DESIGN POTENTIAL CONCEPT FOR RELIABILITY-BASED DESIGN OPTIMIZATION

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ABSTRACT

This paper presents a design potential concept for reliability-based design optimization by integrating the probability analysis into the design optimization process. The reliability-based system parameter design is illustrated by the design potential concept in a unified system space, where design potential surfaces are derived from first-order reliability methods and an adaptive strategy is developed for robust and more efficient probabilistic constraint evaluation. More important, the design potential concept provides an in-depth understanding of reliability-based design optimization and leads to a design potential method that can significantly accelerate the process of robust system parameter design optimization.

NOMENCLATURE

P(●) Probability function  
Φ−1(●) Inverse of the standard normal CDF Φ(●)  
F−1 c (●) Inverse of the CDF of G(x)  
βG General probability index; βG(g) = −Φ−1(Fc(g))  
βs Reliability index; βs = −Φ−1(Fc(0))  
g Target probabilistic performance measure  
βFORM First-order probability index  
u MPP in the u-space  
x MPP in the x-space  
dP DPP in DPM corresponding to the design d^k

1. INTRODUCTION

The existence of uncertainties in engineering simulations and manufacturing processes requires a reliability-based design optimization (RBDO) model to obtain optimum designs with expected confidence level. In the RBDO model for robust system parameter design, the mean values of random system parameters are chosen as design variables, and the cost function is minimized subject to prescribed probabilistic constraints. In the conventional RBDO methodology (Enevoldsen, 1994; Enevoldsen and Sorensen, 1994; Chandu and Grandi, 1995; Grandhi and Wang, 1998), the probabilistic constraint is directly prescribed by the reliability index of the first-order reliability method (FORM). The RBDO problem is then solved by search methods for constrained nonlinear optimization. However, the close coupling of the system probability analysis and the system design optimization was essentially disconnected (Tu and Choi, 1999; Tu et al., 1999b). Consequently, the prohibitive computational cost prevented RBDO from many engineering applications (Frangopol and Corotis, 1996).

This paper proposes to integrate probability analysis into the design optimization process by introducing the design potential concept (DPC) in a unified system space. In DPC, the design potential surface derived from FORM is used to evaluate the probabilistic constraint, and an adaptive strategy is proposed for robust and efficient probabilistic constraint evaluation (Tu et al., 1999a). More important, DPC provides an in-depth understanding of RBDO and leads to a design potential method (DPM) for effective RBDO applications. For DPM, the design potential point (DPP) is defined as the design that renders the probabilistic constraint active and can be obtained as a by-product of the probabilistic constraint evaluation. The probabilistic constraint is then linearized at its DPP in defining the search direction determination subproblem.
Because DPP is on the probabilistic constraint limit-state surface, DPM provides a better constraint approximation than the traditional linearization at the current design and thus significantly improves the RBDO rate of convergence. Therefore, the proposed DPC-based RBDO methodology using DPM with adaptive probabilistic constraint evaluation can be used effectively for broader engineering applications.

2. RBDO MODEL FOR ROBUST SYSTEM PARAMETER DESIGN

The engineering system can be described by a set of continuous random system parameters $X = [X_i]^T$ ($i = 1, \ldots, n$), which represent exhaustive sets of outcomes $x = [x_i]^T$ that can take on any real values over specified tolerance limits, i.e., $x^l \leq x \leq x^u$, and are completely characterized by the system parameter joint probability density function (JPDF) $f_x(x)$ (Ayub and McCuen, 1997).

In the robust system parameter design, mean values of random system parameters are often used as independent design variables $d = [d_i]^T = [\mu_i]^T$. For a given design, a system performance criterion is described by a system performance function $G(x)$, such that the system fails if $G(x) < 0$. The RBDO model for robust system parameter design (Enevoldsen and Sorensen, 1994; Chandu and Grandi, 1995) can be defined as

\begin{align}
\text{minimize} & \quad \text{Cost}(d) \\
\text{subject to} & \quad P(G(x) < 0) \leq \overline{P}_{ij}, \quad j = 1, 2, \ldots, np \quad \text{(1a)} \\
& \quad d^l \leq d \leq d^u \quad \text{(1b)} \\
& \quad \text{(1c)}
\end{align}

where $\overline{P}_{ij}$ is the target failure probability limit for the jth system performance criterion and np is the total number of probabilistic constraints.

