Dimension Reduction Method (DRM) Based RBDO for Highly Nonlinear Systems

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1. Abstract
There are two commonly used reliability analysis methods: linear approximation - First Order Reliability Method (FORM); and quadratic approximation - Second Order Reliability Method (SORM), of the performance functions. The reliability analysis using FORM could be acceptable for mildly nonlinear performance functions, whereas the reliability analysis using SORM is usually necessary for highly nonlinear performance functions of multi-variables. However, SORM requires the second-order sensitivities, and thus, the SORM-based inverse reliability analysis is very difficult to develop.

This paper proposes an inverse reliability analysis method that can be used for multi-dimensional highly nonlinear systems to yield very accurate failure rate calculation without requiring the second order sensitivities and an RBDO method using the inverse reliability analysis result. For this purpose, the univariate dimension reduction method (DRM) is used. Since the FORM-based reliability index ($\beta$) could be inaccurate for the most probable point (MPP) search, a three-step computational process is proposed to carry out the inverse reliability analysis: constraint shift, reliability index update using DRM, and MPP search using the updated reliability index. Using the three steps, a new DRM-based MPP is obtained, which estimates the failure rate of the performance function more accurately than FORM and more efficiently than SORM. The DRM-based MPP is then used for the next design iteration of RBDO, and thus yields an accurate optimum design even for highly nonlinear system. Since the DRM-based RBDO requires more function evaluations, the enriched performance measure approach (PMA+) with new tolerances for constraint activeness and reduced rotation matrix is used to reduce the number of function evaluations.

2. Keywords: dimension reduction method (DRM), inverse reliability analysis, first-order reliability method (FORM), second-order reliability method (SORM), most probable point (MPP) update, reliability-based design optimization (RBDO)

3. Introduction
In recent years, there have been various attempts to develop enhanced reliability analysis methods to accurately compute the probability of failure of a performance function. The most popular reliability analysis methods are (1) MPP-based method, (2) simulation or sampling method, and (3) probability density function (PDF) approximation method. The MPP-based method includes the First Order Reliability Method (FORM) [1,2] and the Second Order Reliability Method (SORM) [3,4]. FORM or SORM computes the failure rate by approximating the performance function $G(X)$ using the first or second order Taylor series expansion at the most probable point (MPP). Since the FORM or SORM-based method requires MPP search, the sensitivities are usually used for the methods. When the sensitivities are not available, the response surface method can be used [5, 6]. The simulation or sampling method, such as Monte Carlo simulation (MCS) [7], importance sampling method [8], and Latin cube sampling method [9], can be readily used for the failure rate
calculation since these methods do not require any analytic derivations. The PDF approximation method [10-12] evaluates PDF of the performance function by assuming a general distribution type and then, using the approximated PDF, the method evaluates the failure rate of the performance function.

Among these methods, the MPP-based method is still very popular since it can be very effectively used for the inverse reliability analysis. However, the reliability analysis using FORM could be very well erroneous if the multi-dimensional performance functions are highly nonlinear. This is because FORM approximates the performance function using a linear function, which cannot reflect the complexity of nonlinear and high dimensional functions. Although the reliability analysis using SORM may be accurate, it is not desirable to use since SORM requires the second-order sensitivities, which are difficult and very expensive to obtain in practical engineering problems. The accuracy of the response surface method is still challenging, especially for the highly nonlinear problems that require high reliability, even though the method could be efficient. The simulation or sampling-based method could be accurate. However, they require a very large number of function evaluations. The PDF approximation can also give accurate results, however the method generally needs to be combined with the response surface method for the design optimization [12], which may have accuracy problem.

The dimension reduction method (DRM) [13,14] has been recently proposed to represent a multi-dimensional function using the sum of one-dimensional functions. Because of its wide applicability, DRM has been applied to robust design [15,16], reliability-based design optimization [17], and PDF approximation [11,12]. For robust design and PDF approximation, DRM is used to calculate the statistical moments of the performance function. However, DRM has not been used for the inverse reliability analysis, which is essential for the performance measure approach (PMA) of reliability-based design optimization (RBDO) [18].

In this paper, DRM is used to calculate the failure rate accurately, which is then used to develop an enhanced inverse reliability analysis method (i.e., MPP search) that is accurate for multi-dimensional highly nonlinear problems. Then PMA of RBDO is developed using the updated MPP. Since the DRM-based inverse reliability analysis still requires MPP search, it can be categorized as a MPP-based method. A three-step computational process is proposed to accurately and efficiently carry out the inverse reliability analysis using DRM: constraint shift, reliability index (β) update, and MPP update. Using the three steps, a new DRM-based MPP is obtained, which is used for the next design iteration of RBDO. Since the DRM-based RBDO requires more function evaluations, the enhanced hybrid mean value (HMV+) [19] is used for the efficient inverse reliability analysis and the enriched performance measure approach (PMA+) [18] with newly proposed efficient methods, which are new tolerances for constraint activeness and reduced rotation matrix, are used for efficient design optimization.

Numerical examples show that the DRM-based reliability analysis can estimate the failure rate of the performance function more accurately than the FORM-based reliability analysis and more efficiently than the SORM-based reliability analysis. The examples also show that an optimum design obtained by DRM can be different from an optimum by FORM depending on the type of constraint functions, which means that the DRM-based RBDO yields an accurate optimum design even for highly nonlinear systems.

4. Failure Rate Calculation Using Dimension Reduction Method (DRM)

4.1. MPP-based Dimension Reduction Method

The DRM [13,14,17,20] is a newly developed technique to accurately and efficiently approximate a multi-dimensional integral. There are several DRM methods depending on the level of dimension reduction: (1) univariate dimension reduction, which is an additive decomposition of N-dimensional performance function into one-dimensional functions; (2) bivariate dimension reduction, which is an additive decomposition of N-dimensional performance function into at most two-dimensional functions; (3) multivariate dimension reduction, which is an additive decomposition of N-dimensional performance function into at most S-dimensional functions, where S ≤ N. In this paper, the univariate DRM is used for computation of probability of failure because of its simplicity and efficiency.

