Efficient Evaluation Approaches for Probabilistic Constraints in Reliability-Based Design Optimization

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1. Abstract

This paper presents new efficient evaluation methods for probabilistic constraints in reliability-based design optimization (RBDO) to substantially improve computational efficiency when applied to large-scale applications. Two different methods are presented: efficient identification of feasible probabilistic constraints and fast reliability analysis using the condition of design closeness. Unlike deterministic design optimization, a significant computational burden is imposed on feasibility check of constraints in the RBDO process due to expensive reliability analysis. Such difficulties can be effectively resolved by using a mean value (MV) first-order method with an allowable accuracy for the purpose of feasibility identification, and by carrying out the refined reliability analysis using the hybrid mean value (HMV) first-order method for \( \varepsilon \)-active and violate constraints in the RBDO process. The fast reliability analysis method is proposed by reusing some of the information obtained at the previous RBDO iteration to efficiently evaluate probabilistic constraints at the current design iteration under the assumption of the design closeness. To demonstrate numerical efficiency of the new RBDO process, two numerical examples including a large-scale multi-crash application are employed.

2. Keywords: Reliability-Based Design Optimization (RBDO), Probabilistic Feasibility Check, Performance Measure Approach (PMA), Hybrid Mean Value (HMV) Method, and Crashworthiness.

3. Introduction

Due to extensive efforts in many engineering disciplines over last three decades, design guidelines and/or standards have been or are being modified to incorporate the concept of uncertainty into product design. In response to these new requirements, various methods have been developed to treat uncertainties in engineering design and, more recently, to carry out design optimization with an additional requirement of reliability. The latter is referred to as reliability-based design optimization (RBDO). A significant computational burden is required to perform a feasibility check of probabilistic constraints, since their feasibility needs to be identified through a reliability analysis. Moreover, the significant information is generated to evaluate probabilistic constraints, which have not been reused in the RBDO process. This paper is thus focused on the development of efficient evaluation methods of probabilistic constraints in RBDO without sacrificing numerical accuracy: an efficient identification of feasible probabilistic constraints and a fast reliability analysis using the condition of design closeness.

To make the RBDO computationally affordable for large-scale applications, it is necessary to develop an efficient method of feasibility identification suitable for probabilistic constraints in RBDO. This can be carried out by employing a mean value (MV) first-order method that provides an allowable accuracy for the purpose of feasibility identification. Once the feasibility of probabilistic constraints is identified by the MV first-order method, the refined reliability analysis is performed using the hybrid mean value (HMV) first-order method to evaluate probabilistic constraints. Consequently, a numerical efficiency of the RBDO process can be substantially improved by using the proposed feasibility check for probabilistic constraints.

During RBDO iterations, significant information is generated to evaluate the cost and probabilistic constraints, and update the design. Some of this information could be reused to evaluate probabilistic constraints efficiently at the next iteration using the condition of design closeness. In other words, under the assumption that two consecutive designs in the RBDO iteration are close enough, a reliability analysis can be carried out efficiently by starting at the most probable point (MPP) obtained from the previous iteration, instead of at the mean value point at the current design iteration. The proposed fast reliability analysis method is integrated with the HMV method of the performance measure approach for efficient constraint evaluation and RBDO process. Large-scale industrial applications are used to demonstrate numerical efficiency of the proposed probabilistic constraint evaluation method in RBDO by comparing with RBDO without the proposed method.
4. Reliability-Based Design Optimization (RBDO)

For any engineering application, the RBDO model [1,2] can be generally formulated as

\[
\text{minimize } \text{Cost}(d)
\]
\[
\text{subject to } P(G(d(X)) \geq 0) - \Phi(-\beta_i) \leq 0, \quad i = 1, 2, \ldots, np
\]

where \( d = \mu(X) \) are the design vector, \( X = [X_i] \in \mathbb{R}^n \) are the random vector, and the probabilistic constraints are described by the performance function \( G(x) \) with \( G(X) \) indicates failure, their probabilistic models, and their prescribed confidence level \( \beta_i \). Performance function failure is statistically defined by a cumulative distribution function \( F_{\beta_i}(0) \) as

\[
P(G(X) \geq 0) = 1 - F_{\beta_i}(0) = 1 - \int_{G(X) \leq 0} f_X(x) \, dx \leq \Phi(-\beta_i)
\]