3. PROBABILITY ANALYSIS OF PERFORMANCE FUNCTION IN RBDO

The probability analysis of a system performance function $G(x)$ is to evaluate its cumulative distribution function (CDF) $F_G$ as (Madsen et al., 1986)

$$F_G(g) = P(G(x) < g) = \int_{x^l}^{x^u} \ldots \int f_x(x)dx_i \ldots dx_n,$$

$$x^l \leq x \leq x^u \quad \text{(2a)}$$

where $g$ is named the probabilistic performance measure and $F_G(g)$ is a monotonically increasing function of $g$. The probability integration domain is bounded by the random system parameter tolerance limits. For convenience, a generalized probability index $\beta_G$ is often used to measure the performance probability level as

$$\beta_G(g) = -\Phi^{-1}(F_G(g)) \quad \text{(2b)}$$

where $\beta_G(g)$ is a monotonically decreasing function of $g$. The target failure probability limit in Eq. (1b) can also be represented by a reliability target index as $\beta_i = -\Phi^{-1}(\overline{P}_i)$. Thus, the probabilistic constraint can be rewritten as, using Eq. (2a),

$$F_G(0) \leq \Phi(-\beta_i) \quad \text{(3a)}$$

which can be further expressed in two ways through one-to-one inverse transformations (Rubinstein, 1981) as

$$-\Phi^{-1}(F_G(0)) \geq \beta_i \quad \text{(3b)}$$

$$F_G^{-1}(x) \geq 0 \quad \text{(3c)}$$

3.1 Probability Analyses in Probabilistic Constraint Evaluation

To date, most researchers have used Eq. (3b) by defining a reliability index, $\beta_i = -\Phi^{-1}(F_G(0))$, to directly prescribe the probabilistic constraint as

$$\beta_i(d) \geq \beta_i \quad \text{(4a)}$$

In the reliability index approach (RIA), at a given design $d' = [d_i']^T = [\mu_i']^T$, the evaluation of $\beta_i(d')$ is performed using the well-developed reliability analysis (Madsen et al., 1986) as

$$\beta_i(d') = -\Phi^{-1}(\int_{x^l}^{x^u} \ldots \int f_x(x)dx_i \ldots dx_n), \quad x^l \leq x \leq x^u \quad \text{(4b)}$$

Tu and Choi (1999) proposed the performance measure approach (PMA) by using Eq. (3c) to define a target probabilistic performance measure $g = F_G^{-1}(\Phi(-\beta_i))$ to prescribe the probabilistic constraint as

$$g^*(d') \geq 0 \quad \text{(5a)}$$

and the evaluation of $g^*(d')$ can be performed in an inverse reliability analysis (Tu et al., 1999a) as

$$g^*(d') = F_G^{-1}(\int_{x^l}^{x^u} \ldots \int f_x(x)dx_i \ldots dx_n), \quad x^l \leq x \leq x^u \quad \text{(5b)}$$

Since the exact probability integrations in Eqs. (4b) and (5b) are in general extremely complicated to compute, FORM is often used to provide efficient approximate solutions (Breitung, 1984; Madsen et al., 1986; Tvedt, 1990).

3.2 FORM for Probabilistic Constraint Evaluation

In FORM, the general one-to-one transformation (Hohenbichler and Rackwitz, 1981; Madsen et al., 1986) between the often dependent, non-normal system parameters $X$ (x-space), and the independent, standardized normal variables $U$ (u-space) at the design $d^c$ can be expressed as

$$u = [u_i]^T = T(x; d^c) = [T_i(x; d^c)]^T, \quad i = 1, \ldots, n \quad \text{(6a)}$$

$$x = [x_i]^T = T^{-1}(u; d^c) = [T_i^{-1}(u; d^c)]^T, \quad i = 1, \ldots, n \quad \text{(6b)}$$

and the performance function $G(x)$ can then be expressed in the $u$-space as

$$G(u) = G(T^{-1}(u; d^c)) = G_0(u) \quad \text{(6c)}$$

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In the \( \mathbf{u} \)-space, the point on surface \( G_\theta(\mathbf{u}) = g \) with the maximum joint probability density is the point with the minimum distance \( \beta \) from the origin and is called most probable point (MPP) \( \mathbf{u}_p^* \) or \( \mathbf{u}_p \). This minimum distance is defined in FORM as the first-order probability index as

\[
\beta_{\text{FORM}}(g) = \beta = \| \mathbf{u}_p \| = \| \mathbf{u}_p^* \| \tag{7}
\]

