New Transformation of Dependent Input Variables Using Copula for RBDO

Yoojeong Noh, K.K. Choi*, and Liu Du

Department of Mechanical & Industrial Engineering
College of Engineering
The University of Iowa
Iowa City, IA 52242, U.S.A.
Email: noh@engineering.uiowa.edu
kkchoi@engineering.uiowa.edu
liudu@engineering.uiowa.edu

1. Abstract
For the performance measure approach (PMA) of RBDO, a transformation between the input random variables and the standard normal random variables is required to carry out the inverse reliability analysis. Since the transformation uses the joint cumulative density function (CDF) of input variables, the joint CDF should be known before carrying out RBDO. In many industrial RBDO problems, even though the input random variables are correlated, they are often assumed to be independent because only marginal distribution and covariance are practically obtained and the joint CDF is very difficult to obtain. With the assumption of independent input variables, it is easy to construct the joint CDF, and Rosenblatt transformation, which transforms the conditional CDF of input variables into the standard normal distribution, has been used for RBDO. However, when input variables are correlated, Rosenblatt transformation cannot be directly used because it is hard to obtain the joint CDF of correlated variables. On the other hand, Nataf transformation can be used for correlated input variables because it only requires marginal distribution and covariance. However, since Nataf transformation uses Gaussian copula, which joins multivariate normal and marginal distributions, it cannot be used for input variables with non-Gaussian joint distribution. In this paper, a new transformation that uses a non-Gaussian copula, such as Clayton copula, as the joint CDF of correlated input variables, which is then followed by Rosenblatt transformation, is proposed for non-Gaussian correlated variables. In addition, it is shown that the correlation coefficient between input variables significantly affect RBDO results and different transformations such as Nataf transformation using Gaussian copula and the new transformation using non-Gaussian copula (Clayton copula) provide different RBDO results.

2. Keywords: Reliability-based design optimization (RBDO), inverse reliability analysis, Rosenblatt transformation, Nataf transformation, copula family.

3. Introduction
The RBDO process requires two optimization procedures: the design optimization in the input random variable space and the inverse reliability analysis in the standard normal random variable space [1]. Thus, a transformation between these two variables is necessary for the inverse reliability analysis in RBDO. Rosenblatt and Nataf transformations are commonly used for the reliability analysis. Rosenblatt transformation [2] is a mathematically exact method and requires complete information of the input variables such as joint CDF or conditional CDF [3, 4]. On the other hand, Nataf transformation is an approximate method that only requires the covariance matrix and marginal CDF [3-6] and used to construct a joint CDF (Nataf model) which is identified as Gaussian copula. In fact, Nataf transformation is a combination of Gaussian copula and Rosenblatt transformation. Hence, if the input variables of the RBDO problem are independent, either method can be used because two methods have the same transformation formulation in this case. However, if not, they may yield different RBDO results depending on the input information.

In many RBDO problems, the input random variables such as the material properties are correlated. To solve the RBDO problems with the correlated input variables, a joint PDF or cumulative distribution function (CDF) of input variables should be available. However, often in industrial applications, only limited information such as the marginal distributions and covariance are practically available, whereas the input joint probability distribution functions (PDF) are very difficult to obtain [4, 6]. Thus, in the literature, most RBDO studies have assumed all input random variables are independent and mostly the Rosenblatt transformation method has been used.
In this paper, it is found that Rosenblatt transformation is impractical for problems with correlated input variables due to difficulty of constructing a joint PDF from the marginal distributions and covariance. On the other hand, Nataf transformation can be used to construct a joint CDF, which is called Nataf model or Gaussian copula in the copula family [7], from the limited information such as marginal distributions and covariance. Since Nataf transformation uses copula which is a joint function of marginal distributions, even if input variables are mixed with different types of distributions, the joint PDF or CDF can be easily constructed. Further, Nataf transformation is applicable for the normal and lognormal distributions with positive correlation that cover majority of practical industrial applications, and thus, it is applicable to a broad class of RBDO problems [8-10]. However, since Nataf transformation is originated from Gaussian copula, and it may not be applicable to non-Gaussian distributions except the normal and lognormal distributions. Even though the normal and lognormal distributions cover majority of practical industrial applications, there are still some applications with non-Gaussian variables, which will be discussed in Section 6.2. In this paper, a new transformation that combines the advantages of Rosenblatt transformation and copulas is developed for RBDO problems with non-Gaussian correlated random input variables. Using a numerical example, it is shown that the correlated random input variables do significantly affect the RBDO result and the proposed transformation is applicable to RBDO problems with correlated input variables with non-Gaussian joint CDF which has a non-Gaussian distribution.