In the univariate DRM, any N-dimensional performance function \( G(\mathbf{X}) \) can be additively decomposed into one-dimensional functions at the MPP of the random vector \( \mathbf{X} \) as

\[
G(\mathbf{X}) \equiv \tilde{G}(\mathbf{X}) = \sum_{i=1}^{N} G(x^*_i, x_{i+1}^*, \ldots, x_N^*) - (N-1)G(\mathbf{x}^*)
\]

where \( \mathbf{x}^* = [x^*_1, x^*_2, \ldots, x^*_N]^T \) is the MPP of the performance function \( G(\mathbf{X}) \) and \( N \) is the number of random variables. For example, if \( G(\mathbf{X}) = G(X_1, X_2) \) with \( N = 2 \), then the univariate additive decomposition of \( G(\mathbf{X}) \) is

\[
G(\mathbf{X}) \equiv \tilde{G}(\mathbf{X}) = G(X_1^*, X_2^*) + G(x^*_1, x^*_2) - G(x^*_1, x^*_2)
\]

As shown in Eq. (1), the univariate DRM approximates the performance function \( G(\mathbf{X}) \) using the sum of one-dimensional functions. Consequently, if there are off-diagonal or mixed terms in the performance function \( G(\mathbf{X}) \), then there should be some error that results from approximating off-diagonal terms using sum of one-dimensional
functions. To reduce this error, the bivariate DRM or multivariate DRM can be used [13,14].

4.2. Rotated Standard Normal V-space

Consider a performance function \( G(X) \) that depends on \( X=[X_1, X_2, \cdots, X_N]^T \) and whose MPP is denoted as \( x^*=[x^*_1, x^*_2, \cdots, x^*_N]^T \). Since the reliability analysis is performed in the standard normal U-space that is obtained using Rosenblatt transformation [21], MPP in U-space is denoted by \( u^*=[u^*_1, u^*_2, \cdots, u^*_N]^T \) and defined as the closest point on the limit state function \( G(U)=0 \) to the origin in U-space (mean in X-space). The distance from MPP to the origin is commonly called the Hasofer-Lind reliability index [1,2] and denoted by \( \beta_{HL} \).

To obtain the rotated standard normal V-space from U-space, construct an orthogonal matrix \( R \in \mathbb{R}^{N \times N} \) whose \( N \)th column is \( \alpha_i = \frac{u^*_i}{\beta_{HL}} \), i.e., \( R = [R_i]_{i=1}^{N} \), where \( R_i \in \mathbb{R}^{N \times N} \) satisfies \( (\alpha^*)^T R_i = 0 \in \mathbb{R}^{1 \times N} \) and \( N \) is the number of design variables [17,20]. Using an orthogonal transformation \( u = Rv \), \( v \) can represent the rotated standard normal V-space with the associated MPP \( v^* = [0, \cdots, 0, \beta_{HL}]^T \). The orthogonal matrix \( R \) can be found, for example, by Gram-Schmidt orthogonalization. However, the orthogonal matrix \( R \) is not uniquely determined. Figure 1 shows U-space and V-space for \( N = 2 \).

![Figure 1. Standard Normal U-space and Rotated Standard Normal V-space [17]](image)

4.3. FORM and SORM

A reliability analysis entails calculation of probability of failure, denoted by \( P_f \), which is defined using a multi-dimensional integral [22]

\[
P_f = P[G(X) > 0] = \int_{G(X)=0} f_X(x) dx
\]

where \( X=[X_1, X_2, \cdots, X_N]^T \) is an N-dimensional random vector, \( G(X) \) is the performance function such that \( G(X) > 0 \) is defined as failure, and \( f_X(x) \) is a joint probability density function (PDF) of \( X \). Since evaluation of Eq. (3) is very difficult or sometimes impossible to carry out in real engineering applications, many methods have been suggested to approximate Eq. (3): representatively, FORM and SORM. In this section, FORM and SORM to approximate Eq. (3) are briefly reviewed and a new method to approximate Eq. (3) using DRM is explained and compared with FORM and SORM in Section 4.4.

4.3.1. Failure Rate Calculation Using FORM and SORM

To calculate the failure rate of the performance function \( G(x) \) using FORM and SORM, it is necessary to find MPP on the limit state function in the standard normal U-space as shown in Fig. 1. The MPP can be found by solving the following optimization problem to
minimize $\|u\|
subject to \quad G(u) = 0 \quad (4)$

After finding MPP, the Hasofer-Lind reliability index $H_L$ can be obtained by measuring the distance between MPP and the origin. Using the reliability index $H_L$, FORM can approximate the probability of failure using a linear approximation of the performance function as

$$P_f^{\text{FORM}} \simeq \Phi(-H_L) \quad (5)$$

where $\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp\left(-\frac{1}{2} \xi^2\right) d\xi$ is the cumulative distribution function (CDF) of the standard Gaussian random variable.

The MPP obtained by solving Eq. (4) is also used for the failure rate calculation using SORM. Using a quadratic approximation of the performance function in and the rotational transform from $U$-space to $V$-space explained in Section 4.2., the failure rate can be obtained using SORM as [3,4,20]

$$P_f^{\text{SORM}} \simeq \Phi(-H_L) \left| I_{N-1} - 2 \frac{\phi(H_L)}{\Phi(-H_L)} \hat{A}_{N-1} \right|^2 \quad (6)$$

where $\hat{A} = \begin{bmatrix} \hat{A}_{N-1} & \hat{A}_{1N} \\ \hat{A}_{N1} & \hat{A}_{NN} \end{bmatrix} = \frac{1}{2} [\nabla G]_0^T \mathbf{R}^T \mathbf{H} \mathbf{R}$, $\mathbf{H}$ is the Hessian matrix evaluated at MPP, $\mathbf{R}$ is the rotation matrix such that $u = \mathbf{R}v$, and $\phi(\cdot)$ is the PDF of a standard Gaussian random variable.