Thus, the reliability analysis of Eq. (4b) in RIA by FORM can be performed by solving a nonlinear constrained optimization problem as (Madsen et al., 1986)

\[
\begin{align*}
\text{minimize} & \quad \| \mathbf{T}(\mathbf{x}; \mathbf{d}') \| \\
\text{subject to} & \quad G(\mathbf{x}) = 0 \\
& \quad \mathbf{x}^* \leq \mathbf{x} \leq \mathbf{x}^U
\end{align*} \tag{8a-8c}
\]

where the optimum \( \mathbf{x}_p^* \) corresponds to MPP of RIA as \( \mathbf{u}_p^* = \mathbf{T}(\mathbf{x}_p^*; \mathbf{d}') \). The first-order reliability index can then be obtained as

\[
\beta_{\text{FORM}}(\mathbf{d}') = \| \mathbf{u}_p \| = \| \mathbf{T}(\mathbf{x}_p^*; \mathbf{d}') \| \tag{9}
\]

The inverse reliability analysis of Eq. (5b) in PMA by FORM can also be performed by solving a nonlinear constrained optimization problem as (Tu et al., 1999b)

\[
\begin{align*}
\text{Minimize} & \quad G(\mathbf{x}) \\
\text{Subject to} & \quad \| \mathbf{T}(\mathbf{x}; \mathbf{d}') \| \leq \beta_i
\end{align*} \tag{10a-10b}
\]

where the optimum \( \mathbf{x}_{p\beta}^* \) corresponds to MPP of PMA as \( \mathbf{u}_{p\beta}^* = \mathbf{T}(\mathbf{x}_{p\beta}^*; \mathbf{d}') \). The first-order probabilistic performance measure can then be obtained as

\[
g_{\text{FORM}}(\mathbf{d}') = G_{\text{U}}(\mathbf{u}_{p\beta}^*) = G(\mathbf{x}_{p\beta}^*) \tag{11}
\]

### 3.3 Example

Consider a system described by two independent, uniformly distributed random system parameters, \( \mathbf{X}_i \sim \text{Uniform}[a_i, b_i] \) \( (i = 1, 2) \), and their PDFs are expressed as

\[
f_{\chi_i}(x_i) = 1/(b_i - a_i), \quad a_i \leq x_i \leq b_i, \quad i = 1, 2 \tag{12a}
\]

where the mean values and variances of system parameters are expressed, respectively, as

\[
\mu_i = \int_{a_i}^{b_i} x_i f_{\chi_i}(x_i) \, dx_i = (b_i - a_i)/2, \quad i = 1, 2 \tag{12b}
\]

\[
\sigma^2_i = \int_{a_i}^{b_i} (x_i - \mu_i)^2 f_{\chi_i}(x_i) \, dx_i = (b_i - a_i)^2/12, \quad i = 1, 2 \tag{12c}
\]

In the system parameter design, both \( \mu_1 \) and \( \mu_2 \) are chosen as design variables, \( \mathbf{d} = [d_1, d_2]^T = [\mu_1, \mu_2]^T \), and their variances are constants as \( \sigma^2_1 = \sigma^2_2 = 1/3 \).

The CDFs of the uniformly distributed system parameters are

\[
F_{\chi_i}(x_i) = \int_{a_i}^{x_i} f_{\chi_i}(x) \, dx_i = (x_i - a_i)/(b_i - a_i), \quad a_i \leq x_i \leq b_i, \quad i = 1, 2 \tag{13a}
\]

which can be rewritten in terms of design variables as

\[
F_{\chi_i}(x_i) = (x_i - d_i - 1)/2, \quad d_i - 1 \leq x_i \leq d_i + 1, \quad i = 1, 2 \tag{14b}
\]

