \varphi_\theta \) is a continuous and strictly decreasing function from \([0,1]\) to \([0,\infty)\) such that \( \varphi_\theta(0) = \infty \) and \( \varphi_\theta(1) = 0 \) and the inverse \( \varphi_\theta^{-1} \) is completely monotonic on \([0,\infty)\), then the Archimedean copula can be defined as,\(^{7}\)
\[
C(u_1, \ldots, u_n | \theta) = \varphi_\theta^{-1} \left[ \varphi_\theta(u_1) + \cdots + \varphi_\theta(u_n) \right]
\]
for \( n \geq 2 \). Each Archimedean copula has a corresponding unique generator function \( \varphi_\theta \), which provides a multivariate copula as shown in Eq. (7). Once the generator function is provided, the Kendall’s tau can be obtained as
\[
\tau = 1 + 4 \int_0^1 \frac{\varphi_\theta(t)}{\varphi_\theta'(t)} dt
\]
Using Eq. (8), the correlation parameter \( \theta \) can be expressed in terms of Kendall’s tau.

The Archimedean copula can be used for a multivariate CDF. But it is hard to expand to an \( n \)-dimensional copula because, as shown in Eq. (7), it has one generator function, and thus has the same correlation parameter even if \( n \) variables are correlated with different correlation coefficients. Hence, most copula applications consider bivariate data. For multivariate data, the data are analyzed pair by pair using a bivariate copula. This paper also considers a bivariate copula. More detailed information on Kendall’s tau is presented in Ref. 10.
Including the Gaussian copula and Archimedean copula, there exist various kinds of copulas; thus, selecting an appropriate copula is necessary to correctly model a joint CDF based on the given experimental data. As mentioned earlier, to model a joint CDF using a copula, the marginal CDFs and correlation parameters need to be obtained. The marginal CDFs are often known to follow specific CDF types; for example, some material properties such as fatigue parameters are known to follow lognormal CDFs. On the other hand, selecting an appropriate copula that best describes the given experimental data is not a simple problem. Since addressing two issues (effect of transformation ordering and identification of the right copula) together is complicated and requires lengthy discussion, in this paper, only the effect of transformation ordering is addressed, and the joint CDFs modeled by copulas are assumed to be exact. The identification of the right copula is addressed in Refs. 11 and 12 in detail.

III. Effect of Transformation Ordering in RBDO

Based on the identified joint CDF, the input variables need to be transformed into independent standard Gaussian variables for the inverse reliability analysis in RBDO using the Rosenblatt transformation. When the input variables are independent or the joint CDF is modeled by a Gaussian copula, the ordering of input variables does not affect the transformation. However, when the joint CDF is modeled by a non-Gaussian copula, different orderings of input variables cause different transformations for the inverse reliability analysis, which leads to different RBDO results. This issue will be addressed in this section.

A. Rosenblatt Transformation for RBDO

The RBDO problem can be formulated to

\[
\begin{align*}
\min \quad & \text{cost}(d) \\
\text{s.t.} \quad & P(G_i(X) > 0) \leq P^\text{Tar}_{G_i}, \quad i = 1, \ldots, nc \\
& d = \mu(X), \quad d_L \leq d \leq d_U, \quad d \in R^{ndv} \text{ and } X \in R^n
\end{align*}
\]

where \( X \) is the vector of random variables; \( d \) is the vector of design variables; \( G_i(X) \) represents the constraint functions; \( P^\text{Tar}_{G_i} \) is the given target probability of failure for the \( i^{th} \) constraint; and \( nc, ndv, \) and \( n \) are the number of probabilistic constraints, number of design variables, and number of random variables, respectively.

The probability of failure is estimated by a multi-dimensional integral of the joint PDF of the input variables over the failure region as

\[
P(G_i(X) > 0) = \int_{G_i(X) > 0} f_X(x)dx, \quad i = 1, \ldots, nc
\]

where \( x \) is the realization of the random vector \( X \). However, since it is difficult to compute the multi-dimensional integral analytically, approximation methods such as the FORM or the SORM are used. The FORM often provides adequate accuracy and is much easier to use than the SORM, and hence it is commonly used in RBDO. Since the FORM and SORM require the transformation of the correlated random input variables into the standard Gaussian variables, the Rosenblatt transformation is used.

Using a performance measure approach (PMA+)\textsuperscript{13}, the \( i^{th} \) constraint can be rewritten, from Eq. (9), as

\[
P[G_i(X) > 0] - P^\text{Tar}_{G_i} \leq 0 \Rightarrow G_i(x^*) \leq 0
\]

where \( G_i(x^*) \) is the \( i^{th} \) constraint function evaluated at the most probable point (MPP) \( x^* \) in \( X \)-space. Using the FORM, Eq. (11) can be rewritten as

\[
P[G_i(X) > 0] - \Phi(-\beta_i) \leq 0 \Rightarrow G_i(x^*) \leq 0
\]

where \( P^\text{Tar}_{G_i} = \Phi(-\beta_i) \) and \( \beta_i \) is the target reliability index.