### 4.3.2. Error in FORM-based Reliability Analysis

Although FORM has been widely used for the reliability analysis and inverse reliability analysis due to its simplicity and efficiency, FORM could be erroneous if the multi-dimensional performance function is highly nonlinear as shown in the following example. Consider

$$G(X) = -\beta - a \sum_{i=1}^{N} X_i^2 + X_N \quad (7)$$

where $X_i \sim N(0,1)$ for $i = 1, N$. The performance function has an MPP at $x^* = \{0, \cdots, 0, \beta\}^T$ and the failure rate by FORM is $P_f^{\text{FORM}} \simeq \Phi(-\beta)$ regardless of $a$ and $N$. If $\beta = 2$, then the failure rate by FORM becomes $P_f^{\text{FORM}} \simeq \Phi(-2) = 2.2750\%$. This failure rate can be compared with the results obtained using Monte Carlo simulation (MCS) for different $a$ and $N$, respectively, as shown in Tables 1 and 2.

<table>
<thead>
<tr>
<th>Table 1. MCS Result of Eq. (7) When $N = 2$</th>
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</thead>
<tbody>
<tr>
<td>$a = 0.2$</td>
</tr>
<tr>
<td>$P_f^{\text{MCS}}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2. MCS Result of Eq. (7) When $a = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 2$</td>
</tr>
<tr>
<td>$P_f^{\text{MCS}}$</td>
</tr>
</tbody>
</table>

From Tables 1 and 2, it can be seen that the failure rate obtained using FORM has significant error when a performance function is highly nonlinear (i.e. larger $a$) and especially for a high dimensional problem (i.e. larger $N$). These errors can be improved by SORM since SORM uses a quadratic approximation of the performance function. However, the Hessian matrix is required to calculate the failure rate in Eq. (6) using SORM, which is very difficult or sometimes impossible to accurately estimate in real engineering applications. For this reason, SORM has been limitedly applied in engineering applications.

### 4.4. Failure Rate Calculation Using DRM

#### 4.4.1. Inverse Reliability Analysis

The reliability analysis presented in Section 4.3.1 is called Reliability Index Approach (RIA) since it finds the reliability index $H_L$ using Eq. (4). The advantage of RIA is that the failure rate for the performance function can be calculated at a given design, for example, using Eqs. (5) and (6). However, it is well known that the inverse reliability analysis in
Performance Measure Approach (PMA) [18] is much more efficient than reliability analysis in RIA. PMA does not calculate the failure rate directly. Instead, PMA judges whether or not a given design satisfies the probabilistic constraint with a given target reliability index \( \beta \) by solving the following optimization problem to maximize \( G(\mathbf{u}) \)
subject to \( |\mathbf{u}| = \beta \) \( (8) \)
Since Eq. (8) is the inverse problem of Eq. (4), this is called the inverse reliability analysis. The optimum point of Eq. (8) is also called MPP and denoted by \( \mathbf{u}^* \). If the constraint function value at the MPP, \( G(\mathbf{u}^*) \), is less than zero, then the probabilistic constraint is satisfied with the given target reliability \( \beta \) and target failure rate \( P_{f \text{tar}} = \Phi(-\beta) \) by FORM. The inverse reliability analysis using SORM is much more difficult and has not been developed yet. Moreover, it requires the second order sensitivity.

4.4.2. Failure Rate Calculation Using DRM and Inverse Reliability Analysis
In the rotated standard normal \( V \)-space, the probability of failure in Eq. (3) can be rewritten as
\[
P_f = P[G(V) > 0] = \int_{G(V) > 0} f_v(v) dv \quad (9)
\]
Since the inverse reliability analysis does not calculate the failure rate, a constraint shift concept is introduced for the failure rate calculation such that
\[
G^*(v) = G(v) - G(v^*) \quad (10)
\]
where \( G^*(v) \) is a shifted performance function and \( v^* = [0, \ldots, 0, \beta]^T \) is the FORM-based MPP in \( V \)-space with a given reliability index \( \beta \). By applying the MPP-based DRM explained in Section 4.2. to \( G^*(v) \), \( G^*(v) \) can be approximated at the MPP \( v^* \) as
\[
G^*(v) \equiv \tilde{G}(v) = \sum_{i=1}^{N} G_i(v_i) - (N-1)G^*(v^*) = G_i^*(v_i) + \sum_{i=1}^{N} G'_i(v_i) - (N-1)G^*(v^*) \quad (11)
\]
where \( G_i(v_i) = G_i(v_i^*, \ldots, v_i^*, v_i, v_i^*, \ldots, v_i^*) \) and \( G'_i(v_i) \) is the gradient of \( G_i(v_i) \) at the MPP. By the definition of \( G^*(v) \) in Eq. (10), \( G^*(v^*) \) is zero, thus, we obtain
\[
G^*(v) \equiv \tilde{G}(v) = G_i^*(v_i) + \sum_{i=1}^{N} G'_i(v_i) \quad (12)
\]
Due to the rotational transformation of the coordinates as shown in Fig. 1, the \( N \)-th univariate component \( G'_i(v_i) \) can be linearly approximated [20]. This linear assumption of \( G'_i(v_i) \) along \( v_i \)-axis is also used for the failure rate calculation in SORM [3, 4]. Using the linear assumption, Eq. (9) can be written as
\[
P_f = P[G(V) > 0] \equiv P[G_i^*(V_i) + \sum_{i=1}^{N} G'_i(v_i) > 0] \equiv P[h + b_i V_i + \sum_{i=1}^{N} G'_i(v_i) > 0] \quad (13)
\]
Since function value and gradient at the MPP are used during the inverse reliability analysis, \( b_i + h_i V_i \) can be rewritten using the first order Taylor series expansion at the MPP as
\[
G_i^*(V_i) \equiv b_i + h_i V_i = G_i^*(V_i^*) + \frac{\partial G_i^*(V)}{\partial V_i} \bigg|_{V_i^*} (V_i - V_i^*) = \frac{\partial G_i(V)}{\partial V_i} \bigg|_{V_i^*} (V_i - \beta) \quad (14)
\]
where \( G'_i(V_i^*) = 0 \), \( V_i^* = \beta \), and \( \frac{\partial G_i^*(V)}{\partial V_i} \bigg|_{V_i^*} = \frac{\partial G_i(V)}{\partial V_i} \bigg|_{V_i^*} = h_i \). Inserting Eq. (14) into Eq. (13) yields
\[
P_f \equiv P[h + b_i V_i + \sum_{i=1}^{N} G'_i(v_i) > 0] \equiv P[h(V_i - \beta) + \sum_{i=1}^{N} G'_i(v_i) > 0] \equiv E[\Phi(-\beta + \sum_{i=1}^{N} G'_i(v_i))] \quad (15)
\]
Since the gradient \( h_i \) at MPP is always positive due to maximization in Eq. (8), Eq. (15) can be rewritten, by dividing both sides by \( h_i \) and using the symmetry of the standard normal distribution since \( V \sim N(0,1) \), as
\[
P_f \equiv P[V_i > \beta - \frac{1}{h_i} \sum_{i=1}^{N} G'_i(v_i)] \equiv P[V_i < -\beta + \frac{1}{h_i} \sum_{i=1}^{N} G'_i(v_i)] = E[\Phi(-\beta + \frac{1}{h_i} \sum_{i=1}^{N} G'_i(v_i))] \quad (16)
\]
where \( E \) is the expectation operator. Since Eq. (16) is a \( N-1 \) dimensional integration, Eq. (16) can be further simplified by applying DRM to the integrand of Eq. (16) as
\[
P_f^{\text{DRM}} \equiv \prod_{i=1}^{N-1} \frac{\int_{-\beta + \frac{1}{h_i} \sum_{i=1}^{N} G'_i(v_i)}^{\infty} \Phi(-\beta + \frac{1}{h_i} \sum_{i=1}^{N} G'_i(v_i)) dv_i}{\Phi(-\beta)^{N-2}}
\]
where $b_i$ can be obtained as $b_i = (\mathbf{u}_i^T \mathbf{u}_i)^{\frac{1}{2}} \frac{\partial G(\mathbf{u})}{\partial \mathbf{u}} \bigg|_{\mathbf{u} = \mathbf{u}_i} = \frac{\mathbf{u}_i^T}{\beta} \frac{\partial G(\mathbf{u})}{\partial \mathbf{u}} \bigg|_{\mathbf{u} = \mathbf{u}_i}$. Detailed derivation of Eq. (17) can be found in Ref. 17. Using the moment-based integration rule (MBIR) [23], which is similar to Gaussian quadrature [24], Eq. (17) is further approximated as