4. Rosenblatt Transformation
Rosenblatt transformation is a well-known transformation method that maps the original variables onto the standard normal variables. It is defined as the following successive conditioning

\[ \Phi(u_i) = F_{x_i}(x_i) \]
\[ \Phi(u_i) = F_{x_i}(x_i | x_i) \]
\[ \vdots \]
\[ \Phi(u_i) = F_{x_i}(x_i | x_1, x_2, \cdots, x_{n-1}) \]

where \( n \) is number of input variables, \( F_{x_i}(x_i | x_1, x_2, \cdots, x_{n-1}) \) is the CDF of \( X_i \) conditional on \( X_1 = x_1, X_2 = x_2, \cdots, X_{i-1} = x_{i-1} \), and \( \Phi^{-1}(\cdot) \) is the inverse CDF of the standard normal variables.

Based on Eq. (1), when the multivariate joint PDF or conditional CDFs are known, the probability of failure can be exactly estimated using Rosenblatt transformation. For independent input variables, the probability of failure can be obtained from the joint PDF, which is simply multiplication of the marginal PDFs.

However, even though Rosenblatt transformation has advantages, it may not be widely applicable to practical engineering problems due to following reasons. First, the joint PDF or conditional CDFs should be available for all variables to estimate the probability of failure, which is often too expensive or difficult to obtain in industrial applications where the marginal CDF and covariance are commonly available. Also, when the distribution types of input variables are mixed, i.e., some of the variables are lognormal and others are exponential or Weibull, it is not possible to express the joint PDF in a mathematical formulation. Thus, Rosenblatt transformation can be used only for limited cases where all input variables have normal distribution and a joint PDF or conditional CDFs are provided. However, if a copula, which is a joint CDF of marginal distributions, is used instead of the multivariate joint CDF, Rosenblatt transformation can be used because copulas only require marginal distributions and correlation parameters that can be practically obtained. It is discussed in Section 7 in detail.

5. Copulas
The copula is originated from a Latin word for “link” or “tie” that connects different things. In statistics, the copulas are functions that join multivariate distribution functions to their one dimensional marginal distribution functions. That is, copulas are multivariate distribution functions whose marginal distributions are uniform on the interval on \([0,1]\). According to Sklar’s theorem [7], if the random variables have marginal distributions, then there exists an n-dimensional copula \( C \) such that

\[ F_{x_1,\ldots,x_n}(x_1,\ldots,x_n) = C(F_{x_1}(x_1),\ldots,F_{x_n}(x_n)) \]  

If marginal distributions are all continuous, then \( C \) is unique. Conversely, if \( C \) is an n-dimensional copula and \( F_{x_1}(x_1),\ldots,F_{x_n}(x_n) \) are marginal CDFs, then the joint distribution is an n-dimensional function with marginal distributions [7]. By taking the derivative of Eq. (2), the joint PDF is defined as

\[ f(x_1,\ldots,x_n) = c(F_{x_1}(x_1),\ldots,F_{x_n}(x_n)) \prod_{i=1}^{n} f_{x_i}(x_i) \]

where \( c(u_1,\ldots,u_n) = \frac{\partial^n C(u_1,\ldots,u_n)}{\partial u_1 \cdots \partial u_n} \) with \( u_i = F(x_i) \) and \( f_{x_i}(x_i) \) is marginal PDF for \( i = 1,\ldots,n \).
5.1. Fréchet-Hoeffding Bounds

Any copula \( C(u_1, \cdots, u_n) \) lies between the Fréchet-Hoeffding lower and upper bounds for every \( (u_1, \cdots, u_n) \) in \( I^n \) and the bounds are themselves copulas which are given as [7]

\[
\max(u_1 + \cdots + u_n - n + 1, 0) \leq C(u_1, \cdots, u_n) \leq \min(u_1, \cdots, u_n)
\]  \( (4) \)

where \( I^n = I \times I \times \cdots \times I \) (\( I = [0,1] \)).

For the two-dimensional case, Eq. (4) can be written as

\[
\max(u_1 + u_2 - 1, 0) \leq C(u_1, u_2) \leq \min(u_1, u_2)
\]  \( (5) \)

Let \( W(u_1, u_2) = \max(u_1 + u_2 - 1, 0) \), \( M(u_1, u_2) = \min(u_1, u_2) \), and consider an independent copula \( \Pi(u_1, u_2) = u_1 \cdot u_2 \).

These copulas \( W(u_1, u_2), M(u_1, u_2), \) and \( \Pi(u_1, u_2) \) are graphically show in Fig. 1 in \( u_1 - u_2 \) space.

![Figure 1. Graph of the Copulas: (a) Fréchet-Hoeffding Upper Bound Copula \( W \) (b) Fréchet-Hoeffding Lower Bound Copula \( M \) (c) Independent Copula \( \Pi \)](image_url)

Three copulas can be easily compared by drawing these copulas along the diagonal direction \( u_1 = u_2 \) as shown in Fig. 2. The graph of any copula is a continuous surface within \( I^n \) (Fig. 1), and along the horizontal, vertical, and diagonal directions, all copulas are nondecreasing functions. Moreover, all copulas are uniformly continuous on \( I \) [7].