In Eq. (2), \( f_X(x) \) is the joint probability density function of all random parameters. Some approximate probability integration methods, such as the first-order reliability method (FORM) [1,2] with a rotationally invariant measure as the reliability, have been widely used to provide efficient and adequately accurate solutions. Through inverse transformations of \( F_{\beta_i}(\bullet) \), the probabilistic constraint in Eq. (2) can be expressed as:

\[
G_{\beta_i}(d(X)) = F_{\beta_i}^{-1}(1 - \Phi(-\beta_i)) = F_{\beta_i}^{-1}(\Phi(\beta_i)) \leq 0
\]

where \( G_{\beta_i} \) is the \( \beta_i \)th probabilistic constraint that replaces the probabilistic constraint in Eq. (1). It is referred to as the performance measure approach (PMA) [1,2].

The first-order probabilistic performance measure \( G_{\beta_i} \) is obtained from a nonlinear optimization problem with an n-dimensional explicit sphere constraint in a standard normalized \( U \)-space, defined as

\[
\text{minimize } G(U), \quad \text{subject to } \|U\| = \beta_i
\]

The optimum point on the target reliability surface is identified as the most probable point (MPP) \( u_{\beta_i} \). In this paper, a hybrid mean value (HMV) first-order method [1,2], which adaptively employs the AMV and conjugate mean value (CMV) first-order methods, is used to solve the inverse problem in PMA. This section introduces the HMV method that enhances numerical stability and efficiency of reliability analysis for PMA.

4.1 Advanced Mean-Value (AMV) Method

In order to minimize the performance function \( G(U) \) (the cost function in Eq.(4)), the normalized steepest descent direction \( n(U) \) of a performance function is introduced. The AMV method iteratively updates the steepest descent direction vector at the probable point \( u_{\beta_i}^{(k)} \). The AMV method can be formulated as [1]

\[
\begin{align*}
\text{AMV}^{(0)} & = 0 \\
\text{AMV}^{(1)} & = \beta_i n(U_{\text{AMV}}^{(0)}) \\
\text{AMV}^{(2)} & = \beta_i n(U_{\text{AMV}}^{(1)}) \\
\end{align*}
\]

The mean value (MV) first-order method is a special case of AMV method with only one iteration as follows

\[
\begin{align*}
\text{AMV}^{(0)} & = 0 \\
\text{AMV}^{(1)} & = \beta_i n(U_{\text{AMV}}^{(0)}) \\
\end{align*}
\]

4.2 Conjugate Mean-Value (CMV) Method

As shown in Ref. 1, the AMV method tends to be slow in the rate of convergence and/or to be divergent for the concave performance function \( G(U) \) due to lack of updated information during iterative reliability analyses. This difficulty can be overcome by using both the current and previous search directions to develop the proposed CMV method [1]. Thus, the new search direction is obtained by combining \( n(U_{\text{CMV}}^{(k-2)}), n(U_{\text{CMV}}^{(k-1)}), \) and \( n(U_{\text{CMV}}^{(k)}) \) with an equal weight, such that it is directed toward the diagonal of the three consecutive steepest descent directions. That is,

\[
\begin{align*}
\text{CMV}^{(0)} & = 0, \quad \text{CMV}^{(1)} = \text{AMV}^{(1)}, \quad \text{CMV}^{(2)} = \text{AMV}^{(2)}, \quad \text{CMV}^{(k)} = \beta_i n(U_{\text{CMV}}^{(k)}) + n(U_{\text{CMV}}^{(k-1)}) + n(U_{\text{CMV}}^{(k-2)}) \quad \text{for } k \geq 2
\end{align*}
\]

where

\[
\begin{align*}
n(U_{\text{CMV}}^{(k)}) & = -\nabla G(U_{\text{CMV}}^{(k)}) / \|\nabla G(U_{\text{CMV}}^{(k)})\|
\end{align*}
\]

Consequently, the conjugate steepest descent direction significantly improves the rate of convergence as well as stability compared to the AMV method for the concave performance function. However, the CMV method is shown to be inefficient for the convex function.