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To satisfy the feasibility of the constraint, the MPP needs to be estimated for each constraint by solving the following optimization problem:

$$\begin{align*}
\text{max.} \quad & g_i(u) \\
\text{s.t.} \quad & \|u\| = \beta_i
\end{align*}$$

(13)

where $g_i(u)$ is the $i^{th}$ constraint function that is transformed from the original space ($X$-space) into the standard Gaussian space ($U$-space), i.e., $g_i(u) \equiv G_i(x(u)) = G_i(x)$. The optimum of Eq. (13) is called the MPP, which is denoted by $u^*$ in $U$-space or $x^*$ in $X$-space. If the constraint function at the MPP, $g_i(u^*)$, is less than or equal to zero, then the $i^{th}$ constraint in Eq. (9) is satisfied for the given target reliability. Thus, Eq. (9) can be rewritten as

$$\begin{align*}
\text{min.} \quad & \text{cost}(d) \\
\text{s.t.} \quad & G_i(x^*) \leq 0, \ i = 1, \cdots, nc \\
& d^l \leq d \leq d^u, \ d \in R^{ndv} \text{ and } X \in R^n
\end{align*}$$

(14)

As shown in Eq. (13), the correlated input variables need to be transformed into the independent standard Gaussian variables using the Rosenblatt transformation, which is defined as the successive conditioning:

$$\begin{align*}
\Phi(u_i) &= F_{x_1}(x_1) \\
\Phi(u_z) &= F_{x_2}(x_2|x_1) \\
& \vdots \\
\Phi(u_i) &= F_{x_i}(x_i|x_1, x_2, \cdots, x_{i-1}) \\
& \vdots \\
\Phi(u_j) &= F_{x_j}(x_j|x_1, x_2, \cdots, x_{j-1}) \\
& \vdots \\
\Phi(u_s) &= F_{x_s}(x_s|x_1, x_2, \cdots, x_{s-1})
\end{align*}$$

(15)

If the ordering of the variable $x_i$ is changed into the variable $x_j$, Eq. (15) can be rewritten as

$$\begin{align*}
\Phi(u_i) &= F_{x_j}(x_i) \\
\Phi(u_z) &= F_{x_3}(x_3|x_1) \\
& \vdots \\
\Phi(u_i) &= F_{x_i}(x_i|x_1, x_2, \cdots, x_{i-1}) \\
& \vdots \\
\Phi(u_j) &= F_{x_j}(x_j|x_1, x_2, \cdots, x_{j-1}, x_{i+1}, \cdots, x_{j-1}) \\
& \vdots \\
\Phi(u_s) &= F_{x_s}(x_s|x_1, x_2, \cdots, x_{s-1})
\end{align*}$$

(16)

Thus, if the number of variables is $n$, there are $n!$ ways of transforming the original variables into independent standard Gaussian variables. Even though there are many different ways to transform the original variables into independent standard Gaussian variables, since the Rosenblatt transformation is exact, if the inverse reliability analysis in the independent standard Gaussian space is exact, then we should obtain the same results. However, if
the FORM is used for the inverse reliability analysis, certain orders of transformation might yield more errors than other orders of transformation according to the input joint CDF type.

B. Effect of Transformation Ordering for Various Input Joint CDF Types

The joint CDFs of input random variables can be categorized as follows: independent joint CDF, joint CDF modeled by a Gaussian copula, and joint CDF modeled by a non-Gaussian copula. For various input joint CDF types, to study the effect of transformation ordering in the inverse reliability analysis, it is necessary to investigate whether the same MPP is obtained for different transformation orderings. However, since the MPPs depend on constraint functions, it is not convenient to compare the MPPs for all constraint functions. Instead of comparing the MPPs, comparing target hyperspheres transformed from $U$- to $X$-space for different transformation orderings is more appropriate because obtaining the same target hyperspheres means obtaining the same MPPs in $X$ space, which lead to same RBDO results.

First, when the input variables are independent, the transformed target hypersphere from $U$ to $X$-space can be obtained using Eq. (15), which means the Rosenblatt transformation with a given ordering as

$$\mathbf{u'} = \mathbf{u} = \left[ \Phi^{-1}(F_{x_1}(x_1)) \right]^2 + \cdots + \left[ \Phi^{-1}(F_{x_n}(x_n)) \right]^2 = \beta_i^2$$ (17)

When the order of the variable $x_i$ is interchanged into the variable $x_j$ for the second ordering, the transformed target hypersphere is obtained as

$$\mathbf{u'} = \mathbf{u} = \left[ \Phi^{-1}(F_{x_j}(x_j)) \right]^2 + \cdots + \left[ \Phi^{-1}(F_{x_n}(x_n)) \right]^2 = \beta_i^2$$ (18)

which results in the same transformed target hypersphere with Eq. (17). Thus, there is no effect of transformation ordering when the input variables are independent.