![Figure 2. Graph of Copulas \( W, M, \) and \( \Pi \) along the Diagonal Direction](image_url)

5.2. Dependence Measures

To measure dependence between two random variables, several types of measures such as Pearson’s rho, Spearman’s rho and Kendall’s tau are used. Pearson’s rho, which is also called a product moment correlation coefficient, is first discovered by Bravais in 1846 [11] and developed by Pearson in 1896 [12]. Pearson’s rho indicates the degree of linear relationship between two random variables as follows.

\[
\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}
\]  \( (6) \)

where \( \sigma_X \) and \( \sigma_Y \) are standard deviations of \( X \) and \( Y \), respectively and \( \text{Cov}(X, Y) \) is the covariance between \( X \) and
Y. Since Pearson’s rho only indicates the linear relationship between two random variables, it is not a good measure for nonlinear relationship between two random variables, which often occurs in practical engineering applications. If given data follows non-Gaussian joint distribution, another measure needs to be introduced to estimate dependency between random variables. Therefore, Pearson’s rho is only valid when the joint PDF is multivariate normal distribution.

Unlike Pearson’s rho, the following two measures, Spearman’s rho and Kendall’s tau, do not require the assumption that the relationship between two random variables is linear. Spearman’s rho and Kendalls tau measures the correspondence of two rankings between random variables in a different way. Since these measures estimate the relationship between two rankings of random variables, they are called rank correlation. The rank correlation coefficient is first introduced by Spearman, who is psychologist, in 1904 [13]. In psychology, more than in any other science, it is hard to find a measure that estimates correlation between two variables because there are some cases where correlation cannot be measured quantitatively. For example, the dependence between hereditary qualities of brothers cannot be quantitatively measured if Pearson’s rho that requires specific values of two variables is used. On the other hand, if children of a school are divided into conscientious and non-conscientious group, the correlation can be measured by counting how much brothers tend to be in the same division. Thus, in that case, comparison (ranking) of two groups is a better way rather than measuring Pearson’s rho. The rank does not change under strictly increasing function; hence it can be expressed as copulas because copulas are invariant under strictly increasing transformation of its margins.

As previously mentioned, since the rank correlation coefficient measures the degree of correspondence between two variables, the correspondence should be mathematically defined. Concordance is one way of expression for correspondence. If large values of one tend to be associated with large values of the other and small values of one with small values of the other, two random variables are said to be concordant. Likewise, if large values of one tend to be associated with small values of the other, two random variables are called discordant. Since the copulas play an important role in concordance and dependence measures are also associated with concordance, a concordance function $Q$ needs to be introduced. The concordance is the difference of the probability of concordance and the probability of discordance for a pair of random vectors $(X_1,Y_1)$ and $(X_2,Y_2)$ and defined as

$$Q = P[(X_i - X_2)(Y_i - Y_2) > 0] - P[(X_i - X_2)(Y_i - Y_2) < 0]$$

where $(X_1,Y_1)$ and $(X_2,Y_2)$ are independent vectors of continuous random variables with joint distribution $H_1(x,y) = C_1(F(x),G(y))$ and $H_2(x,y) = C_2(F(x),G(y))$ with same margins $u = F(x)$ and $v = G(y)$. Equation (7) can be expressed as copulas

$$Q = Q(C_1,C_2) = 4\int_{R^2} C_2(u,v)dC_1(u,v)$$

5.2.1 Spearman’s Rho

Spearman’s rho is first introduced by Spearman in 1904 [13]. Spearman’s rho is defined to be proportional to the probability of concordance minus the probability of discordance between two random vectors $(X_1,Y_1)$ and $(X_2,Y_2)$ with same margins $u = F(x)$ and $v = G(y)$, but with different copulas, $H_1(x,y) = C(F(x),G(y))$ of $(X_1,Y_1)$ and $H_2(x,y) = \Pi(x,y) = F(x)G(y)$ of $(X_2,Y_2)$. The population version of Spearman’s rho is defined as

$$\rho_s = 3\left[ P[(X_i - X_2)(Y_i - Y_2) > 0] - P[(X_i - X_2)(Y_i - Y_2) < 0] \right]$$

Here, the multiplication of 3 is to make Spearman’s rho to have ranges $-1$ to $1$. Equation (9) can be rewritten in terms of a copula as

$$\rho_s = 3Q(C,\Pi) = 12\int_{R^2} uvdC(u,v) - 3 = 12\int_{R^2} C(u,v)dudv - 3$$

The sample version of Spearman’s rho is

$$r_s = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)}$$

where $d_i$ is the difference of two rankings and $n$ is the number of samples.

Using Eq. (10), Spearman’s rho can be estimated from a given copula, or conversely using Eq. (11), the correlation parameter between marginal distributions in a copula can be estimated provided that the copula type is defined. For many copulas, since there is an explicit relationship between correlation parameter and Spearman’s rho, the correlation parameter can be easily obtained.