4.3 Hybrid Mean-Value (HMV) Method

To develop an efficient MPP search method, type of the performance function must first be identified. The function type can be determined by employing the steepest descent direction at three consecutive iterations as [1]
constraint set, in general, is comprised of probabilistic constraints are said to be a strategy of potential probabilistic constraint. The potential probabilistic constraints and their sensitivities. The numerical algorithms that use gradients of only a subset of the evaluated by the HMV method to search a design direction in the RBDO process.

The set of potential probabilistic constraints can be efficiently identified by the MV method and then accurately identified by considering system uncertainties through reliability analyses. This paper proposes an efficient feasibility identification of probabilistic constraints without involving full reliability analysis, while maintaining numerical accuracy. A feasibility check scheme for probabilistic constraint and a method of potential probabilistic constraint are discussed in this section.

5. An Efficient Feasibility Check for Probabilistic Constraints

Unlike a deterministic design optimization, the feasibility of probabilistic constraints at a design point needs to be identified by considering system uncertainties through reliability analyses. This paper proposes an efficient feasibility identification of probabilistic constraints without involving full reliability analysis, while maintaining numerical accuracy. A feasibility check scheme for probabilistic constraint and a method of potential probabilistic constraint are discussed in this section.

6. Fast Reliability Analysis Using Design Closeness

In this section, additional effort to reduce the computational burden in the RBDO process is discussed when evaluating probabilistic constraints. The fast reliability analysis in RBDO is to perform a reliability analysis more efficiently by utilizing some information obtained at the previous design iteration under the design closeness assumption. It is anticipated that the efficient reliability analysis under the design closeness assumption can save numerical computation toward the end of design iterations where consecutive designs tend to be close. The proposed method for reliability analysis is numerically implemented with the HMV method.

6.1 Design Closeness for Fast Reliability Analysis

The fast reliability analysis is carried out only if the design closeness is satisfied, thus evaluating probabilistic constraints more efficiently. The design closeness can be defined as

\[ \sum (x^{(k)} - x^{(k-1)})^T \Sigma (x^{(k)} - x^{(k-1)}) \leq \varepsilon_d \]

where \( x^{(k)} \) and \( x^{(k-1)} \) are designs at the \( k \)-th and \( k-1 \)-th iterations, respectively, \( x^{(k-1)} \) and \( x^{(k-2)} \) are MPPs at the \( k-1 \)-th and \( k-2 \)-th iterations and the covariance matrix \( \Sigma(X) \) of the random vector \( X \) corresponding to the design vector \( d \) is defined as
6.2 Mathematical Proof of Fast Reliability Analysis in RBDO

Assuming that the mean of uncorrelated random vector is regarded as the design vector, it is proved that the first condition of design closeness leads to the MPP closeness in the standard normalized U-space. First, the design closeness can be used to derive the MPP closeness as follows

\[ \| \sum^{-1/2} (d^{(k)} - d^{(k-1)}) \| \leq \| \sum^{-1/2} \left( X^{(k)} - \mu^{(k)} \right) \| \leq E_d, \text{ for } \forall X \in T^{-1}(U; d^{(k)}) \]  

(13)

where \( \mu \) is the mean vector of random vector \( X \) and \( T \) is a transformation matrix from the original random space \( X \) to an independent and standard normal parameter \( U \).

Since Eq. (13) holds for any \( X \), the MPP \( x^{(k)} \) at the \( k \)th design iteration can replace \( X \). To show that the MPP closeness can be derived from the design closeness, Eq. (13) can be rewritten as

\[ \| u^{(k-1)} - u^{(k)} \| \leq \| \sum^{-1/2} \left( x^{(k-1)} - x^{(k)} \right) \| \leq E_d \]  

(14)

Since \( x^{(k)} \) is not available at the current iteration, the information at the previous iteration can be used by assuming the RBDO process to be monotonically convergent as

\[ \| u^{(k-1)} - u^{(k)} \| \leq E_d + \left\| \sum^{-1/2} \left( x^{(k-1)} - x^{(k)} \right) \right\| \]  

(15)

Using the second condition of design closeness in Eq. (11), Eq. (15) can be rewritten to provide the MPP closeness at the current design as

\[ \| u^{(k-1)} - u^{(k)} \| \leq 2E_d \]  

(16)

It is found that the condition of the design closeness leads to the MPP closeness in X-space. Consequently, a reliability analysis can be carried out efficiently by starting at the MPP obtained from the previous design iteration, instead of the mean value point at the current iteration.