$$P_{\text{DRM}}^* \approx \prod_{i=1}^{N-1} \sum_{j=1}^{N} w_j \Phi(-\beta + G'_i(\mathbf{v}'_j))$$

where $w_j$ are weights, $\mathbf{v}'_j$ are quadrature points and $n$ is the number of quadrature points and weights. Since $\mathbf{v}_i$ are standard normal random variables, quadrature points and weights in Table 1 can be used to calculate Eq. (18) [24].

<table>
<thead>
<tr>
<th>$n$</th>
<th>Quadrature Points</th>
<th>Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>3</td>
<td>$\pm \sqrt{3}$</td>
<td>0.166667</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.666667</td>
</tr>
<tr>
<td>5</td>
<td>$\pm 2.856970$</td>
<td>0.011257</td>
</tr>
<tr>
<td></td>
<td>$\pm 1.355626$</td>
<td>0.222076</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.533333</td>
</tr>
</tbody>
</table>

For a special case of Eq. (18), assume $n = 1$, which means one quadrature point and weight, then Eq. (18) can be written as

$$P_{\text{DRM}}^* \approx \prod_{i=1}^{N-1} \Phi(-\beta + G'_i(\mathbf{v}'_j))$$

where $w_1 = 1$ and $\mathbf{v}'_1 = 0$ by Table 3 and $G'_i(\mathbf{v}'_1) = G'_i(0) = G'(\mathbf{v}^*) = 0$. Equation (19) is the same as the failure rate by FORM. Therefore, we can say that the failure rate calculation by FORM is a special case of the failure rate calculation by DRM when one quadrature point and weight is used.

The number of additional function evaluations needed to evaluate Eq. (18) besides MPP search is given by $(N-1)\times(n-1)$. Hence, the total number of function evaluations necessary for Eq. (18) is

$$\text{function evaluations for MPP search} + (N-1)\times(n-1)$$

Since the failure rate calculation using DRM requires integrations in Eq. (18), accuracy of the failure rate estimation can be easily achieved by increasing the number of quadrature points and weights. In this case, the failure rate by DRM requires only function values at the quadrature points, which are $G'_i(\mathbf{v}'_j)$ in Eq. (18). Consequently, the accuracy of the DRM result can be improved by increasing the number of quadrature points if necessary, which does not require any sensitivity. This comparison will be discussed in detail using numerical examples in Section 6.1.

5. DRM-Based Inverse Reliability Analysis and RBDO

The objective of the DRM-based inverse reliability analysis is to find a new DRM-based MPP, denoted by $\mathbf{x}_{\text{DRM}}^*$, using the MPP from the FORM-based inverse reliability analysis denoted by $\mathbf{x}_{\text{FORM}}^*$. As stated in Section 4.4.1, the inverse reliability analysis does not calculate the failure rate directly, instead, it judges whether or not a given design satisfies the probabilistic constraint by checking a performance function value at MPP. However, the probabilistic constraint may not be satisfied even though the constraint value at the FORM-based MPP $G(\mathbf{x}_{\text{FORM}}^*)$ is less than zero. This is because the failure rate calculated by FORM may have significant error especially for the multi-dimensional and highly nonlinear performance function. In this section, a new method is proposed to find a DRM-based MPP $\mathbf{x}_{\text{DRM}}^*$ using the FORM-based inverse reliability analysis, which is used for the next design iteration of RBDO and thus yields an accurate optimum design even for highly nonlinear system.

5.1. DRM-Based MPP

Finding a new DRM-based MPP consists of three steps: constraint shift, reliability index update, and MPP update.