Second, consider when the input variables are correlated with a joint CDF modeled by the Gaussian copula, which is defined in Eq. (4). In the joint CDF modeled by the Gaussian copula, each variable $\Phi^{-1}\left[F_{x_i}(x_i)\right]$ for $i = 1, \cdots, n$ is the standard Gaussian variable with the covariance matrix $\mathbf{P'}$. Let $y_i = \Phi^{-1}\left[F_{x_i}(x_i)\right]$. Since the reduced correlation coefficient $\rho_{ij}$ between $y_i$ and $y_j$ is different from the correlation coefficient $\rho_{ij}$ between $X_i$ and $X_j$, it needs to be calculated from $\rho_{ij}$. The reduced correlation coefficient $\rho_{ij}$ is obtained from the correlation coefficient $\rho_{ij}$ using the following equation:

$$\rho_{ij} = E\left[\Xi_i, \Xi_j\right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi_i \xi_j \phi(y_i, y_j; \rho_{ij}) dy_i dy_j$$ (19)

where $\Xi_i = (X_i - \mu_i)/\sigma_i$ is the normalized random variable of $X_i$ and $\xi_i$ is the realization of $\Xi_i$. However, since the implicit Eq. (19) requires an iterative process that is very tedious to solve, Eq. (19) is approximated by

$$\rho_{ij} = R_y \rho_{ij}$$ (20)

where $R_y = a + bV_i + cV_j^2 + d \rho_{ij} + e \rho_{ij}^2 + f \rho_{ij} V_i' + g V_i' + h V_j' + k \rho_{ij} V_j' + l V_i' V_j'$, $V_i$ and $V_j$ are the coefficients of variation ($V = \sigma/\mu$) for each variable, and the coefficients depend on the types of input variables. When the marginal CDFs are Gaussian so that the joint CDF becomes Gaussian, the reduced correlation coefficient is the same as the original correlation coefficient, which means $R_y = 1$. For various types of input variables, the corresponding coefficients are given in Refs. 14 and 15. The maximum error of the estimated correlation coefficient obtained from Eq. (20) is normally much less than 1%, and even if the exponential marginal CDF or negative correlation is
involved, the maximum error in the correlation coefficient is at most up to 2%.\textsuperscript{15} Therefore, the approximation provides adequate accuracy with less computational effort.

For the joint CDF modeled by a Gaussian copula, the Rosenblatt transformation is linear as

\[ u = L^{-1}y \]  \hspace{1cm} (21)

where \( y \) represents correlated standard Gaussian variables and \( L^{-1} \) is the inverse of a lower triangular matrix \( L \) obtained from the Cholesky decomposition of \( P' \). That is, \( P' = LL^T \) and each entry of the matrix \( L \) is obtained as

\[
l_{ij} = \begin{cases} \sqrt{1 - \sum_{k=1}^{i-1} l_{ik}^2}, & i = j \\ \rho_{ij} - \sum_{k=1}^{i-1} l_{ik} l_{jk}, & i > j \end{cases}
\]  \hspace{1cm} (22)

Using Eq. (21), the transformed target hypersphere can be expressed as

\[ u^T u = y^T (L^{-1})^T L^{-1} y = y^T (P')^{-1} y \]  \hspace{1cm} (23)

If the order is changed, i.e., the order of the \( i^{th} \) and \( j^{th} \) variables are interchanged \( (i < j) \) in the Rosenblatt transformation, the transformed target hypersphere is changed to

\[ u^T u = y_i^T P_i y_i = y_i^T (L_i^{-1})^T L_i^{-1} y_i \]  \hspace{1cm} (24)

where \( y_i = [y_1, \cdots, y_j, \cdots, y_i, \cdots, y_n]^T \) and \( L_i \) is obtained from the Cholesky decomposition of \( P_i' \), which is the reduced covariance matrix of \( y_i \). That is, \( P' \) and \( P_i' \) are

\[
P' = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1i} & \cdots & \rho_{1n} \\ 1 & \rho_{22} & \cdots & \rho_{2j} & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \rho_{ji} & \cdots & 1 \\ \text{sym.} & 1 \end{bmatrix} \quad \text{and} \quad P_i' = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1j} & \cdots & \rho_{1n} \\ 1 & \rho_{22} & \cdots & \rho_{2i} & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & \rho_{ij} & \cdots & 1 \\ \text{sym.} & 1 \end{bmatrix}
\]  \hspace{1cm} (25)

To show that the interchanged ordering provides the same transformed target hyperspheres (\( u^T u = u^T u \)), we need to use \( y \) instead of \( y_i \) in Eqs. (23) and (24). For this, another matrix \( L_2 \) needs to be introduced instead of \( L_i \). The matrix \( L_2 \) can be obtained by interchanging the \( i^{th} \) row and column with the \( j^{th} \) row and column of \( L_i \), respectively, as following.

\[ u^T u = y_i^T P_i y_i = y_i^T (L_i^{-1})^T L_i^{-1} y_i = y_i^T (L_2^{-1})^T L_2^{-1} y = y^T P_2^{-1} y \]  \hspace{1cm} (26)
To show that $P' = P'_2$ or $LL^T = L_2L_2^T$, consider two arbitrary correlation coefficients of $P'_1$. For any arbitrary $k^{th}$ column, the entry at the $i^{th}$ row of $P'_1$, $(P'_1)_{ik}$, and the one in the $j^{th}$ row of $P'_1$, $(P'_1)_{jk}$, are $\rho_{ik}$ and $\rho_{jk}$, respectively, as shown in Eq. (25). Since $\rho_{jk}$ and $\rho_{ik}$ are the entries at the $j^{th}$ row and $k^{th}$ column of $P'$ and at the $i^{th}$ row and $k^{th}$ column of $P'$, respectively, all entries of $P'$ are same as those of $P'_2$ as following:

$$
\begin{align*}
(P'_1)_{ik} &= \rho_{ik} = (P')_{ik} = i^{th} \text{ row of } L_1 \times k \text{ th col. of } L_1^T = j^{th} \text{ row of } L_2 \times k \text{ th col. of } L_2^T = (P'_2)_{ik} \\
(P'_1)_{jk} &= \rho_{jk} = (P')_{jk} = j^{th} \text{ row of } L_1 \times k \text{ th col. of } L_1^T = i^{th} \text{ row of } L_2 \times k \text{ th col. of } L_2^T = (P'_2)_{jk}
\end{align*}
$$

(27)

(28)

This means the transformed target hyperspheres are the same even for different transformation orderings of input variables. Therefore, the Rosenblatt transformation is independent of ordering for the joint CDF modeled by a Gaussian copula.