6.3 The Fast Reliability Analysis in RBDO

Once design closeness and MPP closeness in X-space are verified, a fast reliability analysis can be realized by starting reliability analysis for a new design at the MPP obtained from the previous design, instead of the mean value point. As shown in Fig. 1, the fast reliability analysis is illustrated under the assumption of the design closeness. The proposed reliability analysis method is integrated with the HMV method of the performance measure approach for accurate and efficient reliability analysis and RBDO process.

![Fast Reliability Analysis in RBDO under the Design Closeness](image)

( a ) MPP Search from the Mean Value Point   ( b ) MPP Search from the Previous MPP

Figure 1. A Fast Reliability Analysis in RBDO under the Design Closeness

7. RBDO Example of Multi-Crash Modes [3]
As shown in Figs. 2-5, a large-scale industry design application is used to demonstrate the efficiency of the proposed RBDO methodology. The design objective is to enhance overall performance of multi-crash modes while minimizing the vehicle weight. Simulation conditions of multi-crash modes are summarized in Table 1. The optimal Latin hypercube sampling with a total of 33 runs was used to generate a sample of design points for construction of the stepwise regression response surface [4]. The explicit response and more details used in the RBDO are summarized in Ref. 3. In this study, the explicit approximations of responses are regarded as exact responses of multi-crash modes to demonstrate the efficiency of proposed RBDO methodology.

### Table 1. Simulation Conditions of Multi-Crash Modes

<table>
<thead>
<tr>
<th>Crash mode</th>
<th>No. of element</th>
<th>Barrier velocity</th>
<th>CPU time (SGI Origin)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frontal/Offset</td>
<td>100,000</td>
<td>35 mph</td>
<td>20hrs/CPU</td>
</tr>
<tr>
<td>Roof</td>
<td>120,000</td>
<td>7.5 mph</td>
<td>24hrs/CPU</td>
</tr>
<tr>
<td>Side</td>
<td>100,000</td>
<td>31 mph</td>
<td>20hrs/CPU</td>
</tr>
</tbody>
</table>

### 7.1 RBDO Example for Vehicle Side Impact

A vehicle side impact is employed to demonstrate the efficiency of the proposed RBDO method. The RBDO results for 3-σ reliability are shown in Table 2. Ten performances are used to measure a human safety for vehicle side impact.