A. Constraint Shift

As explained in Section 4, the constraint shift concept is introduced such that
\[ G'(x) = G(x) - G(x') \] or \[ G'(v) = G(v) - G(v') \] (21)
to make the shifted performance function \( G'(x) \) at MPP become zero, that is, \( G'(x) = G'(v') = 0 \). This performance function shift is carried out to calculate the failure rate using the inverse reliability analysis at the current design since the inverse reliability analysis cannot estimate the failure rate directly. Using the constraint shift in Eq. (21) and failure rate calculation in Eq. (18), the failure rate of the shifted performance function is accurately computed and compared with the target failure rate, denoted by \( P_F^{\text{tar}} \). The difference between the computed failure rate \( P_F^{\text{DRM}} \) for \( G'(x) \) and \( P_F^{\text{tar}} \) are used to update the reliability index \( \beta \).

B. Reliability Index Update
After computing the failure rate \( P_F^{\text{DRM}} \) using DRM for the shifted performance function \( G'(x) \), the corresponding reliability index \( \beta^{\text{DRM}} \) is obtained using \( \beta^{\text{DRM}} = -\Phi^{-1}(P_F^{\text{DRM}}) \). It is likely that \( \beta^{\text{DRM}} \) is not the same as the target reliability index \( \beta_t = -\Phi^{-1}(P_F^{\text{tar}}) \) and a new updated reliability index \( \beta_{\text{up}} \) is obtained using linear shift of the reliability index as

\[ \beta_{\text{up}} \approx \beta_{\text{cur}} - (\beta^{\text{DRM}} - \beta_t) \] (22)

where \( \beta_{\text{cur}} \) is the reliability index at the current step, which is the same as \( \beta_t \) at the initial step.

For example, consider a performance function in Fig. 1 and set target failure rate as \( P_F^{\text{tar}} = \Phi^{-1}(P_F^{\text{tar}}) = \Phi^{-1}(\beta_t) \). In this paper, a performance function is defined as concave near MPP if FORM-based reliability analysis overestimates the failure rate and convex near MPP if FORM-based reliability analysis underestimates the failure rate. Since the performance function in Fig. 1 is concave near the MPP, the failure rate calculated by DRM will be smaller than the target failure rate, which means \( \beta^{\text{DRM}} > \beta_t \). Hence, using Eq. (22), a smaller reliability index \( \beta_{\text{up}} \) will be obtained because \( \beta^{\text{DRM}} > \beta_t \). This means that a smaller reliability index than FORM-based reliability index should be used to satisfy the target failure rate for the concave performance function and vice versa for the convex performance function.

C. MPP Update Method
Using this updated reliability index, we can carry out a new inverse reliability analysis to find a better MPP which satisfies the given target failure rate. After finding a new MPP, constraint shift is again used to compute the failure rate by DRM. After iteratively doing this procedure until converged, a DRM-based MPP can be obtained where \( P_F^{\text{DRM}} \) is the same as \( P_F^{\text{tar}} \). However, it will be computationally very expensive if a new MPP search is carried out every time an updated reliability index is obtained. To achieve efficiency, the updated MPP corresponding to the failure rate by DRM is obtained without carrying out a new MPP search as \[ u_{\text{up}} = \frac{\beta_{\text{up}}}{\beta_{\text{cur}}} u_{\text{cur}} \text{ or } v_{\text{up}} = \frac{\beta_{\text{up}}}{\beta_{\text{cur}}} v_{\text{cur}} = \{0, \ldots, 0, \beta_{\text{up}} \}^T \] (23)

That is, it is assumed that the updated MPP \( u_{\text{up}} \) is located along the same radial direction with the current MPP \( u_{\text{cur}} \) in U-space as illustrated in Fig. 2.

![Figure 2. Approximation of Updated MPP](image-url)
For accuracy, the same iterative procedure explained above can be performed until $P_F^{DRM}$ converges to $P_F^{opt}$. The updated MPP obtained through the iterative procedure is called the DRM-based MPP, which will be used to evaluate whether the design satisfies the probabilistic constraint or not. However, this iterative procedure still requires additional function evaluations. To reduce the number of function evaluations, MPP update can be carried out only once at a given design. Even though the MPP update is used once at the given design, the failure rate by DRM will converge to the target failure rate as the design moves near the reliability-based optimum design.

5.2. DRM-Based RBDO
The DRM-based MPP $x_{RM}^{DRM}$ obtained using Eq. (23) is used for the next design iteration of RBDO. Thus, the DRM-based RBDO is formulated to

\[
\text{minimize } \text{Cost}(d) \\
\text{subject to } G_{RM}^i(X(d)) \leq 0, \quad i = 1, \ldots, nc
\]

where $G_{RM}^i$ is the $i^{th}$ probabilistic constraint evaluated at the DRM-based MPP; $d = \{d_i\}^T = \mu(X)$ is the design vector; $X = \{X_i\}^T$ is the random vector; and $nc$, $ndv$ and $nrv$ are the number of probabilistic constraints, design variables, and random variables, respectively. The detailed algorithm of the proposed DRM-based RBDO for Eq. (24) is as following.

**Step 1.** Find a FORM-based MPP using the inverse reliability analysis with a given reliability index $\beta_{cur}$.

**Step 2.** Calculate the failure rate $P_F^{DRM}$ using Eq. (18).

**Step 3.** Using Eq. (22), update the reliability index from $\beta_{cur}$ to $\beta_{up}$.

**Step 4.** Find a new DRM-based MPP using Eq. (23).

**Step 5.** Calculate a function value and sensitivities at the new DRM-based MPP.

**Step 6.** Run a design optimization to find the next design.

**Step 7.** Save the updated reliability index $\beta_{up}$ to $\beta_{cur}$.

**Step 8.** Check convergence.

**Step 9.** If not converged, then repeat Step 1 ~ 8 until converged.

5.3. Strategy for Efficiency of DRM-Based RBDO
As shown in Eq. (18), the number of function evaluations to calculate the failure rate using DRM will increase as the number of design variables increases. However, certain design variables may not affect the performance function. In that case, using a reduced rotation matrix will reduce the number of function evaluation for Eq. (18). In addition, since Eq. (18) requires the FORM-based MPP search, the enhanced hybrid mean value (HMV+) [19] is used for the efficient inverse reliability analysis. Also, since Eq. (24) requires the design optimization, the enriched performance measure approach (PMA+) [18] is used for the efficient design optimization. PMA+ includes three key ideas: launching RBDO at a deterministic optimum, feasibility check using constraint activeness, and design closeness. In this section, new tolerances for constraint activeness is introduced to enhance numerical efficiency of DRM-based RBDO and a deterministic optimum with shift [26] is used instead of using a deterministic optimum directly. Hence, this section discusses two strategies, reduced rotation matrix and new tolerances for constraint activeness, to improve efficiency of DRM-based RBDO.