Finally, consider when the input variables have a joint CDF modeled by a non-Gaussian copula. For example, let two random variables have a joint CDF modeled by the Clayton copula, which is one of the Archimedean copulas, with the marginal Gaussian CDFs $X_1, X_2 \sim N(0,1)$. The Clayton copula is defined as

$$
C(u,v|\theta) = [u^{-\theta} + v^{-\theta} - 1]^{-1/\theta}, \text{ for } \theta > 0
$$

(29)

where the generator is $\phi_{\theta}(t) = \frac{1}{\theta}(t^{-\theta} - 1)$, $u = \Phi(x_1)$, $v = \Phi(x_2)$, and $\theta$ is the correlation parameter of the Clayton copula. In the Clayton copula, using the Kendall’s tau $\tau$ obtained from samples, $\theta$ can be expressed as

$$
\theta = \frac{2\tau}{1-\tau}
$$

(30)

Using the Clayton copula, the Rosenblatt transformation can be carried out in two different ways as

$$
\Phi(u_i) = F_{X_1}(x_i) = \Phi(x_i)
$$

(31)

and

$$
\Phi(u_i) = F_{X_1}(x_1|x_2) = \Phi(x_1)^{-\theta} \left[ \Phi(x_1)^{-\theta} + \Phi(x_2)^{-\theta} - 1 \right]^{-1/\theta-1}
$$

(32)
Using Eqs. (31) and (32), the target hypersphere can be expressed in terms of $x_1$ and $x_2$

$$\mathbf{u}^T \mathbf{u} = u_1^2 + u_2^2 = x_1^2 + \left( \Phi^{-1} \left( \Phi(x_1)^{-1} \left[ \Phi(x_1) - 1 \right] \Phi(x_2) - 1 \right) \right) = \beta^2$$  \hspace{1cm} (33)

and

$$\mathbf{u}^T \mathbf{u} = u_1^2 + u_2^2 = x_2^2 + \left( \Phi^{-1} \left( \Phi(x_1)^{-1} \left[ \Phi(x_1) - 1 \right] \Phi(x_2) - 1 \right) \right) = \beta^2$$  \hspace{1cm} (34)

These different transformation orderings will provide different RBDO results. Further, for the non-Gaussian joint CDF modeled by a non-Gaussian copula, the Rosenblatt transformation becomes highly nonlinear, which cannot be handled accurately by the FORM. A more accurate method than the FORM for the estimation of the probability of failure in the reliability analysis is necessary to reduce the effect of the transformation ordering on the RBDO results.

**IV. Method to Resolve Effect of Transformation Ordering Using MPP-based DRM**

The MPP-based reliability analysis such as the FORM and the SORM has been a very commonly used for reliability assessment. However, when the constraint function is nonlinear or multi-dimensional, the reliability analysis using the FORM could be erroneous because the FORM cannot handle the complexity of nonlinear or multi-dimensional functions. Reliability analysis using the SORM may be accurate, but the second-order derivatives required for the SORM are very difficult and expensive to obtain in industrial applications. On the other hand, the MPP-based DRM achieves both the efficiency of the FORM and the accuracy of the SORM.

The DRM is developed to accurately and efficiently approximate a multi-dimensional integral. There are several DRMs depending on the level of dimension reduction: univariate dimension reduction, bivariate dimension reduction, and multivariate dimension reduction. The univariate, bivariate, and multivariate dimension reduction indicate an additive decomposition of $n$-dimensional performance function into one, two, and $s$-dimensional functions ($s \leq n$), respectively. In this paper, the univariate DRM is used for calculating probability of failure due to its simplicity and efficiency.

The univariate DRM is carried out by decomposing an $n$-dimensional constraint function $G(\mathbf{X})$ into the sum of one-dimensional functions at the MPP as,

$$G(\mathbf{X}) \equiv \hat{G}(\mathbf{X}) = \sum_{i=1}^{n} G(x_i^*, x_{i+1}^*, \ldots, x_n^*) - (n-1)G(\mathbf{X})$$  \hspace{1cm} (35)

where $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ is the FORM-based MPP obtained from Eq. (13) and $n$ is the number of random variables. In the inverse reliability analysis, since the probability of failure cannot be directly calculated in $U$-space, a constraint shift in a rotated standard Gaussian space ($V$-space) needs to be defined as

$$\tilde{G}^s(\mathbf{v}) \equiv \tilde{G}(\mathbf{v}) - \tilde{G}(\mathbf{v}^*)$$  \hspace{1cm} (36)

where $\mathbf{v}^* = [0, \ldots, 0, \beta]^T$ is MPP in $V$-space and $\tilde{G}(\mathbf{v}) \equiv G(\mathbf{X}(\mathbf{v}))$. Then, using the shifted constraint function, the probability of failure using the MPP-based DRM is calculated as,

$$P^\text{DRM}_F = \prod_{i=1}^{n} \Phi(-\beta + \tilde{G}_i(v_i)/b_i) \phi(v_i) dv_i$$  \hspace{1cm} (37)

where $\tilde{G}_i(v_i) \equiv \tilde{G}(0, \ldots, 0, v_i, 0, \ldots, \beta)$ is a function of $v_i$ only and $b_i = \left\| \frac{\partial g(\mathbf{u}^*)}{\partial u_i} \right\|$. 

