Reliability-Based Design Optimization Using Response Surface Method With Prediction Interval Estimation

Since variances in the input variables of the engineering system cause subsequent variances in the product output performance, reliability-based design optimization (RBDO) is getting much attention recently. However, RBDO requires expensive computational time. Therefore, the response surface method is often used for computational efficiency in solving RBDO problems. A method to estimate the effect of the response surface error on the RBDO result is developed in this paper. The effect of the error is expressed in terms of the prediction interval, which is utilized as the error metric for the response surface used for RBDO. The prediction interval provides upper and lower bounds for the confidence level that the design engineer specified. Using the prediction interval of the response surface, the upper and lower limits of the reliability are computed. The lower limit of reliability is compared with the target reliability to obtain a conservative optimum design and thus safeguard against the inaccuracy of the response surface. On the other hand, in order to avoid obtaining a design that is too conservative, the developed method also constrains the upper limit of the reliability in the design optimization process. The proposed procedure is combined with an adaptive sampling strategy to refine the response surface. Numerical examples show the usefulness and the efficiency of the proposed method. [DOI: 10.1115/1.2988476]

Keywords: reliability-based design optimization, response surface method, prediction interval, moving least squares method

1 Introduction

The existence of variances in the input variables of engineering systems due to manufacturing processes and operational conditions causes subsequent variances in the product output performance. To obtain reliable designs, often a probabilistic design method called reliability-based design optimization (RBDO) is used [1–7]. The RBDO formulation may involve the same cost function as deterministic optimization but contains probabilistic performance constraints for considering the probability of the satisfaction/failure of the output performances.

In the practical industry design process, RBDO is necessary to make cost effective quality products, but it requires expensive computational time. Therefore, several metamodeling methods, including the response surface method (RSM) [7–11] and the kriging method [12–14], are used for RBDO problems for computational efficiency.

In RSM-based RBDO, the accuracy of the response surface (RS) is critical. Unfortunately, the conventional measures of the accuracy of RSM are based mostly on the error values at the sampling points. Thus, in the conventional design process using RSM, when the global response surface is constructed using these conventional error estimators, it is used as an accurate function evaluator even though accuracy will depend on the location where the function needs to be estimated. Especially for RBDO, a small response surface error could cause a significant error in the estimation of the reliability, and eventually, the optimized design may not satisfy the target reliability. Thus, these error metrics do not provide good information on the judgment of the accuracy of the response surface for use in RBDO, and a new error metric needs to be developed for a successful application of RSM to RBDO.

The inaccuracy of the metamodels can be interpreted as the metamodel uncertainty, where the true response is unknown except at the sampled points. Several researchers [15–17] studied how to quantify this kind of uncertainty. Hazelrigg [18] studied the estimation of the model error of mathematical models in engineering design. Vittal and Hajela [19] used confidence intervals for the estimation of reliability using RSM but did not extend it to apply to RBDO. Apley et al. [20] proposed the Bayesian prediction interval with the Gaussian random process (GRP) to quantify the effect of metamodel uncertainty (they referred to this as “interpolation uncertainty”) for robust design optimization problems. They developed the closed-form prediction intervals and also discussed additional simulations.

Martin and Simpson [21] studied how to quantify the model uncertainty using the kriging model, and they [22] developed a methodology to quantify the model uncertainty impact with input parameter uncertainty for a system-level decision making. They used the Monte Carlo simulation (MCS) and the coefficient of determination for prediction, $R^2_{prediction}$, as a criterion to assess a model quality. Mahadevan et al. [23] listed several kinds of uncertainties and errors such as the model form error [24,25], the solution approximation error, and the error in reliability analysis. They proposed an iterative method to include model errors in RBDO problems. A random variable that represented the model errors was added to the classical RBDO formulation, and Monte Carlo simulation was used to include the reliability analysis error from a conventional first-order reliability method (FORM).

Many recent papers have quantified the various kinds of uncer-
papers have applied the RBDO problem and also most of them have limited one-sided constraints.

This paper proposes to use the prediction interval [26–28] to estimate the effect of the response surface error on the RBDO results. A number of different methods can be used to estimate the approximation uncertainty. For the response surface, the moving least squares method (MLSM) [29–32], which is a locally weighted approximation method, is used in this paper. The interval prediction can be used as an effective measure of accuracy for RBDO. The proposed interval concept computes the probabilistic interval of the response based on the user-specified confidence level. Using the prediction interval of the response surface, the upper and lower limits of the reliability can be computed. The lower limit of reliability is compared with the target reliability in RBDO, which will provide a conservative optimum design to safeguard against the inaccuracy of the response surface. On the other hand, in order to avoid obtaining a too conservative design that may not be close to the true optimum design, the proposed method constrains the upper limit of the reliability in the RBDO process. The proposed procedure is combined with an adaptive sampling strategy that decides where to sample and how many points to sample. Therefore, this method can give a guideline for the sampling location and the convergence criterion. Numerical examples show the effectiveness and the efficiency of the proposed method.