A. Reduced Rotation Matrix
As explained in Section 4.2., $N \times N$ rotation matrix is used to transform from $U$-space to $V$-space. This rotated standard normal variable $v$ is used to compute the failure rate in Eq. (18). If the random variable $X_i$ does not affect the performance function, that is, $u_i^* = 0$ or $\frac{\partial G_i}{\partial u_i}|_{u_i=0} = 0$, then $G_i'(v_i^*) = \cdots = G_i'(v_i^*) = G_i'(v_i^*) = 0$ since $v_i$ is a function of $u_i$ (i.e., a function of $x_i$) only and $x_i$ does not affect the performance function. Hence, an integral along $i^{th}$ axis can be expressed as $\sum_{j=1}^{m} v_j \Phi(-\beta + \frac{G_i'(v_i^*)}{b_i}) = \Phi(-\beta)$. If there are $N_0$ random variables that does not affect the performance function, then Eq. (18) can be rewritten as
where \( N_e \) is the effective number of random variables defined by \( N_e = N - N_0 \). Since the reduced number of random variables does not change the failure rate as shown in Eq. (25), we can use \( N_e \times N_e \) rotation matrix, which will reduce the number of function evaluations needed to compute the failure rate in addition to the FORM-based MPP search from \((n-1)(N-1)\) to \((n-1)(N_e-1)\). The reduced rotation matrix, which has full rank, can be also generated using the Gram-Schmidt orthogonalization.

### B. New Tolerances for Constraint Activeness

PMA+ provides an efficient feasibility identification using the mean value method (MV), which does not require MPP search. In PMA+, a constraint function is identified as active or violated if \( G_{pi}^M (X_i) + \varepsilon_j \geq 0 \), where the tolerance \( \varepsilon_j \) is a small positive number and \( G_{pi}^M (X_i) \) is the function value at the MV-based MPP. After feasibility identification, if a constraint is identified as active or violated, then an accurate MPP search is carried out using HMV+. However, a single tolerance \( \varepsilon_j \) may not be effective to identify the feasibility of all constraint functions, since each constraint has different gradients. To avoid this difficulty, this paper proposes to adaptively identify feasibility of the constraint functions using the sensitivities at a given design as

\[
G_{pi}^M (X_i) + \frac{\| \nabla G_i \|}{\sqrt{N}} \varepsilon_j \geq 0 \quad \text{(26)}
\]

where \( L_2 \) norm of the sensitivity of \( i^{th} \) constraint is normalized using \( \sqrt{N} \) to eliminate dimensionality of the norm. However, in a case that \( \frac{\| \nabla G_i \|}{\sqrt{N}} \) becomes very large, the feasibility identification using Eq. (26) may be too conservative, which makes the constraint activeness strategy ineffective. Hence, in this paper, a constraint is identified as active or violated based on the normalized \( L_2 \) norm of the sensitivities if

\[
G_{pi}^M (X_i) + \frac{\| \nabla G_i \|}{\sqrt{N}} \varepsilon_j \geq 0, \quad \text{if} \quad \frac{\| \nabla G_i \|}{\sqrt{N}} \leq 1
\]

\[
G_{pi}^M (X_i) + \varepsilon_j \geq 0, \quad \text{otherwise} \quad \text{(27)}
\]

### 6. Numerical Examples

Accuracy of the DRM-based failure rate is verified by comparing it with the FORM and SORM-based failure rate in Section 6.1. For this purpose, the failure rate obtained using MCS is used as a benchmark data. Section 6.2 shows the effectiveness of the reduced rotation matrix in terms of accuracy and efficiency. Section 6.3 compares various RBDO results including FORM-based RBDO, DRM-based RBDO, and DRM-based RBDO with efficiency strategy.

#### 6.1. Comparison of FORM, SORM and DRM for Failure Rate Calculation

For the first example, a highly nonlinear fourth order polynomial function

\[
G_i^M (X) = 0.7361 + (Y - 6)^2 + (Y - 6)^3 - 0.6 \times (Y - 6)^4 + Z
\]

where \( X \) and \( Y \) are normally distributed with means 0.9063 and 0.4226, respectively, and \( X_1 \sim N(4,0.3) \) and \( X_2 \sim N(3,0.3) \), is used for the failure rate computation. The reliability index of \( \beta = 1.645 \) is used for the FORM-based inverse reliability analysis.

Figure 3(a) shows the shifted and original performance functions and Fig. 3(b) shows the approximated functions by FORM, SORM, and DRM at the MPP in \( V \)-space. In Table 4, DRM with three and five quadrature points are used to evaluate Eq. (18). From Table 4, it can be seen that DRM with five quadrature points can be the most accurate method for this example. In fact, this result is even more accurate than the SORM result, compared with the MCS result, which can be considered as exact. In terms of efficiency, FORM shows the best efficiency, which is always true since SORM and DRM require the FORM-based MPP. However, the additional number of function evaluations for DRM besides the MPP search does not require sensitivity analysis. Hence, DRM can estimate the failure rate as accurately as SORM without requiring the second-order sensitivity calculation and as efficiently as FORM without loss of accuracy - the error of the FORM result is about 52% as shown in Table 4.
Table 4. Failure Rate Calculation by Various Methods For 2-D Example

<table>
<thead>
<tr>
<th></th>
<th>FORM</th>
<th>SORM</th>
<th>DRM</th>
<th>MCS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_F, %$</td>
<td>5.000</td>
<td>3.4081</td>
<td>3.5844</td>
<td>3.3676</td>
</tr>
<tr>
<td>FE</td>
<td>7*</td>
<td>7** + Hessian</td>
<td>7** + 2**</td>
<td>7** + 4**</td>
</tr>
</tbody>
</table>