Then the transformations between the \( \mathbf{x} \)-space and the \( \mathbf{u} \)-space at the design \( \mathbf{d}' \) for two independent system parameters can be expressed as

\[
u_i = \Phi^{-1}(F_{\chi_i}(x_i)) = \Phi^{-1}((x_i - d_i + 1)/2), \quad i = 1, 2 \tag{14c}
\]

\[
x_i = 2\Phi(\mathbf{u}_i) + d_i - 1, \quad i = 1, 2 \tag{14d}
\]

and the performance function is then transformed into the \( \mathbf{u} \)-space as

\[
G_{\text{U}}(\mathbf{u}) = 2\Phi(\mathbf{u}_1) + 4\Phi(\mathbf{u}_2) + d_1^2 + 2d_2^2 - 13 \tag{14e}
\]

At the design \( \mathbf{d}' = [4.0, 4.0]^T \), the contours of the performance function, i.e., \( G_{\text{U}}(\mathbf{u}) = g \) for different \( g \) values, and the MPP locus in the \( \mathbf{u} \)-space are illustrated in Fig. 1, where the MPP \( \mathbf{u}_p \) (or \( \mathbf{u}_{p\beta} \)) is found using first-order reliability analysis of RIA, and the MPP \( \mathbf{u}_{p\beta} \) (or \( \mathbf{u}_{p\beta}^* \)) is found using first-order inverse reliability analysis of PMA. The corresponding \( (\beta_{\text{FORM}}, g_{\text{FORM}}) \) curve is then compared with the exact \( (\beta, g) \) curve in Fig. 2, where RIA identifies the point \( (\beta_{\text{FORM}}, 0) \) and PMA identifies the point \( (\beta_{\text{FORM}}, g_{\text{FORM}}) \).

Note that, if the design \( \mathbf{d}' = [4.5, 4.5]^T \) is selected, the performance function of Eq. (14e) is positive everywhere in the probability integration domain as

\[
G_{\text{U}}(\mathbf{u}) = 2\Phi(\mathbf{u}_1) + 4\Phi(\mathbf{u}_2) + 0.5 > 0, \quad d_1^2 - 1 \leq x_i \leq d_i^2 + 1, \quad i = 1, 2 \tag{15}
\]

and thus first-order reliability analysis by Eqs. (8a) - (8c) of RIA yields no solution. In contrast, first-order inverse reliability analysis of PMA will always yield a solution. Although both RIA and PMA can be used for reliability analysis, PMA is meaningful for RBDO since the reliability target \( \beta_i \) is required for PMA.

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4. DESIGN POTENTIAL CONCEPT IN UNIFIED SYSTEM SPACE

In the RBDO model, probabilistic constraints by Eqs. (4a) or (5a) are prescribed in the design variable space (d-space), whereas the system performance function is defined in the system parameter sample space (x-space). For a given design d’, the corresponding probability integration domain, where the system JPDF is defined, is x-space which is defined in FORM. In this section, the design potential concept (DPC) is introduced by integrating probability analysis and the RBDO process.

From a broader perspective, a unified system space for reliability-based system parameter design is defined by mapping the x-space to the d-space as

\[ x = T^{-1}(0;d) \]  \hspace{1cm} (16)

In the unified system space, the design d’ is coincident to the mapping of the corresponding u-space origin \( T^{-1}(0;d) \), and three characteristic design potential surfaces around the design d’ are defined corresponding to a probabilistic constraint as:

- **Surface of reliability potential,** \( \|T(x;d')\| = \beta_i(d') \), where \( \beta_i(d') \) is the first-order reliability index of the probabilistic constraint.

- **Surface of reliability target,** \( \|T(x;d')\| = \beta, \) where \( \beta \) is the reliability target index of the probabilistic constraint.

- **Surface of tolerance limits,** \( \|\bar{T}(x;d')\| \rightarrow \infty \), which encloses the probability integration domain as \( x' \leq x \leq x'' \).

By defining the probabilistic constraint evaluation in the unified system space, performance probability analysis can be integrated into the iterative RBDO process to develop a very effective design potential method (DPM) for RBDO, as illustrated using an example in the following sections.