5.2.2 Kendall’s Tau

Kendall’s tau is first introduced by Kendall in 1938 [14]. Kendall’s tau is the probability of concordance minus the
probability of discordance between two random vectors \((X_i, Y_i)\) and \((X_j, Y_j)\) with same margins \(u = F(x)\) and \(v = G(y)\), but with a common copula, \(H(x, y) = C(F(x), G(y))\) of \((X_i, Y_i)\) and \((X_j, Y_j)\). The population version of Kendall’s tau is expressed using concordance function as

\[
\tau = Q(C, C) = 4\int \int C(u, v)dC(u, v) - 1
\]  

(12)

The sample version of Kendall’s tau is

\[
\hat{\tau} = \frac{c - d}{c + d} = \frac{c - d}{\binom{n}{2}}
\]  

(13)

where \(c\) is the number of concordant pairs and \(d\) is the number of discordant pairs and \(n\) is the number of samples.

Using Eq. (12), Kendall’s tau can be calculated from a given copula or if Kendall’s tau is known, then the correlation parameter of copulas also can be calculated. Like Spearman’s rho, for many copulas, since there are explicit formulations which define the relationship between correlation parameter and Kendall’s tau, the correlation parameter can be easily obtained provided the copula type is determined. More detailed information on Spearman’s rho and Kendall’s tau is presented in Ref. 15.

5.3. Commonly Used Copulas

In this section, two commonly used copulas: the elliptical copulas and Archimedean copulas are introduced. These copulas are most popular copulas because they can be extended to multivariate distributions and easy to handle because of their tractable characteristics. For example, elliptical copulas provide a linear transformation from original variables to standard normal variables, and thus it is easy to transform from one to another. In case of Archimedean copulas, each copula has a unique generator function and the generator is used to calculate the dependence measure, Kendall’s tau. Thus, without using Eq. (12) that require double integration on \(T^2\), Kendall’s tau can easily obtained as will be explained in Section 5.3.2.

5.3.1 Elliptical Copulas

Elliptical copulas are copulas of elliptical distributions. A random variable \(X\) has an elliptical distribution, if and only if there exist \(S\), \(R\), and \(A\) that satisfy

\[
X = \mu + RSA = \mu + AY
\]  

(14)

where \(S\) is \(k\)-dimensional random vector that are uniformly distributed on the unit sphere \(\delta^{d-1} = \{s \in \mathbb{R}^k : s's = 1\}\); \(R\) is a non-negative random variable, independent of \(S\); \(A \in \mathbb{R}^{d \times k}\) with \(AA' = \Sigma\) where \(\Sigma\) is covariance matrix, and \(d\) and \(k\) are the number of random variables in \(X\) and \(S\), respectively [16]. When \(R^2\) has a chi-square distribution with \(d\) degree-of-freedom, \(Y\) has a standard normal distribution \(N(0, 1)\) and accordingly \(X\) has a normal distribution \(N(\mu, \Sigma)\) for \(d\)-dimensional variables. If \(R^2 / d\) has F distribution with \(d\) and \(\nu\) degrees-of-freedom, \(Y\) has a standard \(t\) distribution \(T(\nu, 0, 1)\) and \(X\) has a \(t\) distribution with \(T(\nu, \mu, \Sigma)\) for \(d\)-dimensional variables. The \(t\) distribution and normal distribution provide \(t\) copula and normal (Gaussian) copula, respectively [17].

Gaussian copula is defined as multivariate normal distribution of standard normal variables

\[
C(u_1, \ldots, u_n) = \Phi[\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)]
\]  

(15)

Gaussian copula is used in Nataf transformation to transform original variables \(x\) to correlated standard normal variables \(y\) with \(\Phi(0, P')\) where \(P'\) is the reduced covariance matrix. It is discussed in Section 6 in detail.

The \(t\) copula can be constructed using \(t\) distribution like Gaussian is constructed using normal distribution. The \(t\) copula is defined as multivariate \(t\) distribution of standard \(t\) variables with \(T_{\nu} (0, P', \nu)\) where \(\nu\) is the degree of freedom.

\[
C(u_1, \ldots, u_n) = T_{\nu} \Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)
\]  

(16)

5.3.2 Archimedean Copula

Archimedean copula is constructed in completely different way from the elliptical copula that uses multivariate distribution. An important component of constructing Archimedean copulas is a generator function \(\varphi\) which is a complete monotone decreasing function. The function \(\varphi(t)\) is completely monotonic on an interval \(I = [0, 1]\) if it is continuous and has derivatives of all orders that alternate in sign

\[
(-1)^k \frac{d^k \varphi(t)}{dt^k} \geq 0, \quad k = 1, 2, \ldots
\]  

(17)
If $\phi$ is a continuous and strictly decreasing function from $I$ to $[0, \infty]$ such that $\phi(0) = \infty$ and $\phi(1) = 0$ and the inverse $\phi^{-1}$ is completely monotonic on $[0, \infty)$, then an $n$-copula, which is called Archimedean copula, for all $n \geq 2$ can be defined as

$$C(u_1, \ldots, u_n) = \phi^{-1}\left[\frac{\phi(u_1) + \cdots + \phi(u_n)}{\phi(1)}\right]$$

(18)

Each Archimedean copula has the corresponding unique generator function and the generator function provides copulas as seen in Eq. (18). As stated in the beginning of this section, once the generator is provided, Kendall’s tau can be easily obtained as

$$\tau = 1 + 4\int_0^1 \frac{\phi'(t)}{\phi(t)} \, dt$$

(19)

Unlike Kendall’s tau, there is no explicit formulation that estimates Spearman’s rho using the generator.