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Equation (37) can be approximated as using the moment-based integration rule,\textsuperscript{23}, similar to Gaussian quadrature,\textsuperscript{24}

\[
P_F^{\text{DRM}} = \prod_{i=1}^{n-1} \sum_{j=1}^{N} w_j \Phi \left( -\beta + \frac{\tilde{G}_j(v_j)}{b_j} \right) 
\]

(38)

where \( v_j \) represents the \( j \)th quadrature point for \( v_i \), \( w_j \) denote weights, and \( N \) is the number of quadrature points. The quadrature points and weights for the standard Gaussian random variables \( v_i \) are shown in Table 1. When the quadrature point is \( 1 (N = 1) \) and weight is \( 1 \), Eq. (38) becomes

\[
P_F^{\text{DRM}} \approx \frac{w_1 \prod_{i=1}^{n-1} \Phi(-\beta + \tilde{G}_j(v_j))}{\Phi(-\beta)^{n-2}} = \frac{\prod_{i=1}^{n-1} \Phi(-\beta)}{\Phi(-\beta)^{n-2}} = \Phi(-\beta)
\]

(39)

where \( w_1 = 1 \) and \( v_1 = 0 \) by Table 1 and \( \tilde{G}_j(v_j) = \tilde{G}_j(0) = 0 \). Equation (39) is the same as the probability of failure calculated by the FORM. Therefore, it can be said that the probability of failure calculated by the FORM is a special case of the one calculated by the DRM with one quadrature point and weight.

<table>
<thead>
<tr>
<th>( N )</th>
<th>Quadrature Points</th>
<th>Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>( \pm \sqrt{3} )</td>
<td>0.166667</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.666667</td>
</tr>
<tr>
<td></td>
<td>( \pm 2.856970 )</td>
<td>0.011257</td>
</tr>
<tr>
<td></td>
<td>( \pm 1.355626 )</td>
<td>0.222076</td>
</tr>
<tr>
<td>5</td>
<td>0.0</td>
<td>0.533333</td>
</tr>
</tbody>
</table>

Table 1. Gaussian Quadrature Points and Weights.\textsuperscript{6}

Figure 1. DRM-based MPP for Concave and Convex Functions.\textsuperscript{6}
Using the estimated $P^\text{DRM}_P$ obtained from the MPP-based DRM for the shifted constraint function $G^*(x)$, the corresponding reliability index $\beta^\text{DRM}$ can be defined as

$$\beta^\text{DRM} = -\Phi^{-1}(P^\text{DRM}_P)$$  \hspace{1cm} (40)$$
which is not the same as the target reliability index $\beta_t = -\Phi^{-1}(P^\text{Tar}_P)$ because the nonlinearity of the constraint function is reflected in the calculation of $P^\text{DRM}_P$. Hence, using $\beta^\text{DRM}$, a new updated reliability index $\beta^\text{up}$ can be defined as

$$\beta^\text{up} = \beta^\text{tar} + \Delta \beta = \beta^\text{tar} + (\beta_t - \beta^\text{DRM})$$  \hspace{1cm} (41)$$
where $\beta^\text{tar}$ is the current reliability index. The recursive form of the Eq. (41) is

$$\beta^\text{up}(k+1) = \beta^\text{up}(k) + \Delta \beta = \beta^\text{up}(k) + (\beta_t - \beta^\text{DRM})$$  \hspace{1cm} (42)$$
where $\beta^\text{up}(0) = \beta_t$ at the initial step.

Using this updated reliability index, the updated MPP can be found by using an iterative MPP search or using an approximation. If an iterative MPP search with the updated reliability index is used, the updated MPP is called the true DRM-based MPP and is denoted by $x^\text{up}$, which means the updated MPP is the optimum solution of Eq. (13) using $\beta^\text{up}$ instead of $\beta_t$; however, the procedure will be computationally expensive. Accordingly, to improve the efficiency of the optimization, the updated MPP can be approximated as, \(^25\)

$$u^a_{k+1} \approx \frac{\beta^\text{up}(k+1)}{\beta^\text{up}(k)} u^a_k \text{ or } v^a_{k+1} \approx \frac{\beta^\text{up}(k+1)}{\beta^\text{up}(k)} v^a_k$$  \hspace{1cm} (43)$$
assuming that the updated MPP $v^a_{k+1}$ is located along the same radial direction $v^a_N$ as the current MPP $v^a_k$ in $V$-space, as shown in Fig. 1. The updated MPP obtained from Eq. (43) is called the DRM-based MPP and is used to check whether or not the optimum design satisfies the constraint. The location of the DRM-based MPP for a concave and convex function is shown in Figs. 1.a) and 1.b), respectively.