4.1 An Example of Probabilistic Constraint Evaluation in Unified System Space

Consider a system described by two independent uniformly distributed system parameters \( X = [X_1, X_2]^T \) with constant standard deviations, \( \sigma_1 = 1/2 \) and \( \sigma_2 = 2/5 \). The design variable is chosen as \( d = [d_1, d_2]^T = [\mu_1, \mu_2]^T \), and system parameter CDFs can then be expressed in terms of design variables as

\[ F_{i}(x_i) = (x_i - d_i)/2\sqrt{3}\sigma_i + 1/2, \]

\[ d_i - \sqrt{3}\sigma_i \leq x_i \leq d_i + \sqrt{3}\sigma_i, \hspace{1cm} i = 1, 2 \]  \hspace{1cm} (17a)

The transformations between the u-space and the x-space at the design d’ are nonlinear as

\[ u_i = T_i(x_i; d_i^*) = \Phi^{-1}(F_{i}(x_i)) \]

\[ = \Phi^{-1}(x_i - d_i)/2\sqrt{3}\sigma_i + 1/2, \hspace{1cm} i = 1, 2 \]  \hspace{1cm} (17b)

\[ x_i = T_{i}^{-1}(u_i; d_i^*) = F_{i}^{-1}(\Phi(u_i)) \]

\[ = d_i - \sqrt{3}\sigma_i + 2\sqrt{3}\sigma_i\Phi(u_i), \hspace{1cm} i = 1, 2 \]  \hspace{1cm} (17c)

and the mapping from the x-space to the d-space can be simplified in this example as

\[ x = T^{-1}(0;d) = d \]  \hspace{1cm} (18)

Two probabilistic constraints are defined with the same reliability target index \( \beta = 2 \) as

\[ P(G_j(x) < 0) \leq \Phi(-\beta_i), \hspace{1cm} j = 1, 2 \]  \hspace{1cm} (19a)

where the two nonlinear performance functions are defined in the x-space as

\[ G_1(x) = x_1 x_2 / 20 - 1 \]  \hspace{1cm} (19b)

\[ G_2(x) = (10x_1^2 - x_1^2 x_2 - 2x_1)/10 - 1 \]  \hspace{1cm} (19c)

4.1.1 Illustration of RIA in Unified System Space

At the design \( d'=[3, 4]^T \), the probabilistic constraint evaluation in RIA by FORM is illustrated in the unified system space as shown in Fig. 3. Conceptually, the probabilistic constraint is evaluated in RIA by fitting the surface of the reliability potential, \( \|T(x;d')\| = \beta(d') \), so that it is tangentially contacted with the performance function limit-state surface \( G_i(x)=0, i=1, 2 \).
Note that the first-order reliability index of the first probabilistic constraints is \( \beta_1(d^*) = ||T(x^*; d^*)|| \). However, the distributions of system parameters are often bounded by their finite tolerance limits and the probability integration domain is finite. Therefore, RIA yields singularity for the second probabilistic constraint because the limit-state surface of the performance function, \( G(x) = 0 \), is defined outside of the probability integration domain, which is enclosed by the surface of tolerance limits, \( ||(x - T_0; d)|\rightarrow \infty \). That is, first-order reliability analysis yields no solution because Eq. (8b) conflicts with Eq. (8c) at the design \( d = [3, 4]^T \).

4.1.2 Illustration of PMA in Unified System Space

From another perspective, the probabilistic constraint evaluation in PMA by FORM is illustrated in Fig. 4. Conceptually, if there is no stationary point of \( G(x) \) inside the probability integration domain, the probabilistic constraint is evaluated in PMA by finding the minimum performance function value on the surface of the reliability target.

Note that the target probabilistic performance measures of two probabilistic constraints are \( g_j(d^*) = G(x^*; d^*) \). Because first-order inverse reliability analysis is defined inside the probability integration domain, PMA always yields a solution and thus more robust than RIA in probabilistic constraint evaluations (Tu et al., 1999a & b).

4.2 Design Potential Concept for RBDO

In the design potential concept (DPC) (Tu and Choi, 1999), the limit-state surface of the probabilistic constraint \( \beta_0(d^*) = \beta_1 \) (or \( g(d^*) = 0 \)) can be constructed by tangentially sweeping the surface of the reliability target along the feasible side of the performance function limit-state surface, \( G(x) = 0 \). Thus, the optimum of RBDO is the minimum cost design whose corresponding surface of the reliability target can fit into the feasible side of all performance function limit-state surfaces.