2 Measure of Accuracy of the Response Surface

2.1 Conventional Error Metrics of the Response Surface. The conventional error metrics of the response surfaces are based usually on the error values at the sampling points, such as the mean square error. Even if the mean square error is very small, the response surface results at the nonsampling points may not be good. That is, if the adjusted coefficient of determination, $R^2_{adj}$, which is a representative of the conventional accuracy measure, is very close to 1, then the approximation function passes through very close to all the sampling points. However, that does not guarantee that the response surface is accurate at other nonsampling points, which could be very well candidate points for the optimum design. Therefore, the conventional error metrics may not be effective for design optimization, especially for RBDO since a small response surface error could cause a significant reliability error, and the optimum design result may not satisfy target reliabilities.

It is proposed to use the prediction interval of the response surface using the MLSM [29–32]. This interval concept computes the probabilistic interval of the response surface based on the confidence level that the design engineer specified. This interval is used as a new measure of accuracy for RBDO.

2.2 New Measure of Accuracy for RBDO. This paper considers the effect of the response surface error on the optimum design of RBDO. The error is expressed in terms of the prediction interval, which is then used for RBDO. The prediction interval provides the upper and lower bounds for the given confidence level that the design engineer specified. Using the upper and lower limits and the response surface of the performance function, the upper and lower limits of the reliability can be computed. The lower limit of the reliability is compared with the lower target reliability index to obtain a conservative optimum design. However, in order to avoid obtaining a too conservative design that may not be close to the true optimum design, the upper limit of reliability is bounded above by the upper target reliability index.

Figure 1 shows the error in the estimation of the reliability in RBDO using RSM. Even though the reliability index $\beta_{RSM}$ using the response surface satisfies the target reliability index, the true reliability index $\beta_{true}$ may not satisfy the reliability constraint, and thus the design is not reliable. Note that in this figure and in the following figures, the reliability index $\beta$ (distance from the design point to the most probable failure point (MPP)) is meaningful only for the $U$-space but not for the $X$-space. That is, the reliability index shown in the $X$-space in these figures is used for the purpose of explanation.

Figure 2(a) shows the estimated interval of the response with the upper and lower limit surfaces of the constraint function, $G_U$ and $G_L$, respectively on the standard normal distribution $U_1$-$U_2$ domain. Therefore, the reliability index, which is the distance between the origin and MPP, can be evaluated as an interval $[\beta_{L,RSM}, \beta_{U,RSM}]$. Figure 2(b) shows the upper and lower limits of the reliability index on the original random variable $X_1$-$X_2$ domain.

The conventional RSM-based RBDO method uses $\beta_{RSM}$, whereas the proposed method uses $\beta_{L,RSM}$ and $\beta_{U,RSM}$ to find an optimal design. $\beta_{L,RSM}$ is used to satisfy the lower target reliability index, and $\beta_{U,RSM}$ is used to bind the upper target reliability index in the RBDO process. The proposed design optimization formulation will be described in Sec. 4.
3 Confidence and Prediction Interval for the Moving Least Squares Method

3.1 Confidence Interval of the Response Surface of the Moving Least Squares Method. Suppose that there are \( n \)-response values, \( y_i \), with respect to the changes of \( x_j \), which denote the \( i \)th observation of variable \( x_j \). The relation can be expressed as

\[
y_i = a_0 + a_1 x_{i1} + a_2 x_{i2} + \cdots + a_k x_{ik} + e_i
\]

where \( a_j \) is the coefficient of the \( j \)th variable and \( e_i \) is an approximation error of the \( i \)th observation. The equation can be defined as a matrix form,

\[
y = Xa + \epsilon
\]

where \( \epsilon \) is an \((n \times 1)\) matrix of approximation errors. It is assumed that the approximation error \( \epsilon \) has \( E(\epsilon) = 0 \) and \( \text{Var}(\epsilon) = \sigma^2 \).

For MLSM, the least squares function \( L(x) \) could be defined as in the following equation, which is the weighted sum of squared errors:

\[
L(x) = \sum_{j=1}^{n} w_j e_j^2 = e^T W(x) e = (y - Xa)^T W(x) (y - Xa)
\]

where \( x \) is the location where the approximation is sought. This locally weighted approximation can be performed from the consideration of effective data near the approximation location \( x \), and the data can be weighted according to the distance from the approximation location \( x \).

Note that the diagonal weighting matrix, \( W(x) \), is not a constant matrix in MLSM. In other words, \( W(x) \) is a function of the approximation location \( x \), and it can be obtained by utilizing weighting functions. There are various types of weighting functions among which two functions are defined by the following equation:

\[
w(x - x_j) = w(d) = \begin{cases} 
\text{polynomial} & 1 - 6(d/R)^2 + 8(d/R)^3 - 3(d/R)^4 \\
\text{exponential} & \exp(-d/R) 
\end{cases}
\]

where \( x_j \) is a vector of the \( j \)th sampling point, \( d \) is the distance between \( x = [x_1, x_2, \ldots, x_N]^T \) and \( x_j = [x_{j1}, x_{j2}, \ldots, x_{jN}]^T \) (i.e., \( d = \sqrt{\sum_{j=1}^{N} (x_{ji} - x_{j})^2} / \sqrt{N} \), where \( N \) is the number of variables), \( w(d) \) is a weighting function at distance \( d \), and \( R \) is the size of the approximation region, which is a predefined value in this paper.