Similar to the FORM, using the DRM-based inverse reliability analysis, the RBDO formulation in Eq. (9) can be rewritten as

$$\text{minimize} \quad \text{cost}(d)$$
subject to \quad $G(x^*_{\text{DRM}}) \leq 0, \ i = 1, \cdots, nc$
$$d^L \leq d \leq d^U, \quad d \in \mathbb{R}^{nc} \text{ and } \mathbf{X} \in \mathbb{R}^n$$  \hspace{1cm} (44)$$
where $x^*_{\text{DRM}}$ is the MPP obtained from the DRM while $x^*$ is the one obtained from the FORM.

V. Numerical Examples

To observe how the ordering of transformation affects the RBDO results for the joint non-Gaussian CDFs modeled by non-Gaussian copulas, two- and four-dimensional mathematical problems are tested.

A. Two-dimensional Problem

Assume two input variables are correlated by a Clayton copula with Kendall’s tau $\tau = 0.5$. The RBDO formulation is defined as
\[
\min \text{ cost}(\mathbf{d}) = d_1 + d_2 \\
\text{s.t. } P\left(G_i(\mathbf{X}) \geq 0 \right) \leq \Phi\left(-\beta_i \right), \ i = 1, 2, 3 \\
0 \leq d_1, d_2 \leq 10, \ \beta_i = 2.0 \\
G_1(\mathbf{X}) = 1 - (0.4339X_1 - 0.9010X_2 - 1.5)^2 \left(0.9010X_1 + 0.4339X_2 + 2\right)/20 \\
G_2(\mathbf{X}) = 1 - (X_1 + X_2 - 2.8)^2/30 - (X_1 - X_2 + 12)^2/120 \\
G_3(\mathbf{X}) = 1 - 80/\left(8X_1 + X_2^2 + 5\right) \\
\]

where the marginal CDFs are Gaussian \((X_1, X_2 \sim N(5.0, 0.3))\). Denote the initial ordering as ordering 1 and the interchanged ordering \((x_i \leftrightarrow x_j)\) as ordering 2.

As shown in Fig. 2, the target hyperspheres in \(U\)-space are the same, but the constraint functions in \(X\)-space are differently transformed in \(U\)-space according to the different transformation orderings. Since the transformation of the non-Gaussian copula is highly nonlinear, some transformed constraint functions become highly nonlinear in \(U\)-space. For ordering 1 (Fig. 2.a)), the first constraint function is mildly nonlinear near the MPP \((u_{11}^*)\), but the second constraint function is highly nonlinear near the corresponding MPP \((u_{21}^*)\), which yields a large FORM error. On the other hand, for ordering 2 (Fig. 2.b)), two constraint functions are mildly nonlinear near the MPPs \((u_{12}^* \text{ and } u_{22}^*)\), so that the FORM estimates the probability of failure more accurately than that of the second constraint function with the first ordering. The FORM results for different orderings are shown in Table 2.

In Table 2, Case FORM-1 and Case FORM-2 indicate the FORM with orderings 1 and 2, respectively. As expected, when the FORM is used for the ordering 1 (FORM-1), the probability of failure for the second constraint \((P_{f2})\) is poorly estimated (much less than target probability of 2.275%). As a result, the optimum design points obtained using the FORM with different orderings are indeed different as shown in the third and fourth columns of Table 2. If the MPP-based DRM with three quadrature points, denoted as DRM3-1 and DRM3-2 for orderings 1 and 2, respectively, is used, the difference between optimum design results is reduced from 0.062 to 0.034 and the DRM provides a more accurate estimation of the probabilities of failure (closer to 2.275%) for both orderings. If the number of quadrature points is five (DRM5-1 and DRM5-2), then the optimum design points are much closer to...
each other for both orderings and the probability of failure calculation also becomes more accurate. Thus, the MPP-based DRM indeed reduces the effect of transformation ordering on the RBDO results.

Table 2. RBDO Results Obtained from Clayton Copula.

| Case    | Cost  | Optimum design points | $|d_i^{opt} - d_j^{opt}|$ | $G_1$  | $G_2$  | $G_3$  | $P_{f_1}$ (%) | $P_{f_2}$ (%) |
|---------|-------|-----------------------|---------------------------|--------|--------|--------|---------------|---------------|
| FORM-1  | 3.446 | 1.413, 2.032          | 0.062                     | 0.000  | 0.000  | -1.606 | 2.531         | 1.031         |
| FORM-2  | 3.386 | 1.352, 2.034          |                           | 0.000  | 0.000  | -1.632 | 2.354         | 2.108         |
| DRM3-1  | 3.417 | 1.380, 2.037          | 0.034                     | 0.000  | 0.000  | -1.621 | 2.352         | 1.582         |
| DRM3-2  | 3.383 | 1.347, 2.036          |                           | 0.000  | 0.000  | -1.633 | 2.280         | 2.257         |
| DRM5-1  | 3.400 | 1.364, 2.036          | 0.016                     | 0.000  | 0.000  | -1.630 | 2.320         | 1.881         |
| DRM5-2  | 3.385 | 1.348, 2.037          |                           | 0.000  | 0.000  | -1.633 | 2.276         | 2.266         |

For the same problem in Eq. (45), assume that two input variables are now correlated with a Frank copula, which belongs to the Archimedean copula, with Kendall’s tau $\tau = 0.5$. The Frank copula is given as

$$C(u, v | \theta) = -\frac{1}{\theta} \ln \left( 1 + \frac{(e^{\theta u} - 1)(e^{\theta v} - 1)}{e^{\theta} - 1} \right)$$

(46)

The correlation parameter $\theta$ can be calculated from Kendall’s tau by solving the following equation,

$$\tau = 1 - \frac{4}{\theta} \left( 1 - \frac{1}{\theta} \int e^{-\theta} dt \right)$$

(47)

As observed in the previous example, when the joint CDF modeled by a non-Gaussian copula is used, the MPPs obtained from differently transformed constraint functions provide different RBDO results according to the different ordering of input variables.