If FORM is used for approximate probability integration, the surface of the reliability target, \( ||T(x; d^*)|| = \beta_k \), is determined by JPDF of system parameters at the design \( d^* \) and the system reliability target \( \beta_k \). As shown in Fig. 5, MPP \( x^*_0 \) of the design \( d^* \) on the performance function limit-state surface also corresponds to the design \( d^*_0 \) that renders the probabilistic constraint active. The design potential point (DPP) \( d^*_0 \) is on the limit-state surface of the probabilistic constraint as

\[
\beta_0 = \beta_k \quad \text{(also } g_{\text{form}}(d^*_0) = 0) \quad (20a)
\]

\[
x^*_0 = T^{-1}(u^*; d^*) = T^{-1}(u^*; d^*_0) \quad (20b)
\]

4.3 Design Potential Method for Probabilistic Approximation

In the conventional RBDO methodology (Enevoldsen, 1994; Enevoldsen and Sorensen, 1994; Chandu and Grandi, 1995; Grandhi and Wang, 1998), the probabilistic constraint is prescribed by Eq. (4a) and is evaluated in RIA by FORM. The nonlinear constrained RBDO problem is often solved by search methods (or primal methods) (Arora, 1989), such as SLP, SQP, MFD, and the gradient projection method, where the search direction is evaluated by solving an optimization subproblem defined by the linearized probabilistic constraints at the current design \( d^* \) as

\[
\beta_0(d^*) + \nabla^2 \beta_0(d^*) \cdot (d - d^*) \geq \beta_k
\]

\( (21) \)
where

\[ \nabla \mathbf{d}_i^* = \nabla \mathbf{G}_i (\mathbf{u}_m) / \| \nabla \mathbf{G}_i (\mathbf{u}_m) \| \]  

\[ = \nabla \mathbf{G}_i (\mathbf{x}_m) / \| \nabla \mathbf{G}_i (\mathbf{x}_m) \| \]  

\[ \nabla \mathbf{G}_i (\mathbf{d}_i^*) = \left[ \nabla \mathbf{g}(\mathbf{x}_m) / \partial \mathbf{u}_i \right] \]  

\[ \nabla \mathbf{G}_i (\mathbf{d}_i^*) = \left[ \nabla \mathbf{g}(\mathbf{d}_i) / \partial \mathbf{u}_i \right] \]  

\[ \left( \beta \right) \nabla \mathbf{G}_i (\mathbf{u}_m) / \| \nabla \mathbf{G}_i (\mathbf{u}_m) \| \]  

\[ \left( \beta \right) \nabla \mathbf{G}_i (\mathbf{x}_m) / \| \nabla \mathbf{G}_i (\mathbf{x}_m) \| \]  

However, the close coupling of system probability analysis and the system design optimization, which has been illustrated by DPC in the unified system space, was essentially disconnected in Eq. (21). The DPC leads to a design potential method (DPM) for RBDO, where the probabilistic constraint evaluated in RIA is linearized at its DPP \( \mathbf{d}_i^* \) to define the search direction determination subproblem.

As illustrated in Fig. 5 where DPP \( \mathbf{d}_i^* \) of the probabilistic constraint shares the same MPP \( \mathbf{x}_m \) on the performance function limit-state surface with the current design \( \mathbf{d}_i^* \), its MPP \( \mathbf{u}^* \) in the \( \mathbf{u} \)-space can be directly expressed using FORM as

\[ \mathbf{u}^* = \beta \nabla \mathbf{G}_i (\mathbf{d}_i^*) / \| \nabla \mathbf{G}_i (\mathbf{d}_i^*) \| \]  

\[ \mathbf{u}^* = \beta \nabla \mathbf{G}_i (\mathbf{x}_m) / \| \nabla \mathbf{G}_i (\mathbf{x}_m) \| \]  

If FORM is used in RIA, because \( \mathbf{x}_m \) and \( \nabla \mathbf{G}(\mathbf{x}_m) / \partial \mathbf{x}_i \) have already been computed in the MPP search of RIA, \( \mathbf{d}_i^* \) and \( \mathbf{u}^* \) can be determined by solving the nonlinear equation system comprised of Eqs. (20b) and (23) using the Newton’s method (Atkinson, 1988).