The polynomial weighting function, which is used in this paper, is expressed by a bell-shaped function. The weighting function has the maximum value of 1 at 0, the normalized supporting size, and the minimum value of 0 outside of the normalized support, i.e., \( w(0) = 1 \) and \( w(d/R > 1) = 0 \). Also the function decreases smoothly from 1 to 0. The weighting matrix, \( W(x) \), can be constructed using the weighting function in diagonal terms [29].

\[
W(x) = \begin{bmatrix}
w(x - x_1) & 0 & \ldots & 0 \\
0 & w(x - x_2) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & w(x - x_0)
\end{bmatrix}
\]

For MLSM, minimizing Eq. (3) gives the least squares estimator \( b(x) \) of \( a \) as

\[
b(x) = (X^T W(x) X)^{-1} X^T W(x) y
\]

The variance property of \( b(x) \) is expressed by the covariance matrix

\[
\text{Cov}(b(x)) = \text{Cov}((X^T W(x) X)^{-1} X^T W(x) y) = (X^T W(x) X)^{-1} X^T W(x) \text{Cov}(y) X(X^T W(x) X)^{-1}
\]

The sum of squares of the residuals is

\[
R = \sum_{j=1}^{n} w_j (y_j - \hat{y}_j)^2
\]

where \( \hat{y}_j \) is the predicted value of \( y_j \).
Equation (18) implies that the mean response at point \( x \) will be within the confidence interval with a probability of 100(1 - \( \alpha \)) based on the sampled information. In Eq. (18), \( t_{a/2,n-p} \) is a decreasing function of \( \alpha \) and \( (n-p) \). Generally, as the probability increases (i.e., \( \alpha \) decreases), the interval length increases. As the number of sample points to generate the response surface increases, the interval length decreases. In addition, it can be seen from Eqs. (18) and (25) that \( \text{Var}[\hat{y}(x)] < \text{Var}[y_0 - \hat{y}(x)] \) since \( \text{Var}[y_0 - \hat{y}(x)] = \sigma^2 + \text{Var}[\hat{y}(x)] \). Therefore, the prediction interval is always wider than the confidence interval except when \( \sigma^2 = 0 \). This characteristic can also be observed in the example in Sec. 3.4.

### 3.2 Prediction Interval of the Response Surface of Moving Least Squares Method

The confidence interval is for estimating the interval of the mean response. However, the prediction interval is for predicting the interval of “the value of a single future observation” at a point. Therefore, the prediction interval of the response surface is used for the design optimization in this paper.

A point estimate for the future observation \( y_0 \) at point \( x \) is

\[
\hat{y}(x) = x_0^T b(x)
\]

where \( x_0^T = [1 \ x^T] \). The expected value of the prediction error is

\[
E(y_0 - \hat{y}(x)) = \mu_{y,y_0} - \mu_{\hat{y},x} = 0
\]

and the variance of the prediction error is

\[
\text{Var}[y_0 - \hat{y}(x)] = \sigma^2(1 + x_0^T X^T W X)^{-1} x_0
\]

Using \( \sigma^2 \) instead of \( \sigma^2 \), the quantity

\[
T = \frac{y_0 - \hat{y}(x)}{\sqrt{\text{Var}[y_0 - \hat{y}(x)]}}
\]

has a \( t \) distribution with an \( n-p \) degree of freedom. Thus,

\[
P(-t_{a/2,n-p} \leq T \leq t_{a/2,n-p}) = 1 - \alpha
\]

and

\[
P(\hat{y}(x) - t_{a/2,n-p} \sqrt{\text{Var}[y_0 - \hat{y}(x)]} \leq y_0 \leq \hat{y}(x) + t_{a/2,n-p} \sqrt{\text{Var}[y_0 - \hat{y}(x)]}) = 1 - \alpha
\]

Therefore, a 100(1 - \( \alpha \)% prediction interval for the future observation at point \( x \) is

\[
\hat{y}(x) - t_{a/2,n-p} \sqrt{\text{Var}[y_0 - \hat{y}(x)]} \leq y_0 \leq \hat{y}(x) + t_{a/2,n-p} \sqrt{\text{Var}[y_0 - \hat{y}(x)]}
\]

Equation (25) implies that one future response at point \( x \) will be within the prediction interval with a probability of 100(1 - \( \alpha \))% based on the sampled information. In Eq. (25), \( t_{a/2,n-p} \) is a decreasing function of \( \alpha \) and \( (n-p) \). As before, as the probability increases, the interval length increases. As the number of sample points to generate the response surface increases, the interval length decreases.

### 3.3 On the Assumption of Randomness of Response Surface Approximation Error

This paper assumes that the response surface approximation error \( e \) can be treated as a random variable with a normal distribution, which has \( E(e) = 0 \) and \( \text{Var}(e) = \sigma^2 \). Therefore, it is investigated whether the assumption is reasonable by using test functions. The first test function is

\[
X_1 X_2 / 20 - 1
\]

where \( [X_1, X_2] = [0, 10] \). From the explicit function, 36 points are sampled by a Latin-hypercube design, and the response surface is constructed using MLSM with a full quadratic basis model.