![Figure 3. Transformed Constraints and Target Hypersphere in U-space Using Frank Copula.](image)
Due to the transformation of the joint CDF modeled by a non-Gaussian copula, the constraint functions in X-space are differently transformed into those in U-space for different orderings, and some transformed constraint functions are highly nonlinear. For ordering 1, the first constraint function is highly nonlinear near the MPP \((u_{g,1}^*)\) (Fig. 3.a)), while for ordering 2, the second constraint function is highly nonlinear near the MPP \((u_{g,2}^*)\) (Fig. 3.b)). Therefore, for ordering 1, the FORM error is large at the first constraint (FORM-1), while for the second ordering, it is large at the second constraint (FORM-2). As shown in Table 3, probabilities of failure \(P_{f1}\) for ordering 1 and \(P_{f2}\) for ordering 2 are close to the target probability (2.275%), whereas \(P_{f1}\) for ordering 1 and \(P_{f2}\) for ordering 2 are not.

When the MPP-based DRM with three quadrature points is used, the probability of failure becomes closer to the target probability for both orderings (DRM-1 and DRM-2). The DRM with five quadrature points provides the most accurate calculation of the probability of failure (DRM-3 and DRM-5). The optimum design points obtained from the DRM are indeed similar to each other compared with those obtained from the FORM for different orderings. Thus, the DRM is necessary to reduce the effect of transformation ordering and to provide accurate RBDO results. If the number of correlated variables is larger than two, the effect of transformation ordering and inaccurate estimation of probability of failure might be more significant, and thus using the FORM in the reliability analysis might be more unreliable. In the next section, this issue will be further addressed through a four-dimensional problem.

### Table 3. RBDO Results Obtained from Frank Copula.

| Case     | Cost | Optimum design points | \(|d_{10}^\text{opt} - d_{20}^\text{opt}|\) | \(G_1\)  | \(G_2\)  | \(G_3\)  | \(P_{f1}\)(%) | \(P_{f2}\)(%) |
|----------|------|-----------------------|---------------------------------|--------|--------|--------|--------------|--------------|
| FORM-1   | 3.590  | 1.477, 2.114          | 0.036                           | 0.000  | 0.000  | -1.481 | 1.765        | 2.287        |
| FORM-2   | 3.572  | 1.491, 2.081          |                                  | 0.000  | 0.000  | -1.451 | 2.351        | 1.638        |
| DRM3-1   | 3.541  | 1.455, 2.086          | 0.015                           | 0.000  | 0.000  | -1.503 | 2.171        | 2.255        |
| DRM3-2   | 3.551  | 1.469, 2.081          |                                  | 0.000  | 0.000  | -1.461 | 2.323        | 1.953        |
| DRM5-1   | 3.535  | 1.453, 2.082          | 0.006                           | 0.000  | 0.000  | -1.506 | 2.242        | 2.270        |
| DRM5-2   | 3.539  | 1.459, 2.080          |                                  | 0.000  | 0.000  | -1.466 | 2.285        | 2.093        |

### B. Four-dimensional Problem

This example is the four-dimensional modified Rosen-Suzuki problem,\(^2\) and the RBDO is formulated to

\[
\begin{align*}
\text{min.} & \quad \text{cost}(d) = \frac{d_1 (d_1 - 15) + d_2 (d_2 - 15) + d_3 (2d_3 - 41) + d_4 (d_4 - 3) + 245}{245} \\
\text{s.t.} & \quad P(G_i(X) \geq 0) \leq \Phi(-\beta_i), \ i = 1, 2, 3 \\
& \quad 0 \leq d_1, d_2, d_3, d_4 \leq 10, \ \beta_i = 2.0 \\
& \quad G_1(X) = 1 - \frac{X_1 (9 - X_1) + X_2 (11 - X_2) + X_3 (11 - X_3) + X_4 (11 - X_4)}{68} \\
& \quad G_2(X) = 1 - \frac{X_1 (11 - X_1) + 2X_2 (10 - X_2) + X_3 (10 - X_3) + X_4 (21 - 2X_4)}{151} \\
& \quad G_3(X) = 1 - \frac{2X_1 (9 - X_1) + X_2 (11 - X_2) + X_3 (10 - X_3) + X_4}{95}
\end{align*}
\]

Assume that the first and second variables are correlated with the Gumbel copula and the third and fourth variables are correlated with the A12 copula, in which the Gumbel and A12 copula belong to the Archimedean copula. The Gumbel copula is defined as

\[
C(u, v; \theta) = \exp \left\{ - \left[ (-\ln u)^\theta + (-\ln v)^\theta \right]^{1/\theta} \right\}
\]
where \( u = \Phi(x_1) \) and \( v = \Phi(x_2) \) with \( X_1, X_2 \sim N(5.0, 0.3) \). Kendall’s tau \( \tau = 0.5 \) is assumed for both copulas and the correlation parameter is obtained as \( \theta = \frac{1}{1-\tau} \). The A12 copula is defined as

\[
C(u,v|\theta) = \left\{ 1 + \left[ (u^{-1} - 1)^\theta + (v^{-1} - 1)^\theta ight]^{-1} \right\}^{-1} \tag{50}
\]

Likewise, \( u = \Phi(x_3) \), \( v = \Phi(x_4) \) with \( X_3, X_4 \sim N(5.0, 0.3) \) and \( \theta = \frac{2}{3(1-\tau)} \).