6. Nataf Transformation

Nataf transformation is used to construct a normal CDF, which is identified as Gaussian copula (Nataf model), by transforming original variables into standard normal variables. Nataf transformation contains two steps: transformation from original variables $X$ to correlated standard normal variables $Y$ and transformation from correlated normal variables $Y$ to independent standard normal variables $U$. The first step approximates the joint CDF using Gaussian copula and the second step requires a linear transformation which is same with Rosenblatt transformation for normal variables [18].

6.1. Nataf Transformation Using Gaussian Copula

Using Sklar’s theorem, Gaussian copula, which is also called Nataf model, can be obtained as

$$F_{x_1, \ldots, x_n}(x_1, \ldots, x_n) = \Phi\left[\Phi^{-1}\left[\frac{F_{x_1}(x_1) + \cdots + F_{x_n}(x_n)}{\Phi(1)}\right]\right]$$

(20)

where $F_{x_i}$ is the marginal CDF of $x_i$. Consider the transformation from $X$ to $Y$ as

$$y_i = \Phi^{-1}\left[F_{x_i}(x_i)\right], \quad i = 1, \ldots, n$$

(21)

Then, the joint PDF is defined by

$$f_{y_1, \ldots, y_n}(y_1, \ldots, y_n) = \frac{\partial^n F}{\partial y_1 \cdots \partial y_n} = \frac{\partial y_1}{\partial x_1} \cdots \frac{\partial y_n}{\partial x_n} \frac{\partial^n F}{\partial y_1 \cdots \partial y_n} = \frac{f_{x_1}(x_1) \cdots f_{x_n}(x_n)}{\varphi(y_1) \cdots \varphi(y_n)} \varphi(y, P')$$

(22)

where $\frac{\partial y_i}{\partial x_1} = f(x_i), \quad \frac{\partial^n F}{\partial y_1 \cdots \partial y_n} = \varphi(y, P')$. If the marginal PDFs $f_{x_i}(x_i)$ for $i = 1, \ldots, n$ are available, then the reduced covariance matrix $P' = \{\rho'_{ij}\}$ can be estimated. Thus, $Y$ is an n-dimensional standard normal variable with a joint PDF $\varphi(y, P')$. Therefore, the joint PDF can be calculated using the marginal PDFs of the input variables and the reduced covariance matrix.

The reduced correlation coefficient between two variables can be estimated by an iterative process using the double integral as

$$\rho_{ij} = E[Z_i Z_j] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} z_i z_j \varphi(y_i, y_j; \rho_{ij}) \, dy_i \, dy_j$$

(23)

where $Z_i = (X_i - \mu_i)/\sigma_i$. However, since the iterative process is very tedious and unknowns are within the double integral, Eq. (23) is approximated by

$$\rho_{ij} = R_{\rho_{ij}}$$

(24)

to obtain the reduced coefficient. In Eq. (24), $R_{\rho_{ij}}$ is approximated by a polynomial as

$$R_{\rho_{ij}} = a + b V_i + c V_j d + e \rho_{ij} + f \rho_{ij}^2 + g V_i V_j + h V_i^2 + k \rho_{ij} V_j + l V_i V_j$$

(25)

where $V_i$ and $V_j$ are the coefficient of variation ($V = \sigma / \mu$) for each variable and the coefficients depend on the types of input variables. For different types of input variables, the corresponding coefficients are given in Refs. 3, 4 and 6. The maximum error of the estimated correlation coefficient obtained from Eq. (24) is normally much below 1%, and even if the exponential distribution or negative correlation is involved, the maximum error in the correlation coefficient is at most up to 2% [4, 6]. Therefore, the approximation provides adequate accuracy with less computational effort.

The inverse reliability analysis is carried out using the transformed standard uncorrelated normal variables $U$. Since the relationship between the original correlated variables $X$ and the correlated standard normal variables $Y$ is given in
Eq. (21), the next step is to transform the correlated standard normal variables $Y$ to the uncorrelated standard normal variables $U$ using a linear transformation.