Figure 3 shows a frequency histogram and a normal quantile plot (or quantile-quantile plot) of the residuals from 36 sampling points using JMP, which is a statistical software [35]. The residual is the difference between the approximated response and the sampled value at the sampled location. If the normal quantile plot is a straight line, the distribution is a normal distribution [27], and the normal quantile plot in Fig. 3 shows that it is close to a straight line. In addition, the Shapiro–Wilk test [33,34] is used to test the normality of the approximate error. Generally, if a \( p \)-value [27,35] of the test statistic is greater than 0.05, we can say that the test variable has a normal distribution. For this example, the test statistic \( W \) is 0.9865 and the \( p \)-value is 0.9510. Thus, we can say that the approximation error of the given example behaves like a normal random variable.

After the construction of the response surface, the error is calculated at 10201 points (101 \times 101 points) to check the global behavior. The sample variance \( \hat{\sigma}^2 \) from the 36 sampling points is 0.117589, and the variance \( \sigma^2 \) from the 10201 sampling points is 0.112380. This means that the sample variance is very close to the true variance, and the sample variance can be used to estimate the system characteristic.

The second normality test function is [36]

\[
f = 1 - (Y - 6)^2 - (Y - 6)^3 + 0.6(Y - 6)^4 - Z
\]

where

\[
Y = 0.9063 X_1 + 0.4226 X_2
\]

\[
Z = 0.4226 X_1 - 0.9063 X_2
\]

\[
[X_1, X_2] = [0, 10]
\]

The first test function is mildly nonlinear, but the second one is highly nonlinear. Figure 4 shows the graphical result of the normality test, whereas according to the Shapiro–Wilk test, \( W \) is
0.9691 and the p-value is 0.4866. Thus, we can say that the approximation error of the given example behaves like a normal random variable.

3.4 Test of the Interval Estimation. The confidence and prediction intervals are tested for a system defined by

\[ f(X) = 0.5X^5 - 1.5X^4 - 2.5X^3 + 0.53X^2 + 1.3X + 2.0 \]  

where \( X = [-1, 1] \). Evenly distributed 11 points are sampled to construct the response surface using MLSM. The intervals are computed for the 95%, 90%, 80%, and 70% confidence levels. Figure 5 shows the true response, \( Y_{true} \); the response surface value, \( Y_{rsm} \); the lower and upper limits of the confidence interval, \( CIL \) and \( CIU \), respectively; the lower and upper limits of the prediction interval, \( PIL \) and \( PIU \), respectively; and the sampled points, \( Experi \). These figures show that as the confidence level increases, the confidence/prediction intervals become wider, which should be the case.

After constructing the response surface using MLSM, 1001 test points are sampled from the true function shown in Fig. 5. For these 1001 test points, the number of test points that are inside the confidence/prediction interval is counted. Table 1 shows the test results for different confidence levels. These test results show that the confidence interval may not be suitable for RBDO since the percentage of test points within the confidence interval is less than the required confidence level, and thus the prediction interval is recommended.

The sample variance \( \hat{\sigma}^2 \) from the 11 sampling points is 0.060493, and the variance \( \sigma^2 \) from the 1001 test points is 0.042172. The sample variance \( \hat{\sigma}^2 \) is larger than the variance \( \sigma^2 \), which could be treated as the true variance. Since the sampling points have larger variance than the true variance, the prediction interval will cover the true responses up to the required confidence level. If the sample variance is smaller than the true variance, the prediction interval may not contain enough of the true responses.

4 Design Improvement of RBDO Using Response Surface Method Considering Prediction Interval

4.1 Main Concept of the Proposed Method. The conventional RBDO problem is defined by

Minimize \( f(d) \)

Subject to \( P(G_j(X) \leq 0) \geq 1 - \Phi(-\beta_j), \quad j = 1, \ldots, n_{con} \)
where $X$ is the vector of random variables, $d$ is the vector of mean values of $X$, $G_j$ is the limit state function (performance function) ($G_j<0$ is regarded as failure), $\beta_{ij}$ is the required target reliability index of the $j$th constraint, $\text{ndv}$ is the number of design variables, and $\text{ncon}$ is the number of constraints. The RBDO problem can be reformulated, using the reliability index approach (RIA) [1-7], as

\[
\begin{aligned}
& \min \ f(d) \\
\text{Subject to} \quad & \beta_{ij} \leq \text{Reliability}[G_j(X)=0], \quad j = 1, \ldots, \text{ncon} \\
& d_i^l \leq d_i \leq d_i^U, \quad i = 1, \ldots, \text{ndv}
\end{aligned}
\]  

(30)

If RSM is used, the following problem is solved, instead of the problem in Eq. (30):

\[
\begin{aligned}
& \min \ f(d) \\
\text{Subject to} \quad & \beta_{ij} \leq \beta_{ij}^{\text{RSM}}, \quad j = 1, \ldots, \text{ncon} \\
& d_i^l \leq d_i \leq d_i^U, \quad i = 1, \ldots, \text{ndv}
\end{aligned}
\]  

(31)

where $\beta_{ij}^{\text{RSM}}$ is the reliability index that is computed using the response surface. However, this problem may yield an unreliable optimum design because of the approximation error of the response surface.