Since the number of input variables is four and two pairs of variables are correlated, four different orderings are possible in the transformation. In Table 4, FORM-1 indicates FORM with the initial ordering, which means the ordering is not changed. FORM-2 is the case with the interchanged ordering of \( x_1 \) and \( x_2 \), and FORM-3 is the one with interchanged ordering of \( x_3 \) and \( x_4 \). FORM-4 is the case where the orderings of all variables are interchanged, which means \( x_1 \) and \( x_3 \) are interchanged and \( x_2 \) and \( x_4 \) are interchanged. As seen in Table 4, for all orderings, the probabilities of failure \( P_{f_2} \) and \( P_{f_3} \) are poorly estimated when the FORM is used. Even though the calculation of probability of failure for the fourth ordering is the most accurate, \( P_{f_2} \) and \( P_{f_3} \) are still much larger than the target probability, 2.275%. Compared with the two-dimensional example, the FORM error for the four-dimensional case is more significant. When the MPP-based DRM with three quadrature points is used (DRM3-1, 2, 3, and 4), the difference between the probabilities of failure becomes smaller than when the FORM is used. When five quadrature points are used (DRM5-1, 2, 3, and 4), the MPP-based DRM estimates the probabilities of the failure more accurately than the case with three quadrature points. Thus, the MPP-based DRM is necessary to reduce the ordering effect on RBDO results.

<table>
<thead>
<tr>
<th>Case</th>
<th>Cost</th>
<th>Optimum design points</th>
<th>( G_i )</th>
<th>( G_j )</th>
<th>( G_k )</th>
<th>( P_{f_2} ) (%)</th>
<th>( P_{f_3} ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FORM-1</td>
<td>-0.144</td>
<td>4.620, 5.623, 6.618, 4.024</td>
<td>-0.561</td>
<td>0.000</td>
<td>0.000</td>
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<td>FORM-2</td>
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<td>4.644, 5.591, 6.618, 4.020</td>
<td>-0.561</td>
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<td>0.000</td>
<td>8.192</td>
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<tr>
<td>FORM-3</td>
<td>-0.145</td>
<td>4.634, 5.561, 6.639, 4.065</td>
<td>-0.561</td>
<td>0.000</td>
<td>0.000</td>
<td>7.110</td>
<td>4.022</td>
</tr>
<tr>
<td>FORM-4</td>
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<td>0.000</td>
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<td>4.636, 5.586, 6.574, 4.119</td>
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<td>0.000</td>
<td>0.000</td>
<td>3.000</td>
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</tr>
<tr>
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<td>4.636, 5.561, 6.582, 4.123</td>
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<td>0.000</td>
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<td>DRM5-1</td>
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<td>0.000</td>
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<td>DRM5-2</td>
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<td>0.000</td>
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<td>DRM5-3</td>
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<td>0.000</td>
<td>2.468</td>
<td>2.306</td>
</tr>
<tr>
<td>DRM5-4</td>
<td>-0.138</td>
<td>4.612, 5.544, 6.591, 4.149</td>
<td>-0.563</td>
<td>0.000</td>
<td>0.000</td>
<td>2.359</td>
<td>2.133</td>
</tr>
</tbody>
</table>

Method-1: Ordering 1 (original ordering)
Method-2: Ordering 2 (\( X_1 \leftrightarrow X_2 \))
Method-3: Ordering 3 (\( X_3 \leftrightarrow X_4 \))
Method-4: Ordering 4 (\( X_1 \leftrightarrow X_2 \) and \( X_3 \leftrightarrow X_4 \))

VI. Conclusion

In RBDO problems, the joint CDF needs to be used in the Rosenblatt transformation for the inverse reliability analysis. However, since the joint CDFs are difficult to obtain, copulas are proposed to model the joint CDFs in this paper. Incorporating the copula concept, the joint CDF can be categorized as independent joint CDF, joint CDF
modeled by Gaussian copula, and joint CDF modeled by non-Gaussian copula. When the input variables are independent or correlated with Gaussian copula, the inverse analysis results are the same for different transformation orderings of the input variables. However, when the correlated input variables with joint CDFs are modeled by a non-Gaussian copula, different transformation orderings could lead to highly nonlinear constraint functions. Thus, it becomes a significant challenge to accurately carry out the inverse reliability analysis using the FORM. Thus, the MPP-based DRM, which can handle the nonlinear constraints, is proposed to be used in this paper for the RBDO of problems with correlated input variables with joint CDFs modeled by non-Gaussian copulas. Numerical examples show that when the MPP-based DRM is used, the difference between the RBDO results using different transformation orderings is reduced as well as the accurate estimation of probability of failure is achieved.

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References