Using DPM, the probabilistic constraint for RIA is linearized at its DDP \( \mathbf{d}_i^* \) (instead of \( \mathbf{d}_i^* \)) as

\[ \nabla \mathbf{G}_i (\mathbf{d}_i^*) (\mathbf{d} - \mathbf{d}_i^*) \geq 0 \]  

\[ \nabla \mathbf{G}_i (\mathbf{d}_i^*) = \nabla \mathbf{G}_i (\mathbf{x}_m) / \| \nabla \mathbf{G}_i (\mathbf{x}_m) \| \]  

The key point of DPM is that, since DDP \( \mathbf{d}_i^* \) is on the limit-state surface of the probabilistic constraint, the probabilistic constraint approximation in Eq. (24a) by DPM is exact at \( \mathbf{d}_i^* \), and its advantage over Eq. (21) becomes significant if the probabilistic constraint is highly nonlinear. Consequently, a higher rate of convergence in solving the RBDO problem can be achieved by using DPM (Tu and Choi, 1999; Tu et al., 1999b).

It is important to emphasize that solving the nonlinear equation system in DPM does not require any additional evaluation of \( \mathbf{G}(\mathbf{x}) \) that often requires costly large-scale numerical simulation, such as structural FEA and durability analysis (Yu et al., 1997 and 1998). Terms that need to be evaluated are \( \mathbf{T} (\mathbf{x}^*; \mathbf{d}_i^*) \) and \( \mathbf{T} (\mathbf{x}^*; \mathbf{d}_i^*) / \partial \mathbf{u}_i \), which can be easily computed numerically. Therefore, DPM provides better constraint approximation at minimal additional cost because probability analysis for the probabilistic constraint evaluation already provides the important design information.

4.4. Adaptive Probabilistic Constraint Evaluation Strategy

In RBDO, either Eq. (4a) (for RIA) or Eq. (5a) (for PMA) can be used to prescribe the probabilistic constraint. Tu et al. (1999a) showed that PMA is inherently robust and more efficient in evaluating inactive probabilistic constraints, while RIA is more efficient for active or violated probabilistic constraints. In addition, it is shown in Section 3.3 that RIA yields no solution when the probability integration domain does not intersect any of the design constraints \( G_j(x)=0 \), \( j=1-np \), regardless of whether the current design \( \mathbf{d}_i \) is a feasible design or not in deterministic sense.

In addition, the proposed DPM can be used to significantly accelerate the convergence of RBDO if RIA is used to evaluate the active or violated probabilistic constraint (Tu et al., 1999b). Thus, to take advantage of these properties, an adaptive probabilistic constraint evaluation strategy is proposed (Tu and Choi, 1999) by choosing RIA and PMA depending on the marginal status of the probabilistic constraint in RBDO iterations. The adaptive strategy represents a balance of the overall RBDO rate of convergence and the computational costs and robustness for each probabilistic constraint evaluation.

Since PMA is used for when the probability integration domain does not intersect any of the design constraints \( G_j(x)=0 \), \( j=1-np \), for PMA, the DPP for the design \( \mathbf{d}_i \) is the current design \( \mathbf{d}_i^* \). Thus, for PMA, the linearization of the probabilistic constraint is obtained at the current design \( \mathbf{d}_i^* \) as

\[ \nabla \mathbf{g}(\mathbf{d}_i^*) + \nabla \mathbf{G}_i (\mathbf{d}_i^*) (\mathbf{d} - \mathbf{d}_i^*) \geq 0 \]  

\[ \nabla \mathbf{G}_i (\mathbf{d}_i^*) = \nabla \mathbf{G}_i (\mathbf{x}_m^*) / \| \nabla \mathbf{G}_i (\mathbf{x}_m^*) \| \]  

5. RBDO EXAMPLE

Consider the same example described in Section 4.1. With the design variable \( \mathbf{d} = [d_1, d_2]^T = [\mu_1, \mu_2]^T \), an RBDO problem with reliability target \( \beta = 2 \) is defined as

\[ \text{minimize} \quad \text{Cost}(\mathbf{d}) = d_1 + d_2 \]  

\[ \text{subject to} \quad P(G_j(x) < 0) \leq \Phi(\beta), \quad j = 1, 2, 3 \]  

\[ 1 \leq d_1 \leq 10, \quad 1 \leq d_2 \leq 10 \]  

where the three nonlinear constraints are

\[ G_j(x) = x_j - 20 \leq 0 \]  

\[ G_j(x) = (x_1 + x_2 - 5)^2 + (x_1 - x_2 - 12)^2 \leq 0 \]  

\[ G_j(x) = 80/(x_1 + x_2 + 5) - 1 \leq 0 \]  