### Table 2. Deterministic Optimization and RBDO Results for Side Impact

<table>
<thead>
<tr>
<th>CONSTRAINT</th>
<th>BASE DESIGN</th>
<th>TARGET</th>
<th>DET. OPT.</th>
<th>RBDO</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>P(G&lt;sub&gt;1&lt;/sub&gt; ≤ 1000) ≥ 3σ</td>
<td>510.0</td>
<td>654.0 612.0</td>
</tr>
<tr>
<td>G&lt;sub&gt;2&lt;/sub&gt;</td>
<td>VC&lt;sub&gt;2&lt;/sub&gt;</td>
<td>P(G&lt;sub&gt;2&lt;/sub&gt; ≤ 0.32) ≥ 3σ</td>
<td>0.300</td>
<td>0.316 0.314</td>
</tr>
<tr>
<td>G&lt;sub&gt;3&lt;/sub&gt;</td>
<td>VC&lt;sub&gt;3&lt;/sub&gt;</td>
<td>P(G&lt;sub&gt;3&lt;/sub&gt; ≤ 0.32) ≥ 3σ</td>
<td>0.210</td>
<td>0.291 0.292</td>
</tr>
<tr>
<td>G&lt;sub&gt;4&lt;/sub&gt;</td>
<td>VC&lt;sub&gt;4&lt;/sub&gt;</td>
<td>P(G&lt;sub&gt;4&lt;/sub&gt; ≤ 0.32) ≥ 3σ</td>
<td>0.240</td>
<td>0.300 0.301</td>
</tr>
<tr>
<td>G&lt;sub&gt;5&lt;/sub&gt;</td>
<td>Def&lt;sub&gt;1&lt;/sub&gt;</td>
<td>P(G&lt;sub&gt;5&lt;/sub&gt; ≤ 32) ≥ 3σ</td>
<td>30.39</td>
<td>20.01 19.68</td>
</tr>
<tr>
<td>G&lt;sub&gt;6&lt;/sub&gt;</td>
<td>Def&lt;sub&gt;2&lt;/sub&gt;</td>
<td>P(G&lt;sub&gt;6&lt;/sub&gt; ≤ 32) ≥ 3σ</td>
<td>28.88</td>
<td>16.49 16.54</td>
</tr>
<tr>
<td>G&lt;sub&gt;7&lt;/sub&gt;</td>
<td>Def&lt;sub&gt;3&lt;/sub&gt;</td>
<td>P(G&lt;sub&gt;7&lt;/sub&gt; ≤ 32) ≥ 3σ</td>
<td>31.66</td>
<td>32.00 32.00</td>
</tr>
<tr>
<td>G&lt;sub&gt;8&lt;/sub&gt;</td>
<td>F&lt;sub&gt;pubic symphysis&lt;/sub&gt;</td>
<td>P(G&lt;sub&gt;8&lt;/sub&gt; ≤ 4.0) ≥ 3σ</td>
<td>3.900</td>
<td>4.000 4.000</td>
</tr>
<tr>
<td>G&lt;sub&gt;9&lt;/sub&gt;</td>
<td>Vel&lt;sub&gt;top-pillar&lt;/sub&gt;</td>
<td>P(G&lt;sub&gt;9&lt;/sub&gt; ≤ 10) ≥ 3σ</td>
<td>9.510</td>
<td>10.00 10.00</td>
</tr>
<tr>
<td>G&lt;sub&gt;10&lt;/sub&gt;</td>
<td>Vel&lt;sub&gt;front door&lt;/sub&gt;</td>
<td>P(G&lt;sub&gt;10&lt;/sub&gt; ≤ 15.7) ≥ 3σ</td>
<td>15.70</td>
<td>15.70 15.23</td>
</tr>
<tr>
<td>Weight (Kg)</td>
<td>29.05</td>
<td>Minimize</td>
<td>24.12</td>
<td>25.39 25.11</td>
</tr>
</tbody>
</table>

Using the PMA, the proposed method (PMA+) for evaluation of probabilistic constraints is compared with the conventional RBDO method (PMA−). It has been found that the proposed method for efficient evaluation of probabilistic constraints reduces computational effort by about 38% (276 to 172) in terms of the number of analyses. Two major reduction of computational cost results from a set of well feasible constraints through the efficient feasibility check, which has been activated toward the later part of the design iterations. The proposed method for
evaluation of probabilistic constraints remarkably enhances numerical efficiency by about 38% (276 to 172) in terms of the number of analysis, while minimizing the vehicle weight.

7.2 RBDO Example for Multi-Crash Modes

A large-scale example of multi-crash modes is employed to show numerical benefits of the proposed method with emphasis on numerical efficiency. The design objective is to enhance overall performance of multi-crash modes while minimizing vehicle weight and satisfying a target reliability, 3σ. The RBDO results for 3-σ reliability are shown in Table 3.