Thus, it is proposed to use the prediction interval in the design process. That is, the new RBDO formulation, using the prediction interval, is defined as

\[
\begin{aligned}
& \min \ f(d) \\
\text{Subject to} \quad & \beta_{ij} \leq \beta_{ij}^{\text{RSM}} \quad \text{for the confidence level of } (1-\alpha) \\
& j = 1, \ldots, \text{ncon} \\
& d_i^l \leq d_i \leq d_i^U, \quad i = 1, \ldots, \text{ndv}
\end{aligned}
\]  

(32)

where $\beta_{ij}^{\text{RSM}}$ is the lower limit of the reliability using the prediction interval obtained for the confidence level of $(1-\alpha)$. This formulation will yield a reliable optimum design. However, this formulation could yield a too conservative design that may not be close to the true optimum. Thus, it is proposed to bind the upper limit of the reliability $\beta_{ij}^{\text{RSM}}$ in the design optimization formulation. The upper limit of the reliability of the performance function is considered only when the lower limit of the reliability becomes an active constraint. The proposed formulation is integrated with an adaptive sampling strategy that is performed at MPP in this paper.

The proposed RSM-based RBDO formulation is to adaptively add sampling points until the following optimization process is completed:

\[
\begin{aligned}
& \min \ f(d) \\
\text{Subject to} \quad & \beta_{ij} \leq \beta_{ij}^{\text{RSM}} \\
& \text{if } \beta_{ij}^{\text{RSM}} \text{ is active impose the constraint } \beta_{ij}^{\text{RSM}} = \beta_{ij}^{\text{Uj}} \\
& \text{for the confidence level of } (1-\alpha), \quad j = 1, \ldots, \text{ncon} \\
& d_i^l \leq d_i \leq d_i^U, \quad i = 1, \ldots, \text{ndv}
\end{aligned}
\]  

(33)

where $\beta_{ij}^{\text{Lj}}$ and $\beta_{ij}^{\text{Uj}}$ are the lower and upper target reliability indices, respectively, and $\beta_{ij}^{\text{RSM}}$ and $\beta_{ij}^{\text{Uj}}$ are the lower and upper limits of the reliability using the prediction interval, respectively. For the specified confidence level, if the interval $[\beta_{ij}^{\text{Lj}}, \beta_{ij}^{\text{Uj}}]$ is small, the uncertainty of the response surfaces is low (i.e., accurate response surface). However, the smaller interval may require a larger number of sampling points. On the other hand, if the interval is large, the uncertainty of the response surfaces is high and the optimum design obtained will not be close to the true optimum design. Therefore, there is a tradeoff between the uncertainty of the response surface and the number of sampling points.

4.2 Computational Process of the Proposed RSM-Based RBDO Method. Figure 6 shows the overall computational process to solve the proposed formulation, which is described below.

(a) Take the initial sampling based on a conventional or a user-defined design of experiment.

(b) Construct the response surfaces from the analysis results at the sampling points.

(c) Solve the RBDO problem using the constraints $\beta_{ij} \leq \beta_{ij}^{\text{RSM}}$, $j = 1, \ldots, \text{ncon}$.

(d) If the RBDO of step (c) is converged, go to the next step; otherwise, iterate the RBDO process.

(e) Check step (f) to step (g) for all constraints.

(f) If the $j$th constraint $\beta_{ij} \leq \beta_{ij}^{\text{RSM}}$ is active, go to the next step; otherwise, go to step (e).

(g) If $\beta_{ij}^{\text{RSM}} \leq \beta_{ij}^{\text{Uj}}$ is satisfied, go to step (e); otherwise, save the value $\beta_{ij}^{\text{RSM}}$ and go to step (e).

(h) If all constraints are satisfied, finish the optimization; otherwise, sample an additional point at the MPP of the dominant constraint. The constraint for which $(\beta_{ij}^{\text{RSM}} - \beta_{ij}^{\text{Uj}})$ is the maximum is the dominant constraint. Note that this step can be applied only for the constraint where $\beta_{ij} \leq \beta_{ij}^{\text{RSM}}$ is active. After the additional sampling at the MPP of the dominant constraint, go to step (c).