Consider the following linear equation.

$$Y = A + BU$$

(26)

where $Y \sim N(0, I)$ has the reduced correlation matrix $\Sigma_Y = P'$ and $U \sim N(0, I)$ has the covariance matrix $\Sigma_U = I$. The mean of $Y$ can be calculated as


(27)

In the same way, the covariance matrix of $Y$ can be calculated as

$$P' = \Sigma_Y = Var[A + BU] = Var[BU] = BU^T = B\Sigma_U B^T$$

(28)

Since the covariance matrix of $Y$ is positive definite, $P'$ can be decomposed into the lower and upper triangular matrix, $B$ and $B^T$ using Cholesky factorization. Each element $b_{ij}$ in $B$ can be calculated as

$$b_{ij} = \begin{cases} \sqrt{1 - \sum_{k=1}^{i-1} b^2_{ik}}, & i = j \\ \rho_{ij} - \sum_{k=1}^{i-1} b_{ik} b_{kj}, & i > j \end{cases}$$

(29)

Using Eqs. (21) and (26), the relationship (Nataf transformation) between original variables $X$ and independent standard variables $U$ can be obtained as

$$x_1 = F^{-1}_{x_1}(\Phi(b_{11}u_1))$$

$$x_2 = F^{-1}_{x_2}(\Phi(b_{12}u_1 + b_{22}u_2))$$

$$\vdots$$

$$x_n = F^{-1}_{x_n}(\Phi(b_{1n}u_1 + b_{2n}u_2 + \cdots + b_{nn}u_n))$$

(30)

Since the original variables can be expressed in terms of the independent standard normal variables, the reliability analysis can be carried out by using Eq. (30). If only the covariance matrix and marginal distribution are available, Nataf transformation is only possible way to construct the joint CDF of the input random variables while Rosenblatt transformation is not. Further, since Nataf transformation can accurately approximate joint PDF for lognormal variables with positive correlation as well as for normal variables and for combined normal and lognormal variable, which cover majority of engineering applications, it can be widely used [10]. For multivariate normal distribution, Rosenblatt transformation is same with Nataf transformation, and thus, Nataf transformation is a combination of Gaussian copula and Rosenblatt transformation.

6.2. Limitation of Nataf Transformation

Nataf transformation is applicable to normal and lognormal variables, but it may not be applicable to other distributions which have non-symmetric association between two random variables. In practical engineering applications, there are cases with non-symmetric association between two random variables. For instance, the following example shows 30 pairs of correlated data from an exhaust manifold used on a Chrysler 5.2L engine. Two random variables $X$ and $Y$ are shown in Fig. 3 and the data are collected from a machine capability study performed on the machine builder’s floor in Chrysler Corporation [19].

![Figure 3. Exhaust Fold [19]](image-url)
Figure 4 (a) shows the scatter plot of the data. Using the data, the mean values and standard deviations of $X$ and $Y$ are calculated as $\mu_X = 8.945$, $\mu_Y = 0.817$, $\sigma_X = 1.569E-3$, $\sigma_Y = 1.514E-3$; and Pearson’s rho and Kendall’s tau are 0.441 and 0.401, respectively. As seen in the Fig. 4, the joint distribution does not fit the bivariate normal (Gaussian) copula because the data are not symmetrically distributed. The data are densely distributed in the left-lower end, but they are widely spread out in the right-upper end. As seen in Fig. 4 (b) and (c), the PDF contours using Clayton copula are dense near the area where data are clustered. On the other hand, the PDF contours using Gaussian copula evenly fall apart from each other regardless of the density of the data. Therefore, Clayton copula better describes the data than Gaussian copula.

![Figure 4. (a) Scatter Plot of Data (b) PDF Contour Using Gaussian Copula (c) PDF Contour Using Clayton Copula](image)

In this example, since the true joint distribution may not be known and the data has lack of information, it is not clear to figure out which copula describes the data better. In fact, in case of the data, how many samples are required to select an appropriate copula type for a given data is an important issue, which has not been investigated in this paper. In this paper, it is assumed that we have enough data whose distributed shape is clearly shown and it can be figured out which type of copulas is appropriate to describe the given data.

Suppose that we have a data set with 500 samples as shown in Fig. 5, which are generated from one of Archimedean copulas (true model) which is given as

$$C(u, v) = \left[\ln\left( \exp(u^{-\theta}) + \exp(v^{-\theta}) - e \right) \right]^{-1/\theta}$$

where the generator is $\varphi\theta(t) = \exp(t^{-\theta}) - e$, and $\theta$ is the correlation parameter of the true model, which is 0.4. Then the corresponding Kendall’s tau can be calculated as 0.326 for given $\theta = 0.4$ using Eq. (19). Two random variables $X_1$ and $X_2$ are supposed to have standard normal distribution $u = \Phi(x_1)$ and $v = \Phi(x_2)$. When 500 pairs of data are used, Kendall’s tau is calculated as $\tau = 0.350$ and the mean values of $X_1$ and $X_2$ are $-0.085$ and $-0.045$, standard deviations of $X_1$ and $X_2$ are estimated as 1.008 and 1.007, respectively.