<table>
<thead>
<tr>
<th>Modes</th>
<th>CONSTRAINT</th>
<th>BASE DESIGN</th>
<th>TARGET</th>
<th>DET. OPT.</th>
<th>RBDO</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>P(σ27 ≤ 3σ)</td>
<td>298.8</td>
<td>361.54 361.54</td>
</tr>
<tr>
<td>Frontal Crash</td>
<td>G1</td>
<td>HIC</td>
<td>366</td>
<td>361.54</td>
<td></td>
</tr>
<tr>
<td></td>
<td>G2</td>
<td>Chest G</td>
<td>39.9</td>
<td>37.695</td>
<td></td>
</tr>
<tr>
<td></td>
<td>G3</td>
<td>Ptotal (%)</td>
<td>8.0</td>
<td>8.100 8.100</td>
<td></td>
</tr>
<tr>
<td>Roof Crash</td>
<td>G4</td>
<td>Resistant F (kN)</td>
<td>34.7</td>
<td>50.914 50.914</td>
<td></td>
</tr>
<tr>
<td>Offset Crash</td>
<td>G5</td>
<td>Intr 1 (in.)</td>
<td>9.9</td>
<td>10.053 10.053</td>
<td></td>
</tr>
<tr>
<td></td>
<td>G6</td>
<td>Intr 2 (in.)</td>
<td>10.9</td>
<td>10.585 10.585</td>
<td></td>
</tr>
<tr>
<td></td>
<td>G7</td>
<td>Intr 3 (in.)</td>
<td>10.5</td>
<td>10.343 10.343</td>
<td></td>
</tr>
<tr>
<td></td>
<td>G8</td>
<td>Intr 4 (in.)</td>
<td>9.7</td>
<td>9.694 9.694</td>
<td></td>
</tr>
<tr>
<td></td>
<td>G9</td>
<td>Intr 5 (in.)</td>
<td>10.3</td>
<td>10.263 10.263</td>
<td></td>
</tr>
<tr>
<td>Side Impact</td>
<td>G10</td>
<td>Disp 1 (mm)</td>
<td>23.1</td>
<td>23.144 23.144</td>
<td></td>
</tr>
<tr>
<td></td>
<td>G11</td>
<td>Disp 2 (mm)</td>
<td>26.0</td>
<td>25.909 25.909</td>
<td></td>
</tr>
<tr>
<td></td>
<td>G12</td>
<td>Disp 3 (mm)</td>
<td>26.9</td>
<td>26.768 26.768</td>
<td></td>
</tr>
<tr>
<td></td>
<td>G13</td>
<td>V*c 1</td>
<td>0.48</td>
<td>0.473 0.473</td>
<td></td>
</tr>
<tr>
<td></td>
<td>G14</td>
<td>V*c 2</td>
<td>0.53</td>
<td>0.524 0.524</td>
<td></td>
</tr>
<tr>
<td></td>
<td>G15</td>
<td>V*c 3</td>
<td>0.55</td>
<td>0.546 0.546</td>
<td></td>
</tr>
<tr>
<td>System</td>
<td>Weight (Kg)</td>
<td>1740.5</td>
<td>Minimize</td>
<td>1723.7 1738.3 1738.3</td>
<td></td>
</tr>
</tbody>
</table>

The RBDO for multi-crash modes requires three design optimization iterations, having two active constraints (G1 and G11) and none of active side constraints. Probabilistic constraints at 3-sigma optimal design are shown in Table 3. The proposed method for evaluation of probabilistic constraints significantly reduces the number of analyses by about 62% (458 to 176) in this example. Compared to the previous example, numerical efficiency is more substantially improved, since it involves more probabilistic constraints but has less number of active probabilistic constraints.

8. Conclusion

This paper is focused on the enhancement of numerical efficiency in the RBDO process by proposing a new method to evaluate probabilistic constraints efficiently. Two different methods are presented: efficient feasibility check of probabilistic constraints and a fast reliability analysis using the design closeness. It has been demonstrated that computational burden on the feasibility check of constraints in the RBDO process is relieved by using the mean value (MV) method with an allowable accuracy for the purpose of feasibility identification, and by carrying out the refined reliability analysis using the HMV method for ε-active and violate probabilistic constraints. In addition, a reliability analysis is proved to become more efficient by using some information generated at the previous design when designs at two consecutive iterations are close enough.

9. Acknowledgement

Research is partially supported by the Automotive Research Center sponsored by the U.S. Army TARDEC and by Ford Research Laboratory providing the example of multi-crash modes.

10. References