During the sequential adaptive sampling procedure, one additional (sequential) point is sampled at the MPP of the dominant constraint of the current design. This additional (sequential) point will be different since each sequence yields a different design. However, a more efficient sampling strategy should be investigated thoroughly in the future.
5 Numerical Example

5.1 Mathematical Design Problem. The given RBDO problem is defined to

Minimize \( \cos (d_1, d_2) = d_1^2 + d_2^2 \)

Subject to \( P[G_j(X) \geq 0] \geq 1 - \Phi(\beta_j), \quad j = 1, 2, 3 \)

\( G_1(X) = X_1^2X_2/20 - 1 \)
\( G_2(X) = (X_1 + X_2 - 5)^2/30 + (X_1 - X_2 - 12)^2/120 - X_1X_2/300 - 1 \)
\( G_3(X) = 80/(X_1^2 + 8X_2 + 5) - 1 \)

\( 0 \leq d_i \leq 10, \quad i = 1, 2 \)

where \( X_i \sim N(d_i, 0.3), \quad i = 1, 2. \) and \( \beta_j = 3.0, \quad j = 1, 2, 3. \)

The problem has two normal random variables \( X_1 \) and \( X_2 \) with the corresponding mean values \( d_1 \) and \( d_2 \) and a variance of 0.3 for both variables. The number of the probabilistic constraints is 3, and the target reliability index is 3.0 for all constraints. Figure 7(a) shows the design domain and the contour plot of the objective and the constraints.

5.2 RBDO Results of Different Formulations for the Mathematical Problem. Four different design formulations are used to solve the given problem: (a) RBDO formulation using the true function as defined in Eq. (30), (b) RBDO formulation using the response surface as defined in Eq. (31), (c) RBDO formulation using only the lower limit of the prediction interval of reliability as defined in Eq. (32), and (d) the proposed RBDO formulation using both the upper and lower limits of the prediction interval of reliability as defined in Eq. (33), with the adaptive sampling strategy.

Sixteen points are initially sampled by the Latin-hypercube design to construct the response surface and DOT is used as an optimizer [37]. The predefined parameters are \( \alpha = 0.1 \) (90% confidence level), \( \beta_{L, i} = \beta_{U, i} = 3.0, \) and \( \beta_{U, j} = 4.0. \)

Table 2 and Fig. 7 show the optimum design results for differ-
ent formulations. In Table 2, “Estimated $\beta(G_i)$ by RS” means that the predicted values or intervals of reliability index of $G_i$ using the obtained RSs at the optimal design for each formulation. “True $\beta(G_i)$” means the true reliability of $G_i$ at the optimal design for each formulation. True $\beta(G_i)$ is computed using the true functions to get the true reliability of the design for the purpose of comparison. Therefore, true $\beta(G_i)$ is computed to test the performance of each formulation. Formulation (a) provides the accurate result but requires a sensitivity analysis. Formulation (b) may provide a solution that is not reliable enough to satisfy the target reliability because this formulation provides a point estimation using RS. The estimated reliability of $G_1$ using RS is 3.00449, and the true reliability of $G_1$ turns out to be 3.58429. Thus, the design is reliable with respect to this performance measure because the target reliability index is 3.0. However, this first constraint happens to be reliable due to the positive error, which we have no control of. The estimated reliability of $G_2$ using RS is 3.00211, and the true reliability of $G_2$ is 2.90484, which is not reliable due to the negative error. Therefore, formulation (b) cannot provide a confident reliability of the optimal design.

Formulation (c) provides the 90% confidence level that the optimal design satisfies the target reliability of 3.0. Indeed Table 2 shows that the estimated reliability intervals of $G_1$, $G_2$, and $G_3$ are [2.999, 4.333], [3.031, 3.171], and [4.460, 12.300], respectively, and the true reliabilities are 4.16928, 3.09381, and 8.50783, respectively. We can see that all the true reliabilities are within the prediction intervals. Note that the upper limit of the prediction interval can be computed, but formulation (c) does not control the upper limit. As a result, the reliability of $G_2$ is reasonable, but the reliability of $G_1$ is too high, and thus, the optimum design is too conservative.

Formulation (d) provides 90% confidence level that the reliability indices of active constraints of the optimum design are between 3.0 and 4.0. Table 2 shows that the estimated reliability intervals of $G_1$, $G_2$, and $G_3$ are [3.000, 3.981], [2.999, 3.197], and [4.362, 11.942], respectively, and the true reliabilities are 3.48106, 3.06378, and 9.21225, respectively. The result of formulation (d) shows reasonable reliabilities for $G_1$ and $G_2$, which are 3.48106 and 3.06378, respectively. Formulation (d) is carried out using additional 14 sample points adaptively at the MPPs of the optimum design obtained using the previous response surface, and therefore a total of 30 sampling points are used during the whole optimization process. Table 3 shows the additional sample points, and Table 4 shows the optimization history with the adaptive sampling.

Figure 8 shows the contour plots of the design result of formulation (d) at different sampling stages: (a) initial 16 sampling (total 16 points), (b) initial 16+ additional 5 (total 21 points), (c) initial 16+ additional 10 (total 26 points), and (d) initial 16+ additional 14 (total 30 points). The small circles on the figures are the sample points at each stage. These figures show that the prediction interval is getting narrower as additional sample points are used, and Table 4 shows that the optimum point is converged to the final result. Figure 8(a) is identical with Fig. 7(c) and 8(d) is the same with Fig. 7(d). The final result shows that at least 30 points are necessary to satisfy the target reliability interval under the given confidence level. It is not easy to determine the number of sampling points that is required to successfully carry out the RSM-based RBDO. However, the proposed method provides an error metric to effectively estimate the confidence level of the optimized design of the RBDO problem. This is the major advantage of the proposed method. Obviously, the initial sampling and the additional sampling strategy can affect the result significantly. Since this paper used the additional sampling strategy at the dominant MPP, as explained in Sec. 4.2, all the added points are located at the previous-step MPP of the first constraint. This adaptive sampling strategy may not be effective, and therefore further research is required.