5.1. RBDO Starting at Arbitrary Initial Design \( \mathbf{d}^0 = [5.00, 5.00]^T \)

Using sequential linear programming (SLP) with adaptive probabilistic constraint evaluation, the RBDO history is illustrated in Fig. 6 and listed in Table 1, where RIA and PMA are used adaptively depending on the status of the probabilistic constraints. In the iterative optimization process, the adaptive MPP search algorithm (Tu and Choi, 1999) chooses PMA or RIA to evaluate probabilistic constraints and DPM is used to accelerate the convergence of RBDO. The optimum of this RBDO problem is \( \mathbf{d}^* = [3.52, 3.19]^T \).
As shown in Fig. 6 and iterations 1 and 2 of Table 1, RIA yields no solutions at some intermediate feasible designs in evaluating some probabilistic constraints because their limit-state surfaces are not defined in the finite probability integration domain bounded by the surface of tolerance limits, \( |\mathbf{F}(\mathbf{x}; \mathbf{d}^0)| \to \infty \). Thus, the conventional RBDO methodology cannot be used directly to solve this RBDO problem. Note that design sensitivity analysis of the probabilistic constraint is by-products of constraint evaluations (Tu and Choi, 1999). The SLP is attractive for RBDO because it is usually more efficient in terms of the total number of probabilistic constraint evaluations (Arora, 1989). More robust search methods, such as SQP and MFD that require the line search, can be used if SLP fails to converge.

5.2 RBDO Starting from Deterministic Optimum

Design \( \mathbf{d}^0 = [3.11, 2.06]^T \)

The optimum of a deterministic optimization problem often provides an effective initial design for RBDO. The initial design \( \mathbf{d}^0 = [3.11, 2.06]^T \) is obtained from

minimize \( \text{Cost}(\mathbf{d}) \)  \hspace{1cm} (28a)

subject to \( g_j(\mathbf{d}) \geq 0, \quad j = 1, 2, 3 \)  \hspace{1cm} (28b)

\[ 1 \leq d_1 \leq 10, \quad 1 \leq d_2 \leq 10 \]  \hspace{1cm} (28c)

which can be obtained by using SLP, SQP, or MFD.

Using DPM and the adaptive probabilistic constraint evaluation, the RBDO problem can be solved by SLP, and the RBDO history is shown in Table 2. Because potential probabilistic constraints at the initial design are also active probabilistic constraints at the RBDO optimum, RBDO can take full advantage of DPM and therefore converge quicker than starting from an arbitrary initial design, such as \( \mathbf{d}^0 = [5.00, 5.00]^T \).

For comparison, it is also solved using the adaptive probabilistic constraint evaluation without DPM, which takes significantly more iteration to converge as shown in Table 3. Note that RIA does not yield solutions for all three probabilistic constraints at \( \mathbf{d}^0 = [4.51, 3.83]^T \) and \( \mathbf{d}^0 = [3.97, 3.31]^T \) in Table 1, and \( \mathbf{d}^0 = [3.34, 3.90]^T \) in Table 3.

![Figure 6. RBDO History in Unified System Space](image)

**Table 1. RBDO History with DPM \( (\mathbf{d}^0 = [5.00, 5.00]^T) \)**

<table>
<thead>
<tr>
<th>Iteration</th>
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<th>( \mathbf{d}^0 )</th>
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<th>PMA</th>
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6. SUMMARY

The DPC provides an in-depth understanding of RBDO by integrating performance function probability analysis into the design optimization process in the unified system space. By taking advantage of the inherent design information from probability analysis, this concept leads to a highly effective new methodology for RBDO that includes robust and efficient adaptive probabilistic constraint evaluation strategy by FORM and the novel DPM for probabilistic constraint approximation. Extension of DPC for RBDO to non-smooth and extreme cases has been developed by Choi et al. (1999). Future research is focused on practical applications of this development for large-
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REFERENCES

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