![Figure 5. 500 Pairs of Data](image)
To characterize the input variables, we need to select a copula which best describes the data given in Fig. 5. Of course, if the Archimedean copula in Eq. (31) is selected, then the true copula is selected. However, for this given data set, we could select Clayton copula, which is one of Archimedean copulas, because the PDF contour using Clayton copula is similar with the data set as shown in Fig. 5. Of course, it is clear that Gaussian copula cannot describe the data set in Fig. 5. Thus, suppose we have selected the following Clayton copula as a joint PDF,

\[ C(u, v) = \left[ u^{-\theta} + v^{-\theta} - 1 \right]^{-1/\theta} \text{ for } \theta > 0 \]  

(32)

where the generator is \( \phi_\theta(t) = \frac{1}{\theta} (t^{-\theta} - 1) \) and \( \theta \) is the correlation parameter of Clayton copula. In Clayton copula, there is a relationship between \( \theta \) and Kendall’s tau as

\[ \theta = \frac{2\tau}{1 - \tau} \]  

(33)

If the data are infinite, the correct correlation parameter for Clayton copula \( \theta \) is 0.968, which is obtained from Eq. (33) using the true Kendall’s tau \( \tau = 0.326 \). If Kendall’s tau is calculated from the 500 pairs of data, i.e., \( \tau = 0.350 \), then the correlation parameter of Clayton copula is \( \theta = 0.996 \) from Eq. (33). Using the estimated correlation parameter \( \theta \), Clayton copula can be obtained.

As stated in Section 6.1, in Nataf transformation, Gaussian copula is used to describe the data. Hence, it is interesting to verify how accurately Gaussian copula describe the data with non-symmetric association by comparing with other non-Gaussian copulas. Gaussian copula is defined as

\[ C(u, v) = N\left( \Phi^{-1}(u), \Phi^{-1}(v) \right) = \frac{1}{2\pi \sqrt{1 - \rho^2}} \int_{-\infty}^{\Phi^{-1}(v)} \exp \left[ -\frac{s^2 - 2\rho st + t^2}{2(1 - \rho^2)} \right] ds dt \]  

(34)

In Gaussian copula, Pearson’s rho \( \rho \) is calculated from the data using Eq. (6). If the data are infinite, the correct Pearson’s rho would have been 0.497. However, when 500 pairs of data are used, Pearson’s rho is calculated as \( \rho = 0.526 \). Using the estimated Pearson’s rho, Gaussian copula is obtained.

Figure 6 shows that the PDF contour using true model; and Clayton copula and Gaussian copula, with infinite data and 500 data, respectively. As seen in Fig. 6, PDF contour using Clayton copula is similar with the true model, which fits well with the data in Fig. 5. On the other hand, the joint PDF contour using Gaussian copula is quite different from the true copula. The difference can be clearly shown by comparing CDFs using each copula as seen in Fig. 7 (a) and (b). As expected, the CDF using Clayton copula is close to the one using true model whereas the CDF using Gaussian copula is different from one using true model especially at the left tail which is an important region to estimate failure rate. The differences of CDFs from these copulas can be clearly shown if CDFs are magnified in the region \( x = -3.1 \) to \(-2.0 \) as shown in Figs. 8 (a) and (b). Consider a specific CDF value \( F(x, x) = 0.135\% \) (i.e., \( \beta = 3.0 \) or 3-\( \sigma \) design), which is indicated by the horizontal lines in Figs. 8 (a) and (b). When 500 data sets are used, Clayton copula and Gaussian copula identify \( x = -2.89 \) and \( x = -2.42 \), respectively for the CDF value of 0.135%. When these values are compared with the true value \( x = -2.96 \), the result from Clayton copula is indeed better. When data sets are infinite, Clayton copula and Gaussian copula identify \( x = -2.78 \) and \( x = -2.31 \) respectively for the CDF value of 0.135%. Thus, the result of Clayton copula is better. From this observation, if we used Clayton copula to characterize the input distribution, we will find the optimum point that is near the true optimum design in RBDO while Gaussian copula fails. The RBDO results using different copulas are shown in Section 7.
7. New Transformation

As stated in Section 6, if the data follow non-Gaussian copula, Nataf transformation that uses Gaussian copula may generate incorrect joint PDF, which could lead to an incorrect RBDO result. For correlated input variables with non-Gaussian joint distribution, a non-Gaussian copula which provides best fit to the data needs to be selected.
7.1. New Transformation Using Copulas

Nataf transformation transforms the original variables into the correlated standard normal variables using Gaussian copula, and then transforms correlated standard normal variables to independent standard normal variables using linear transformation that is the same as Rosenblatt transformation for correlated normal variables. Likewise, for non-Gaussian copula, once a copula which captures the data is selected, and then Rosenblatt transformation can be used to transform to the normalized Gaussian joint distribution.