### Table 2 Comparison of design results

<table>
<thead>
<tr>
<th>Method</th>
<th>Response</th>
<th>Opt. $d_1$</th>
<th>Opt. $d_2$</th>
<th>Cost</th>
<th>Estimated $\beta(G_1)$ by RS</th>
<th>True $\beta(G_1)$</th>
<th>True $\beta(G_2)$</th>
<th>True $\beta(G_3)$</th>
<th>No. of sample points</th>
</tr>
</thead>
<tbody>
<tr>
<td>RDBO using true function</td>
<td>Real function</td>
<td>3.275454</td>
<td>3.00449</td>
<td>24.44493</td>
<td>N/A</td>
<td>3.00890</td>
<td>3.00554</td>
<td>3.09834</td>
<td>N/A</td>
</tr>
</tbody>
</table>

### Table 3 Additional sample points

<table>
<thead>
<tr>
<th>Total sample point</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>Cost at sample point</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>2.78562</td>
<td>3.46751</td>
<td>19.7833</td>
</tr>
<tr>
<td>18</td>
<td>2.73810</td>
<td>3.42071</td>
<td>19.1985</td>
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<tr>
<td>19</td>
<td>2.71037</td>
<td>3.39029</td>
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<tr>
<td>20</td>
<td>2.69212</td>
<td>3.38548</td>
<td>18.7090</td>
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<tr>
<td>21</td>
<td>2.67865</td>
<td>3.37551</td>
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<td>22</td>
<td>2.66799</td>
<td>3.36779</td>
<td>18.4602</td>
</tr>
<tr>
<td>23</td>
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<td>3.35916</td>
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</tr>
<tr>
<td>24</td>
<td>2.65307</td>
<td>3.35390</td>
<td>18.2874</td>
</tr>
<tr>
<td>25</td>
<td>2.64655</td>
<td>3.34939</td>
<td>18.2226</td>
</tr>
<tr>
<td>26</td>
<td>2.64089</td>
<td>3.34531</td>
<td>18.1654</td>
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<tr>
<td>27</td>
<td>2.63723</td>
<td>3.33934</td>
<td>18.1062</td>
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<td>28</td>
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<td>30</td>
<td>2.62454</td>
<td>3.33074</td>
<td>17.9820</td>
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</table>
tion spring stiffness (x-directional translational stiffness). The MEMS device has special characteristics like the scaling effect and etching process, which means that the system is under various uncertainties and therefore needs to apply RBDO.

Figure 9 shows four selected design variables, and Table 5 shows the design bounds. The shape RBDO problem is defined to

\[
\begin{align*}
\text{Maximize} & \quad k_x \\
\text{Subject to} & \quad P(G_j(X) \geq 0) \geq 1 - \Phi(-\beta_{ij}), \quad j = 1, 2, 3, 4 \\
& \quad G_1 = k_x - 200 \\
& \quad G_2 = 300 - k_x
\end{align*}
\]

where \( \beta_{ij} = 1.5 \), \( X_i \sim N(\mu_i, \sigma_i^2) \), \( i = 1, \ldots, 4 \)

<table>
<thead>
<tr>
<th>Sample</th>
<th>( X_1 ) (opt)</th>
<th>( X_2 ) (opt)</th>
<th>Cost</th>
<th>( g(1) )</th>
<th>( g(2) )</th>
<th>( g(3) )</th>
<th>Beta1L</th>
<th>Beta1U</th>
<th>Beta2L</th>
<th>Beta2U</th>
<th>Beta3L</th>
<th>Beta3U</th>
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<td>-0.0001</td>
<td>-0.0835</td>
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<td>12.1976</td>
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<td>-0.0005</td>
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<tr>
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<td>25.8583</td>
<td>0.0000</td>
<td>0.0000</td>
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<td>3.9806</td>
<td>2.9993</td>
<td>3.1965</td>
<td>4.3624</td>
<td>11.9422</td>
</tr>
</tbody>
</table>

Fig. 8 Contour plots of result of the formulation (d) for different sampling stages
Three different design formulations described in Sec. 5.2 is applied for the design confidence of the RBDO result. The response surface method using MLSM is applied, and the proposed automatically updated. For the computational efficiency, the re-changed, the shape of the FE model is changed and the meshes are

5.4 RBDO Results of Different Formulations for the DFS System. Three different design formulations described in Sec. 5.2 are used to solve the given problem: (b) RBDO formulation using the response surface, (c) RBDO formulation using only the lower limit of the prediction interval of reliability, and (d) the proposed RBDO formulation using both the upper and lower limits of the prediction interval of reliability with the adaptive sampling strategy. Initially 40 points are sampled by the Latin-hypercube design for the entire design domain. The predefined parameters are $\alpha = 0.1$ (90% confidence level), $\beta_{y,i} = \beta_{L,i} = 1.5$, and $\beta_{U,i} = 3.0$.