* 7 means the number of function and sensitivity analysis for MPP search
** 2 and 4 function evaluations for DRM do not require sensitivity analysis

For the second example, a four dimensional quadratic function

$$G_i(X) = -X_1^2 - X_2^2 - X_3^2 - X_4^2 + 9X_1 + 11X_2 + 11X_3 + 11X_4 - 95.75$$

where $X_i \sim N(5, 0.4)$ for $i=1,2,3,4$, is used for the failure rate computation. The reliability index of $\beta = 1.645$ is used for the FORM-based inverse reliability analysis. As described in Eq. (20), the total number of function evaluations for the DRM-based reliability analysis will increase as the number of random variables increases. Since the performance function in Eq. (29) has four random variables, $4 \times (4 - 1) = 6$ function evaluations are required for DRM with three quadrature points; and $(4 - 1) \times (5 - 1) = 12$ function evaluations are required for DRM with five quadrature points as shown in Table 5. Again, these function evaluations do not require sensitivity analysis. Even though the number of function evaluations increases for DRM, DRM still shows the best accuracy compared with the MCS results regardless of the number of input random variables.

Table 5. Failure Rate Calculation By Various Methods For 4-D Example

<table>
<thead>
<tr>
<th></th>
<th>FORM</th>
<th>SORM</th>
<th>DRM</th>
<th>MCS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_F, %$</td>
<td>5.0000</td>
<td>14.0396</td>
<td>11.8049</td>
<td>11.8976</td>
</tr>
<tr>
<td>FE</td>
<td>2*</td>
<td>2* + Hessian</td>
<td>2* + 6**</td>
<td>2* + 12**</td>
</tr>
</tbody>
</table>

* 2 means the number of function and sensitivity analysis for MPP search
** 6 and 12 function evaluations for DRM do not require sensitivity analysis

6.2. Effectiveness of Reduced Rotation Matrix

One of the constraint functions

$$G_i(X) = 0.4511 - 0.61X_2 - 0.163X_3 X_4 + 0.001232X_3 X_{10} - 0.166X_4 X_5 + 0.227X_3^2$$

from the side impact problem [15, 27] is used to test the effectiveness of the reduced rotation matrix. Since only six random variables out of eleven are included in the constraint function, $6 \times 6$ rotation matrix is used to compute the failure rate in Eq. (18). This reduced rotation matrix reduces the number of function evaluations from $(3 - 1) \times (11 - 1) = 20$ to $(3 - 1) \times (6 - 1) = 10$ when three quadrature points are used, while maintaining the accuracy as shown in Table 6. The
reliability index of $\beta = 2$ is used for the FORM-based inverse reliability analysis.

<table>
<thead>
<tr>
<th>Table 6. Effectiveness of Reduced Rotation Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>FORM</td>
</tr>
<tr>
<td>$P_r$, %</td>
</tr>
<tr>
<td>FE</td>
</tr>
</tbody>
</table>

* 3 means the number of function and sensitivity analysis for MPP search
** 10 function evaluations for DRM do not require sensitivity analysis

6.3. Comparison of Various RBDO Methods
Consider the following two-dimensional mathematic model for RBDO. The RBDO problem is formulated to minimize $C(\mathbf{d})$
subject to $P(G_i(\mathbf{X};\mathbf{d}) \geq 0) \leq P^\text{tar}_i, \quad i = 1, \cdots, nc$
where $\mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \quad \mathbf{d} \in R^{ndf}$ and $\mathbf{X} \in R^{nvn}$

$$
C(\mathbf{d}) = - \frac{(d_1 + d_2 - 10)^2}{30} - \frac{(d_1 - d_2 + 10)^2}{120}
$$

$$
G_1(\mathbf{X}) = 1 - \frac{X_1^2X_2}{20}
$$
$$
G_2(\mathbf{X}) = -1 + (0.9063X_1 + 0.4226X_2 - 6)^2 + (0.9063X_1 + 0.4226X_2 - 6)^3
- 0.6(0.9063X_1 + 0.4226X_2 - 6)^4 - (0.4226X_1 + 0.9063X_2)
$$
$$
X_i \sim N(d_i, 0.5) \text{ for } i=1,2
$$

where the target failure rate for each constraint is $P^\text{tar}_i = 5.0\%$, for $i=1\sim3$. Since the given target failure rate is 5.0%, the initial reliability index $\beta = -\Phi^{-1}(P^\text{tar}_i) = 1.645$ is selected, and during the design iteration the reliability index is updated using the DRM-based failure rate.

![Figure 4. Feasibility Identification Using PMA+ and New Tolerances for Constraint Activeness](image)

Figure 4 (a) shows the approximated feasible region for PMA+ with the tolerance $\varepsilon_r = 0.5$ and Fig. 4 (b) shows the approximated feasible region when the new tolerances for constraint activeness in Eq. (27) is used with the same
From Fig. 4 (a), we can see that even though the true MPP is far from the third constraint, the PMA+ will identify the third constraint as active because $G_{p_i}^{SM}(X(d)) + \varepsilon_j = -0.26 + 0.5 > 0$. This happens because PMA+ does not consider the magnitude of gradients of each constraint, which may not be close to each other, but uses single tolerance $\varepsilon_j$ for all constraints. In this example, we can see that $\varepsilon_j = 0.5$ is rather large for the third constraint compared to the other two constraints. However, using Eq. (27), the approximated feasible region can describe the true feasible region more reasonably. In Fig. 4 (b), since the third constraint is identified as inactive because $G_{p_i}^{SM}(X(d)) + \|V_{G_{p_i}}\|_{\sqrt{2}} \varepsilon_j < 0$, and thus MPP search for the third constraint is not carried out at the deterministic optimum design with shift, which saves the number of function evaluations from 109+73 to 83+59 as shown in Table 8. Table 8 also shows that even though the optimum of FORM-based RBDO seems to be close to the optimum of DRM-based RBDO, the failure rate computed at the optimum shows significant difference, especially for the highly nonlinear second constraint, compared to the MCS results. For the second constraint, three quadrature points may not be sufficient to detect the high nonlinearity of the constraint. In this case, five quadrature points can be used to enhance the accuracy as shown in Table 8. Table 7 illustrates the history of DRM-based RBDO with three quadrature points when new tolerances for constraint activeness are used.