In reliability analysis, since the sensitivities of the constraints with respect to the standard normal variables is required, we need to take derivatives of Eq. (1). The sensitivity of the constraint \( G_i \) with respect to \( u_j \) is

\[
\frac{\partial G_i}{\partial u_j} = \sum_{k=1}^{n} \frac{\partial G_i}{\partial x_k} \frac{\partial x_k}{\partial u_j} = \sum_{k=1}^{n} \frac{\partial G_i}{\partial x_k} \cdot J_{ij}
\]

(35)

where \( J_{ij} \) is given by

\[
J_{ij} = \begin{cases} 
\frac{\partial x_i}{\partial u_k} = \frac{f(x_i, \ldots, x_{k-1})}{f(x_i, \ldots, x_k)} & \text{for } k = j \\
-\frac{f(x_i, \ldots, x_{k-1})}{f(x_i, \ldots, x_k)} \sum_{i \neq j}^{n} \frac{\partial F(x_i | x_{i+1}, x_{i+2}, \ldots, x_n)}{\partial x_i} \frac{\partial x_i}{\partial u_j} & \text{for } k > j
\end{cases}
\]

and each component of Jacobian matrix \( J_{ij} \) can be calculated using derivatives of Eq. (1). Thus, once the explicit formulation of joint PDF and CDF (i.e., copula) is given, Eq. (35) can be used to carry out reliability analysis. In the following example, the independent copula, Gaussian copula, and Clayton copula are used for comparison in RBDO.

7.2 Numerical Example

Consider the RBDO formulation of a two dimensional mathematical example,

\[
\begin{align*}
\text{min.} & \quad \text{cost}(d) = d_i + d_j \\
\text{s.t.} & \quad P(G_i(X) \geq 0) \leq \Phi(\beta_i), \quad i = 1, 2, 3 \\
& \quad 0 \leq d_i \leq 10, \quad 0 \leq d_j \leq 10, \quad \beta_i = 3.0 \\
& \quad G_1(X) = 1 - X_1^2 X_2 / 20 \\
& \quad G_2(X) = 1 - (X_1 + X_2 - 5)^2 / 30 - (X_1 - X_2 - 12)^2 / 120 \\
& \quad G_3(X) = 1 - 80 / (X_1^2 + 8X_2 + 5)
\end{align*}
\]

(36)

Assume that the true input model is Archimedean copula given in Eq. (31) with \( \theta = 0.4 \) (\( \tau = 0.326 \)). As PDF contours and CDFs using different copulas are shown in Figs. 6, 7, and 8 for 500 sample data and infinite data, RBDO results using these copulas are compared. In this example, since input variables has normal distribution, \( X_1, X_2 \sim N(5,0.3) \), the data generated from standard normal distributions in Section 6.2 can be linearly transformed to the data with normal distributions. This is to use same CDFs and PDF contours for a given data because CDFs and PDF contours are not changed during linear transformation. Thus, from the new data, Kendall’s tau is \( \tau = 0.350 \); Pearson’s rho is \( \rho = 0.526 \); and mean values and standard deviations of \( X_1 \) and \( X_2 \) are \( \mu_1 = 4.975 \), \( \mu_2 = 4.987 \), \( \sigma_1 = 0.302 \), and \( \sigma_2 = 0.302 \) respectively. As shown in Figs. 6, 7, and 8, since PDF contours and CDFs are similar for 500 data and infinite data, RBDO results are close as shown in Fig. 9. Figures 9 (a) and (b) shows the optimal design points and target contours (\( \| \bar{u} - \bar{\beta} \| = 0 \)) of the independent copula, Gaussian copula, Clayton copula, and true model, for 500 data and infinite data, respectively. As can be seen in Figs. 9 (a) and (b), the optimum design obtained using Clayton copula is close to the optimum design obtained using the true input model, which is also confirmed in Tables 1 and 2.

On the other hand, if Gaussian copula is used, the RBDO result is far from the optimum design obtained using the true copula. Moreover, when independent copula is used, i.e., when it is assumed that two input variables are independent, the RBDO results are very different from the optimum design obtained using the true copula. Thus, the input correlation, as well as the copula types, significantly affects the RBDO results.
8. Discussions and Conclusion

In this paper, an RBDO method that deals with the correlation of input variables using Gaussian copula and non-Gaussian copula is proposed. For this, two representative transformation methods, Rosenblatt transformation combined with non-Gaussian copula such as Clayton copula and Nataf transformation using Gaussian copula, are investigated for applicability to RBDO problems with correlated input variables. Rosenblatt transformation is a mathematically exact transformation method, but it has limited applications since the transformation of original random variables to standard normal variables can be carried out only when a joint CDF or conditional CDFs are available. Thus, it is not applicable to practical applications where only the covariance matrix and marginal distribution are available. On the other hand, Nataf transformation, which uses Gaussian copula in the copula family, is found to be practically applicable where only covariance matrix and marginal distributions are valid. Nataf transformation can construct an exact joint CDF when the input variables are normal or when the normal and lognormal variables are combined. Further, when the input variables are lognormal, it can accurately construct a joint CDF for positive or even some negative correlations. However, Nataf transformation is not applicable for non-Gaussian correlated variables. In this paper, a new transformation method that combines Rosenblatt transformation with non-Gaussian copula is proposed for RBDO of problems with non-Gaussian
correlated input variables to demonstrate that the correlation and joint distribution types of correlated input variables significantly influence the optimum results of RBDO.

9. Acknowledgement
Research is supported by the Automotive Research Center that is sponsored by the U.S. Army TARDEC.

10. References