Table 5 shows the design results of three formulations. Formulation (b) provides the reliability of the active constraint $G_2$ to be $1.50021$, but the true reliability turns out to be 1.10647. Thus, this formulation provides a design that is not reliable. Formulation (c) gives an optimal design that satisfies the target reliability of 1.5 with 90% confidence level, while formulation (d) gives a design whose active reliabilities are between 1.5 and 3.0 under the same confidence level. Table 6 shows that true reliabilities are within the corresponding prediction interval for formulations (c) and (d). Formulation (d) is performed using additional 23 sample points according to the proposed sampling method, and therefore a total of 63 sample points are used during the optimization process. Comparing the results of (c) and (d) gives an interesting observation. Since formulation (d) uses more sample points, the confidence interval length is smaller than formulation (c) as expected. For the second constraint, (c) gives the prediction interval as $[1.500, 4.626]$ and (d) gives $[1.500, 2.986]$. However, the true reliabilities are $1.71623$ and $2.32588$, respectively, and the true reliability of case (c) is close to its lower bound. As a result, the design of case (c) is better than case (d) because (c) is less conservative than (d) in this example. This could be due to the sampling method, which needs to be improved.

Table 7 shows the optimization history with the adaptive sampling for formulation (d). “X(opt)” is the optimal point of the design variables, and “X(add)” is the additional sample point at each stage. Another observation is the history of the length of the prediction interval in Table 7. In some stage of the history, the interval length becomes longer than the previous stage even though an additional sample point is used. When a sample point is added, if the sample variance is increased, then the prediction interval length could be increased. From this observation, the future study about a more effective sampling method is required. It is noted that all the prediction intervals in the examples successfully include the true reliability indices, and thus the developed prediction interval is applicable to RBDO. Figure 10 shows the finite element model of the optimal design shape for the formulation (d) case.

### Table 5 Comparison of design results

<table>
<thead>
<tr>
<th>Design variable (µm)</th>
<th>Lower limit</th>
<th>Upper limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$ (inner length)</td>
<td>100</td>
<td>250</td>
</tr>
<tr>
<td>$X_2$ (outer length ratio)</td>
<td>0.5</td>
<td>2.0</td>
</tr>
<tr>
<td>$X_3$ (spring length)</td>
<td>100</td>
<td>150</td>
</tr>
<tr>
<td>$X_4$ (spring thickness)</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

### Table 6 Comparison of design results

<table>
<thead>
<tr>
<th>Method</th>
<th>RBDO (b)</th>
<th>Prediction interval</th>
<th>Proposed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Response</td>
<td>MLSM</td>
<td>MLSM</td>
<td>MLSM</td>
</tr>
<tr>
<td>Opt. $d_1$</td>
<td>249.993</td>
<td>249.992</td>
<td>249.998</td>
</tr>
<tr>
<td>Opt. $d_2$</td>
<td>0.697926</td>
<td>0.500046</td>
<td>0.679818</td>
</tr>
<tr>
<td>Opt. $d_3$</td>
<td>124.912</td>
<td>130.491</td>
<td>110.307</td>
</tr>
<tr>
<td>Opt. $d_4$</td>
<td>2.80122</td>
<td>2.90288</td>
<td>2.41748</td>
</tr>
<tr>
<td>Cost</td>
<td>$1.35566 \times 10^{10}$</td>
<td>$1.34131 \times 10^{10}$</td>
<td>$1.32654 \times 10^{10}$</td>
</tr>
<tr>
<td>Estimated $\beta(G_1)$ by RS</td>
<td>6.76482</td>
<td>[3.461, 9.143]</td>
<td>[4.081, 6.233]</td>
</tr>
<tr>
<td>Estimated $\beta(G_2)$ by RS</td>
<td>1.50021</td>
<td>[1.500, 4.626]</td>
<td>[1.500, 2.986]</td>
</tr>
<tr>
<td>True $\beta(G_1)$</td>
<td>8.0928</td>
<td>7.85957</td>
<td>5.70475</td>
</tr>
<tr>
<td>True $\beta(G_2)$</td>
<td>1.10647</td>
<td>1.71623</td>
<td>2.32588</td>
</tr>
<tr>
<td>No. of sample points</td>
<td>40</td>
<td>40</td>
<td>63 (=40+23)</td>
</tr>
</tbody>
</table>
The proposed procedure is combined with an adaptive sampling strategy to sample at MPP additionally to refine the response surface until both limits are satisfied. Therefore, this method can give a guideline for the sampling location and the convergence criterion. The most important advantage of the proposed method is that the method gives how much the design is reliable quantitatively (i.e., $[\hat{P}^{\text{RSM}}, \hat{P}^{\text{RSM}}]$) under the required confidence level and how many additional sampling points are necessary when using RSM. The numerical example shows the usefulness and the computational efficiency of the proposed method. To refine the proposed method, further research on the efficient additional sampling strategy and the accuracy of the prediction interval is required.

Acknowledgment

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References