Table 7. DRM-Based RBDO (3pts) with New Tolerances for Constraint Activeness

<table>
<thead>
<tr>
<th>Iter.</th>
<th>Cost</th>
<th>Optimum Design</th>
<th>$G_1(X)$</th>
<th>$G_2(X)$</th>
<th>$G_3(X)$</th>
<th># of FE</th>
</tr>
</thead>
<tbody>
<tr>
<td>D.O.</td>
<td>-1.28</td>
<td>4.621, 3.091</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.39</td>
<td>11+11</td>
</tr>
<tr>
<td>0.1*</td>
<td>-1.28</td>
<td>4.621, 3.091</td>
<td>-0.83</td>
<td>-0.94</td>
<td>-0.26</td>
<td>11+11</td>
</tr>
<tr>
<td>0.2</td>
<td>-2.11</td>
<td>4.925, 1.123</td>
<td>0.65</td>
<td>0.71</td>
<td>-0.52</td>
<td>26+22</td>
</tr>
<tr>
<td>1.1</td>
<td>-1.83</td>
<td>4.604, 1.730</td>
<td>0.14</td>
<td>0.00</td>
<td>-0.50</td>
<td>41+33</td>
</tr>
<tr>
<td>1.2</td>
<td>-1.83</td>
<td>4.606, 1.727</td>
<td>0.14</td>
<td>0.00</td>
<td>-0.50</td>
<td>50+38</td>
</tr>
<tr>
<td>2.1</td>
<td>-1.77</td>
<td>4.678, 1.859</td>
<td>-0.01</td>
<td>-0.01</td>
<td>-0.45</td>
<td>65+49</td>
</tr>
<tr>
<td>2.2</td>
<td>-1.77</td>
<td>4.674, 1.853</td>
<td>0.00</td>
<td>-0.01</td>
<td>-0.45</td>
<td>74+54</td>
</tr>
<tr>
<td>3.1</td>
<td>-1.77</td>
<td>4.682, 1.849</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.45</td>
<td>83+59</td>
</tr>
<tr>
<td>Opt.</td>
<td>-1.77</td>
<td>4.682, 1.849</td>
<td>Active</td>
<td>Active</td>
<td>Inact.</td>
<td>83+59</td>
</tr>
</tbody>
</table>

* 0,1 means 0th iteration and 1st line search.

Table 8. Comparison of Various RBDO Results with the Target Failure Rate $P_{F_{10}}^{Tar} = 5.0%$

<table>
<thead>
<tr>
<th>Method</th>
<th>Cost</th>
<th>Optimum Design</th>
<th>MCS</th>
<th># of FE</th>
</tr>
</thead>
<tbody>
<tr>
<td>FORM***</td>
<td>-1.77</td>
<td>4.580,1.863</td>
<td>5.8128</td>
<td>2.5794</td>
</tr>
<tr>
<td>DRM*</td>
<td>3 pts</td>
<td>4.682,1.849</td>
<td>4.9857</td>
<td>3.8030</td>
</tr>
<tr>
<td></td>
<td>5 pts</td>
<td>4.717,1.833</td>
<td>4.9616</td>
<td>4.5010</td>
</tr>
<tr>
<td>DRM**</td>
<td>3 pts</td>
<td>4.682,1.849</td>
<td>4.9857</td>
<td>3.8030</td>
</tr>
<tr>
<td></td>
<td>5 pts</td>
<td>4.717,1.833</td>
<td>4.9616</td>
<td>4.5010</td>
</tr>
</tbody>
</table>

* means DRM with PMA+.
** means DRM with PMA+ and new tolerances for constraint activeness.
*** means FORM without PMA+.

Since the DRM-based RBDO updates the reliability index for each constraint, the reliability index at the optimum design will be different from the initial reliability index, for this example $\beta = -\Phi^{-1}(P_{F_{10}}^{Tar}) = 1.645$. As shown in Table 9, when DRM with five quadrature point is used, the reliability index for the second constraint at the optimum is 1.379, which is significantly reduced from the initial reliability index since the second constraint is highly nonlinear and concave near the MPP, and the reliability index for the first constraint at the optimum is 1.715, which is increased slightly since the first constraint is mildly nonlinear and convex near the MPP as shown in Fig. 5.

Table 9. Updated Reliability Index at the Optimum

<table>
<thead>
<tr>
<th>Method</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_{initial}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DRM with 3 pts</td>
<td>1.717</td>
<td>1.462</td>
<td>1.715</td>
<td>1.379</td>
<td>1.645</td>
</tr>
<tr>
<td>DRM with 5 pts</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
7. Discussions and Conclusion

Three methods to evaluate the probability of failure using FORM, SORM, and DRM are compared in terms of efficiency and accuracy. In terms of efficiency, the failure rate calculation by FORM is the best since the failure rate calculation by SORM and DRM uses the MPP of the FORM-based inverse reliability analysis. However, as shown using the examples in this paper, the failure rate calculation by FORM could be very erroneous in particular when the multi-dimensional performance function is highly nonlinear. Even though SORM can evaluate the probability of failure more accurately than FORM, SORM has limited application since SORM requires the second-order sensitivities. On the other hand, the failure rate calculation by DRM is as accurate as SORM, and sometimes even better than SORM, without requiring the second-order sensitivities.

The DRM-based accurate failure rate is used to find the DRM-based MPP through the DRM-based inverse reliability analysis, which can identify failure region of the performance function better than FORM-based MPP. A three-step computational process is proposed to find out the DRM-based MPP using the inverse reliability analysis: constraint shift, the reliability index update, and the MPP update. The DRM-based MPP is used in the design iteration of RBDO. Since the DRM-based RBDO requires a number of function evaluations, especially when the number of design variables is large, PMA+ with new tolerances for constraint activeness and reduced rotation matrix is used to enhance the efficiency. The design example shows that the optimum design of DRM-based RBDO is indeed different from the optimum design of FORM-based RBDO, and the failure rate by FORM at the optimum is significantly erroneous compared to the failure rate by DRM.

8. Acknowledgement

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9. References